Homologische Algebra

Encyclopedia of Mathematics

https://encyclopediaofmath.org/wiki/Homological_algebra

Homological algebra – The branch of algebra whose main study is derived functors on various categories of algebraic objects (modules over a given ring, sheaves, etc.).

AN INTRODUCTION TO HOMOLOGICAL ALGEBRA

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Learning homological algebra is a two-stage affair. First, one must learn the language of Ext and Tor and what it describes. Second, one must be able to compute these things, and, often, this involves yet another language: spectral sequences. The following exercise appears on page 105 of Lang's book, "Algebra": "Take any book on homological algebra and prove all the theorems without looking at the proofs given in that book."

AN INTRODUCTION TO HOMOLOGICAL ALGEBRA

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Homological algebra is a tool used to prove nonconstructive existence theorems in algebra (and in algebraic topology). It also provides obstructions to carrying out various kinds of constructions; when the obstructions are zero, the construction is possible. Finally, it is detailed enough so that actual calculations may be performed in important cases. The following simple ques-

Bsp: $G \curvearrowright A$

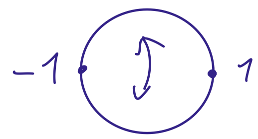
$$A^G := \{ a \in A \mid g \cdot a = a \}$$

Frage: $G \curvearrowright A, B$ $\qquad A \twoheadrightarrow B$

Ist dann auch $A^G \to B^G$ surjektiv ?

Bsp: $C_2 = \{\pm 1\}$ $\qquad C_2 \curvearrowright \mathbb{R}, \mathbb{Z}, \mathbb{Z}/4$

$C_2 \curvearrowright S^1$



$$\begin{array}{ccc} \mathbb{Z}/15 & \twoheadrightarrow & \mathbb{Z}/3 \\ \cup\!\mid & & \cup\!\mid \\ \{\underline{0}\} & \twoheadrightarrow & \{\bar{0}\} \end{array} \qquad \begin{array}{ccc} \mathbb{Z} & \twoheadrightarrow & \mathbb{Z}/4 \\ \cup\!\mid & & \cup\!\mid \\ \{0\} & \to & \{\bar{0}, \bar{2}\} \end{array} \qquad \begin{array}{ccccc} \mathbb{R} & \twoheadrightarrow & \mathbb{R}/\mathbb{Z} & \cong & S^1 \\ \cup\!\mid & & \cup\!\mid & & \cup\!\mid \\ \{0\} & & \{0, \tfrac{1}{2}\} & & \{\pm 1\} \end{array}$$

✓ $\qquad$ ✗ $\qquad$ ✗

Satz: $A \overset{\pi}{\twoheadrightarrow} B$

Falls $\quad H^1(G; \ker \pi) = 0,$

dann

$$A^G \overset{\pi}{\twoheadrightarrow} B^G$$

Bsp: spaltende Sequenzen

kurze exakte Sequenz abelscher Gruppen $\leadsto$ $A \hookrightarrow B \twoheadrightarrow B/A$

sie spaltet $\leadsto$ $B \cong A \oplus B/A$

Bsp: $\mathbb{Z} \hookrightarrow \mathbb{Z} \oplus \mathbb{Z}/2 \twoheadrightarrow \mathbb{Z}/2$ spaltet

$x \mapsto (x, 0)$

$(0, x) \mapsto x$

$2\mathbb{Z} \hookrightarrow \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$ spaltet nicht

$\mathbb{Z} \overset{?}{\cong} \mathbb{Z} \oplus \mathbb{Z}/2 \leftarrow \exists x \neq 0 : 2x = 0$

Satz: Genau dann spaltet jede exakte Sequenz

$$A \hookrightarrow ? \twoheadrightarrow C$$

wenn gilt:

$$\mathrm{Ext}^1(C, A) = 0$$

Bsp (Weibel):

$$\begin{array}{l} B \\ \cup\!\!\mid \\ A \end{array} \quad \text{abelsche Gruppe} \qquad n \in \mathbb{N}$$

Wann ist $\quad nA = A \cap nB$?

Bsp: $\begin{array}{l} B = \mathbb{Z} \\ \cup\!\!\mid \\ A = 2\mathbb{Z} \end{array}$ $\qquad nB = 3\mathbb{Z}$

$n = 3$

$$nB \cap A = 3\mathbb{Z} \cap 2\mathbb{Z} = 6\mathbb{Z} = nA \checkmark$$

$$B/A = \{b \in \mathbb{Z}/2 \mid 3b = 0\} = \{0\}$$

Bsp: $\begin{array}{l} B = \mathbb{Z} \\ \cup\!\!\mid \\ A = 6\mathbb{Z} \end{array}$ $\qquad nB = 3\mathbb{Z}$

$n = 3$

$$nB \cap A = 6\mathbb{Z}$$
$$\neq$$
$$nA = 18\mathbb{Z}$$

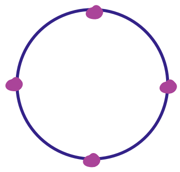
$$\underbrace{\{b \in \mathbb{Z} \mid 3b = 0\}}_{\{0\}} \longrightarrow \underbrace{\{b \in \mathbb{Z}/6 \mid 3b = 0\}}_{\{\bar{0}, \bar{2}, \bar{4}\}} = B/A$$

Bsp: $B = S^1$ $\qquad nB = (S^1)^3 = S^1$

$A = C_4$ $\qquad nB \cap A = C_4$

$\qquad\qquad\qquad\qquad\qquad\qquad \|$

$n=3$ $\qquad nA = (C_4)^3 = C_4$

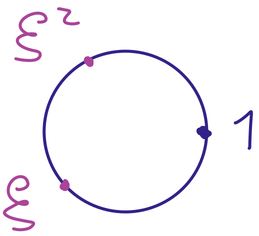


$$\{b \in S^1 \mid b^3 = 1\} \longrightarrow \{b \in B/A \mid b^3 = 1\}$$

$$\| \qquad\qquad\qquad\qquad \cong \qquad\qquad \cong$$

$$\{1, \xi, \xi^2\} \qquad \times \qquad \{b \in S^1 \mid b^3 = 1\} = \{1, \xi, \xi^2\}$$

$\times 4$



Satz: $nA = A \cap nB$

genau dann, wenn

$$\{b \in B \mid nb = 0\} \longrightarrow \{\bar{b} \in B/A \mid n\bar{b} = 0\}$$

surjektiv ist. Insbesondere ist

$$nA = A \cap nB$$

falls $\{\bar{b} \in B/A \mid n\bar{b} = 0\} = 0$.

[Beweis des Satzes:

①

$$\begin{tikzcd} nA \arrow[r, hook] \arrow[d] & A \arrow[r, two heads] \arrow[d, "i"] & A/nA \arrow[d, "\bar{i}"] \\ nB \arrow[r, hook] & B \arrow[r, two heads] & B/nB \end{tikzcd}$$

$nA = nB \cap A \iff \bar{i}$ injektiv

(Diagrammjagd)

②

$$A \hookrightarrow B \twoheadrightarrow B/A$$

induziert ...

$$\mathrm{Tor}(A, \mathbb{Z}/n) \to \mathrm{Tor}(B, \mathbb{Z}/n) \xrightarrow{p} \mathrm{Tor}(B/A, \mathbb{Z}/n) \to A/nA \xrightarrow{\bar{i}} B/nB \to \frac{B/A}{n\, B/A}$$

lange exakte Sequenz

Daher: $\bar{i}$ injektiv $\iff p$ surjektiv

③

$$\begin{array}{ccc} \mathrm{Tor}(B, \mathbb{Z}/n) & \xrightarrow{p} & \mathrm{Tor}(B/A, \mathbb{Z}/n) \\ \| & & \| \\ \{b \in B \mid nb = 0\} & & \{\bar{b} \in B/A \mid n\bar{b} = 0\} \end{array}$$

]

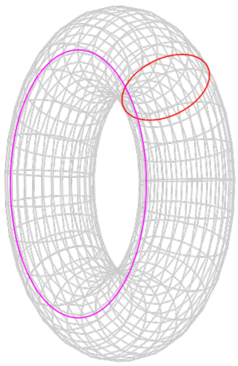
Meine Definition:

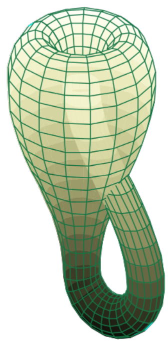
Homologische Algebra ist alles, was Sie brauchen, um meine Vorlesungen zur Algebraische Topologie (Topologie I & Topologie II) zu verstehen.

$$\text{Topologie} \xrightarrow{\text{Übersetzung}} \text{Algebra}$$

Bsp: $S^1 \times S^1 \cong K$?

Kleinsche Flasche





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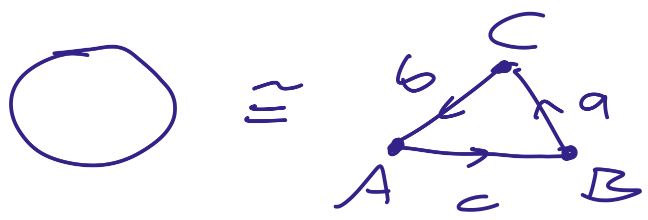
Ansatz:

$$\begin{array}{ccc} S^1 \times S^1 & \longmapsto & H_i(S^1 \times S^1) \\ \not\cong & \Longleftarrow & \not\cong \\ K & \longmapsto & H_i(K) \end{array}$$

Homologie-gruppen

$i = 0, 1, 2, \ldots$

Vorübung: $H_*(S^1)$



zellulärer Komplex

$$\mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c \xrightarrow{\delta} \mathbb{Z}\cdot A \oplus \mathbb{Z}\cdot B \oplus \mathbb{Z}\cdot C$$

$$a \longmapsto C - B$$
$$b \longmapsto A - C$$
$$c \longmapsto B - A$$

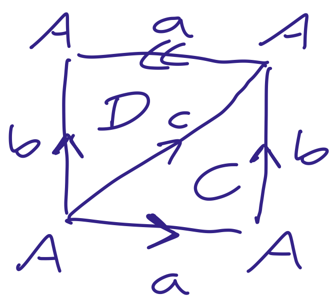
$$\mathbb{Z}^3 \xrightarrow{\delta} \mathbb{Z}^3$$

$$\begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$
$$\wr$$
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$H_1(S^1) = \ker(\delta) \cong \mathbb{Z}$$

$$H_0(S^1) = \mathbb{Z}^3 / \operatorname{im}\delta \cong \mathbb{Z}$$

$H_*(K)$



zellulären Komplex:

$$\mathbb{Z}\cdot C \oplus \mathbb{Z}\cdot D \xrightarrow{\delta_1} \mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c \xrightarrow{\delta_0 = 0} \mathbb{Z}\cdot A$$

$$\begin{aligned} C &\longmapsto a+b-c \qquad & a &\longmapsto 0 \\ D &\longmapsto c+a-b & b &\longmapsto 0 \\ & & c &\longmapsto 0 \end{aligned}$$

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix}} \mathbb{Z}^3 \xrightarrow{0} \mathbb{Z}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{pmatrix} \wr \cdots \wr \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$H_2(K) = \ker(\delta_1) = 0$$

$$H_1 = \ker(\delta_0) \big/ \operatorname{im}(\delta_1) = \mathbb{Z}^3 \big/ 2\mathbb{Z} \oplus \mathbb{Z} \oplus 0 = \mathbb{Z}/2 \oplus \mathbb{Z}$$

$$H_0(K) = \mathbb{Z} / \operatorname{im} \delta_0 = \mathbb{Z}$$

$H_*(S^1 \times S^1)$

Algebraic Topology

Allen Hatcher

top. spaces

Theorem 3B.6. *If X and Y are ~~CW complexes~~ and ~~R is a principal ideal domain~~, then there are natural short exact sequences*

$$0 \longrightarrow \bigoplus_i \left(H_i(X\cancel{;R}) \otimes_R H_{n-i}(Y\cancel{;R})\right) \longrightarrow H_n(X \times Y\cancel{;R}) \longrightarrow \bigoplus_i \mathrm{Tor}_R\left(H_i(X\cancel{;R}), H_{n-i-1}(Y\cancel{;R})\right) \longrightarrow 0$$

and these sequences split.

In unserem Fall $X = Y = S^1$ ist jeweils $\mathrm{Tor}(\ldots,\ldots) = 0$ und wir erhalten

$$\text{“} H_*(S^1 \times S^1) = H_*(S^1) \otimes H_*(S^1) \text{”}$$

$$= \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix} \otimes_{\mathbb{Z}} \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{Z} \\ \mathbb{Z} \oplus \mathbb{Z} \\ \mathbb{Z} \end{pmatrix}$$

$2 = 1 + 1$

$1 = 1 + 0 = 0 + 1$

$0 = 0 + 0$

$(\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Z})$

$H_2(S^1 \times S^1) \cong \mathbb{Z}$ $\quad$ $H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$ $\quad$ $H_0(S^1 \times S^1) = \mathbb{Z}$

Offenbar $H_2(K) \not\cong H_2(S^1 \times S^1)$

(und $H_1(K) \not\cong H_1(S^1 \times S^1)$),

also $K \not\cong S^1 \times S^1$ (nicht homöomorph)