THE T^1-LIFTING THEOREM IN POSITIVE CHARACTERISTIC

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Abstract. Replacing symmetric powers by divided powers and working over Witt vectors instead of ground fields, I generalize Kawamata's T^1-lifting theorem to characteristic $p > 0$. Combined with the work of Deligne–Illusie on degeneration of the Hodge–de Rham spectral sequences, this gives unobstructedness for certain Calabi–Yau varieties with free crystalline cohomology modules.

Introduction

According to a Theorem of Tian [23], Todorov [24], and Bogomolov [2], each infinitesimal deformation of a compact complex Kähler manifold with trivial canonical class extends to arbitrarily high order. In other words, the base of each versal deformation is a power series $\mathbb{C}$-algebra in finitely many variables. This is a remarkable fact, because the obstruction group for the problem is usually nonzero.

Generalizing Ran's work [20], Kawamata [14] proved a general result for functors of Artin $\mathbb{C}$-algebras called the *T^1-lifting Theorem*, and then deduced the result on Calabi–Yau manifolds using Deligne's theorem on cohomological flatness of Kähler differentials [3].

The proofs for these results work in characteristic zero only. The reason is particularly visible in Kawamata's approach: At some point in the proof he needs invertibility of the binomial coefficient $\binom{n}{1}$ in the binomial expansion of $(T+\epsilon)^n$. This seems to lie at the heart of the matter. In fact, Hirokado [11] constructed a Calabi–Yau manifold in characteristic $p = 3$ with obstructed deformations.

The goal of this paper is to extend, under suitable assumptions, the T^1-lifting Theorem to characteristic $p > 0$. The idea is simple: To get rid of the annoying binomial coefficient, I replace the power expansion $(T+\epsilon)^n = \sum \binom{n}{i} T^{n-i}\epsilon^i$ by a *divided power* expansion $\gamma^n(T+\epsilon) = \sum \gamma^{n-i}(T)\gamma^i(\epsilon)$. To have enough divided power algebras, we must work over truncated Witt vectors W_m, $m \geq 0$ instead of a fixed ground field k.

The main result is that, roughly speaking, a semihomogeneous cofibered groupoid over the category of Artin algebras over the Witt ring $W = W(k)$ with residue field k is smooth if it satisfies a suitable T^1-lifting property, and admits a formal object over the Witt ring $W = \varprojlim W_m$.

As an application, I deduce that Calabi–Yau manifolds X_0 in characteristic $p > 0$ with $\dim(X_0) \leq p$ that admit a formal deformation over $\mathrm{Spf}(W)$ are unobstructed, provided that, for certain divided power W-algebras A, the crystalline cohomology groups $H^r(X_0/A_{\mathrm{cris}}, \mathcal{O}_{X_0/A})$ are free A-modules. This relies on the Deligne–Illusie

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Theorem [4] on the degeneration of the Hodge–de Rham spectral sequence for manifolds liftable to Witt vectors of length two.

The paper has four sections. In the first section, I collect some elementary results on power series over discrete valuation rings. In the next section, I characterize smooth algebras in mixed characteristics in terms of lifting conditions using divided powers. In Section 3, we come to the T^1-lifting property and prove our main result for semihomogeneous cofibered groupoids. The last section contains the application to Calabi–Yau manifolds.

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1. Power series over discrete valuation rings

It is well-known that a k-algebra $R = k[[T_1, \ldots, T_r]]/I$ over an algebraically closed field k is formally smooth if and only if each k-map $R \to k[T]/(T^n)$ lifts to $k[T]/(T^{n+1})$. What can be said over more general ground rings? In this section, I collect some results valid over complete discrete valuation rings.

Let W be a complete discrete valuation ring, $\mathfrak{m}_W \subset W$ its maximal ideal, and $k = W/\mathfrak{m}_W$ the residue field. Choose a uniformizer $u \in W$. To avoid endless repetition, we say that a W-algebra R is *formal* if it is the quotient of some formal power series algebra $W[[T_1, \ldots, T_r]]$, and the map on residue fields $W/\mathfrak{m}_W \to R/\mathfrak{m}_R$ is bijective, where $\mathfrak{m}_R \subset R$ is the maximal ideal. Note that formal W-algebras are complete local rings. The following is well-known:

Lemma 1.1. *Each W-map of formal W-algebras is local and continuous.*

Proof. Let $\phi : R \to S$ be such a homomorphism. Write $R = W[[T_1, \ldots, T_r]]/I$ and $S = W[[X_1, \ldots, X_s]]/J$. Each $T_i - \phi(T_i)(0) \in R$ is a nonunit because it maps to a nonunit in S. So $\phi(T_i)(0) \in \mathfrak{m}_W$, and ϕ is a local homomorphism. This implies $\phi^{-1}(\mathfrak{m}_S) = \mathfrak{m}_R$, and ϕ is a continuous homorphism as well. □

Let me clarify what smoothness should mean in our context:

Lemma 1.2. *Let R be a formal W-algebra. Then the following are equivalent:*

(i) *We have $R \simeq W[[T_1, \ldots, T_r]]$ for some $r \geq 0$.*
(ii) *Given a W-map of formal Artin W-algebras $A \to A'$, each W-map $R \to A'$ lifts to A.*
(iii) *Given a W-map of W-algebras $A \to A'$ with nilpotent kernel, each W-map $R \to A'$ annihilating some power of $\mathfrak{m}_R$ lifts to A.*

Proof. The implications (i) ⇒ (iii) and (iii) ⇒ (ii) are trivial, and (ii) ⇒ (i) is explained in [22], Proposition 2.5. □

Condition (iii) is called $\mathfrak{m}_R$*-smoothness* in [18], Section 10, and *formal smoothness* in [7], Definition 19.3.1. For simplicity, we call a formal W-algebra *smooth* if it satisfies the equivalent conditions in Lemma 1.2. The task now is to relate properties of a formal W-algebra R to the existence of nice presentations $R = W[[T_1, \ldots, T_r]]/I$. For a power series $f = f(T_1, \ldots, T_r)$ with homogeneous components $f = \sum_{n=0}^{\infty} f_n$, let $\operatorname{ord}(f)$ be the smallest number $n \geq 0$ with $f_n \neq 0$.

Proposition 1.3. *Let R be a formal W-algebra. Then there is a W-map $R \to W$ if and only if for each presentation $R = P/I$ for some $P = W[\![T_1, \ldots, T_r]\!]$, there is a W-automorphism $\phi\colon P \to P$ so that* $\operatorname{ord}(f) \geq 1$ *for all $f \in \phi(I)$.*

Proof. Suppose there is a W-map $R \to W$. Lift it to a W-map $\psi\colon P \to W$, say with $\psi(T_i) = a_i$. Then $a_i \in \mathfrak{m}_W$. Hence $T_i \mapsto T_i + a_i$ defines a W-automorphism $\phi\colon P \to P$. We have $g(a_1, \ldots, a_r) = 0$ for all $g \in I$. Each $f \in \phi(I)$ is of the form $f = g(T_1 + a_1, \ldots, T_r + a_r)$ with $g \in I$, hence $f(0) = 0$.

Conversely, if $R = P/I$ with $\operatorname{ord}(f) \geq 1$ for all $f \in \phi(I)$, then $T_i \mapsto 0$ defines the desired W-map $R \to W$. □

This takes care of the constant terms. Next, we cope with the linear terms.

Lemma 1.4. *Let R be a formal W-algebra, and $R = W[\![T_1, \ldots, T_r]\!]/I$ a presentation. If $r \geq 0$ is minimal and* $\operatorname{ord}(f) \geq 1$ *for all $f \in I$, then each $f \in I$ has no linear term modulo $\mathfrak{m}_W$.*

Proof. Suppose to the contrary that some $f \in I$ has linear term $\sum_{i=1}^r \lambda_i T_i$ with at least one invertible coefficient λ_j. Rearranging the variables T_i, we may assume $j = 1$. Write $f = \sum_{n=0}^\infty g_n T_1^n$ with $g_n \in W[\![T_2, \ldots, T_r]\!]$. Then $g_0(0) = 0$ because $\operatorname{ord}(f) \geq 1$. Rewrite $f = g_0 + T_1(g_1 + g_2T_1 + \ldots)$. Now g_1 is a unit, so $h = g_1 + g_2T_1 + \ldots$ is a unit, therefore $T_1 \mapsto f$ defines an automorphism of P over $W[\![T_2, \ldots, T_r]\!]$. The inverse automorphism $\phi : P \to P$ satisfies $T_1 \in \phi(I)$, contradicting the minimality of $r \geq 0$. □

For each $m \geq 0$, define $W_m = W/(u^m)$. In what follows, the symbol ϵ shall denote an indeterminate satisfying $\epsilon^2 = 0$. For example, $W_m[\epsilon] = W_m[T]/(T^2)$.

Proposition 1.5. *Let R be a formal W-algebra, and $R = W[\![T_1, \ldots, T_r]\!]/I$ a presentation with $r \geq 0$ minimal, and* $\operatorname{ord}(f) \geq 1$ *for all $f \in I$. If each W-map $R \to W_{m+1}[\epsilon]/(u^m\epsilon)$ with $m \geq 0$ lifts to $W_{m+1}[\epsilon]$, then* $\operatorname{ord}(f) \geq 2$ *for all $f \in I$.*

Proof. Set $P = W[\![T_1, \ldots, T_r]\!]$. Seeking a contradiction, we assume that some $f \in I$ has a nonzero linear part $\sum_{i=1}^r \lambda_i T_i$. Choose such a power series $f \in I$ minimizing the integer $\min\{\operatorname{ord}_W(\lambda_1), \ldots, \operatorname{ord}_W(\lambda_r)\}$. Rearranging the variables T_i, we may assume that $\operatorname{ord}_W(\lambda_1) \leq \ldots \leq \operatorname{ord}_W(\lambda_r)$.

Set $m = \operatorname{ord}_W(\lambda_1)$, and consider the W-map $P \to W_{m+1}[\epsilon]/(u^m\epsilon)$ given by $T_1 \mapsto \epsilon$, and $T_i \mapsto 0$, $i \geq 2$. This induces a W-map $R \to W_{m+1}[\epsilon]/(u^m\epsilon)$, and by assumption there is a lifting $R \to W_{m+1}[\epsilon]$. Such a lifting is induced by a W-map $\phi\colon P \to W_{m+1}[\epsilon]$ annihilating $f \in I$, necessarily of the form

$$T_i \longmapsto \begin{cases} \epsilon + a_1 u^m \epsilon & \text{if } i = 1, \\ a_i u^m \epsilon & \text{if } i \geq 2, \end{cases}$$

for certain $a_i \in W$. We compute

$$0 = \phi(f) = \lambda_1(\epsilon + a_1 u^m \epsilon) + \sum_{i=2}^r \lambda_i a_i u^m \epsilon = \lambda_1 \epsilon,$$

because $\lambda_i \in \mathfrak{m}_W$ by Lemma 1.4. But $\lambda_1 \epsilon \neq 0$ in $W_{m+1}[\epsilon]$, a contradiction. □

The following tells us that smoothness is detectable on infinitesimal arcs.

Proposition 1.6. *Suppose k is algebraically closed. A formal W-algebra R is smooth if and only if given $g \in W[\![T]\!]$ that is either a unit or zero, and integers $n, d \geq 0$, each W-map $R \to W[\![T]\!]/(u - gT^n, T^d)$ lifts to $W[\![T]\!]/(u - gT^n, T^{d+1})$.*

Proof. The condition is clearly necessary. Suppose that such liftings exist. Write $R = P/I$ for some power series ring $P = W[\![T_1, \ldots, T_r]\!]$ with $r \geq 0$ minimal. Using the liftings in the special case $g = 1, n = 1$, we obtain a W-map $R \to W$. By Proposition 1.3, we may assume $\operatorname{ord}(f) \geq 1$ for all $f \in I$.

If $I = 0$ we are done. Seeking a contradiction, we assume $I \neq 0$. By Milnor's curve selection lemma ([19], Lem. 3.1), there is a discrete valuation ring A and a finite map $\phi\colon P \to A$ with $\phi(I) \neq 0$. Milnor's proof is complex algebraic, but the arguments in [5], Lemma 3.1 apply in our situation. The induced map on residue fields is bijective, because k is algebraically closed; if $W \to A$ is finite, then W is totally ramified because A is complete. If $W \to A$ is not finite, then $A = k[\![T]\!]$. In both cases we have $A = W[\![T]\!]/(u - gT^n)$ for some $n \geq 0$ and some power series $g \in W[\![T]\!]$ which is either a unit or zero, by [10], Theorem 1.

Note that $T \in A$ is a uniformizer. The map $\phi\colon P \to A$ is of the form $T_i \mapsto \lambda_i T^{d_i}$ for certain $d_i \geq 0$ and $\lambda_i \in A$. Choose $f \in I$ minimizing $d = \operatorname{ord}_A(\phi(f))$. Then $d \geq 1$ because $\operatorname{ord}(f) \geq 1$. The map $P \to A/\mathfrak{m}_A^d$ induces a map $R \to A/\mathfrak{m}_A^d$. By assumption, there is a lifting $R \to A/\mathfrak{m}_A^{d+1}$. The corresponding mapping $\phi\colon P \to A/\mathfrak{m}^{d+1}$ annihilates $f \in I$ and is of the form $T_i \mapsto \lambda_i T^{d_i} + \mu_i$ for certain $\mu_i \in \mathfrak{m}_A^d$. Write $f = \sum_{n \in \mathbb{N}^r} a_n T^n$, where $T^n = T_1^{n_1} \ldots T_r^{n_r}$ and $a_0 = 0$. Then

$$\phi(f) = f(\lambda_1 T^{d_1} + \mu_1, \ldots, \lambda_r T^{d_r} + \mu_r) = f(\lambda_1 T^{d_1}, \ldots, \lambda_r T^{d_r}),$$

because $a_n \in \mathfrak{m}_W$ whenever $a_n T^n$ is linear, by Lemma 1.4. By the choice of d, we have $f(\lambda_1 T^{d_1}, \ldots, \lambda_r T^{d_r}) \neq 0$ in $A/\mathfrak{m}_A^{d+1}$, a contradiction. □

2. Mixed characteristic and divided powers

In this section we shall encounter another family of Artin rings to test smoothness. The idea is to impose additional structure, namely *divided powers*. Recall that a PD-*ring* is a triple (R, I, γ), where R is a ring, $I \subset R$ is an ideal, and γ is a sequence of maps $\gamma^n\colon I \to R$ satisfying certain axioms. These axioms are listed in [1], Definition 3.1. They imply $(n!)\gamma^n(x) = x^n$. Indeed, $\gamma^n(x)$ serves as a substitute for $x^n/(n!)$, the latter making no sense if $n! \in R$ is not a unit. In our applications, $I \subset R$ is usually the maximal ideal of a local ring, and we simply say that R is a PD-ring.

In the following, we assume that our discrete valuation ring W is of characteristic zero, and that its residue field k is of characteristic $p > 0$. Choose a uniformizer $u \in W$, and let $e > 0$ be the absolute ramification index, defined by $u^e W = pW$. Note that the fraction field $W[u^{-1}]$ has a unique PD-structure $\gamma(x) = x^n/(n!)$, such that W has at most one PD-structure. According to [1], Example 3.2, the inclusion $W \subset W[u^{-1}]$ induces a PD-structure on the subring W if and only if $e < p$. Henceforth, we shall assume this, and regard W as a PD-ring. This automatically holds if $e = 1$, that is, if W is its own Cohen subring.

For each $m > 0$, define $W_m = W/(u^m)$, and consider the free PD-algebra in one variable $W_m\langle T\rangle$. Then

$$W_m\langle T\rangle = \bigoplus_{n \geq 0} W_m \cdot \gamma^n(T)$$

as abelian group. Note that $\operatorname{Spec}(W_m\langle T\rangle)$ contains but one point; the ring $W_m\langle T\rangle$, however, is nonnoetherian, because $T^n = (n!)\gamma^n(T)$ is zero if $\operatorname{ord}_W(n!) \geq m$. To obtain Artin W-algebras with compatible PD-structure, we have to divide by non-noetherian PD-ideals. Indeed,

$$(\gamma^d(T), \gamma^{d+1}(T), \gamma^{d+2}(T), \ldots) \subset W_m\langle T\rangle, \quad d \geq 0$$

is such an ideal, so the quotient

$$W_{m,d} = W_m\langle T\rangle/(\gamma^d(T), \gamma^{d+1}(T), \gamma^{d+2}(T), \ldots)$$

is a formal Artin W-algebra endowed with a compatible PD-structure.

Theorem 2.1. *Let R be a formal W-algebra. Then R is smooth if and only if the following three conditions holds:*

(i) *There is a W-map $R \to W$.*
(ii) *Given an integer $m \geq 0$, each W-map $R \to W_{m+1}[\epsilon]/(u^m\epsilon)$ lifts to $W_{m+1}[\epsilon]$.*
(iii) *Given $m, d > 0$, each W-map $R \to W_{m,d}$ lifts to $W_{m,d+1}$.*

Proof. The conditions are clearly necessary. For the converse, write $R = P/I$ for some power series algebra $P = W[\![T_1, \ldots, T_r]\!]$ with $r \geq 0$ minimal. Using conditions (i) and (ii) and Propositions 1.3 and 1.5, we may assume $\operatorname{ord}(f) \geq 2$ for all $f \in I$. If $I = 0$ we are done. Seeking a contradiction, we assume $I \neq 0$.

I claim that there is a sequence of integers $e_1, \ldots, e_r > 0$ so that $\varphi\colon P \to W[\![T]\!]$, $T_i \mapsto T^{e_i}$ has $\varphi(I) \neq 0$. To see this, choose a nonzero power series $f \in I$. Write $f = \sum_{n\in\mathbb{N}^r} a_n T_1^{n_1} \ldots T_r^{n_r}$, such that

$$\varphi(f) = f(T^{e_1}, \ldots, T^{e_r}) = \sum_{m=0}^{\infty}\Big(\sum_{n_1e_1+\ldots+n_re_r=m} a_n\Big)T^m.$$

As is the case of ground fields (compare [5], proof of Lemma 5.6), it is now easy to see that for some sequence $e_1, \ldots, e_r > 0$, there is an integer $m > 0$ so that there is precisely on $a_n \neq 0$ with $n_1e_1 + \ldots + n_re_r = m$. Then $\varphi(f) \neq 0$, hence $\varphi(I) \neq 0$.

Now fix such a W-map $\varphi\colon P \to W[\![T]\!]$ given by $T_i \mapsto T^{e_i}$. Let $d \geq 0$ be the smallest order occurring in $\varphi(I)$. Then $\varphi(I) \subset (T^d)$, and $d \geq 2$ because $\operatorname{ord}(f) \geq 2$ for all $f \in I$. By construction, φ induce a W-map

$$R \longrightarrow W\langle T\rangle/(\gamma^d(T), \gamma^{d+1}(T), \gamma^{d+2}(T), \ldots).$$

Next, choose $f \in I$ so that $\varphi(f)$ contains a nonzero monomial $\lambda_d T^d$, and choose an integer $m > \operatorname{ord}_W(\lambda_d \cdot d!)$. Consider the composite map

$$R \longrightarrow W\langle T\rangle/(\gamma^d(T), \gamma^{d+1}(T), \gamma^{d+2}(T), \ldots) \longrightarrow W_{m,d}.$$

By condition (iii), this lifts to $W_{m,d+1}$. The induced mapping $\phi\colon P \to W_{m,d+1}$ annihilates $f \in I$ and is of the form $T_i \mapsto T^{e_i} + \mu_i\gamma^d(T)$ for certain $\mu_i \in W$. We calculate

$$\varphi(f) = f(T^{e_1} + \mu_1\gamma^d(T), \ldots, T^{e_r} + \mu_r\gamma^d(T)) = f(T^{e_1}, \ldots, T^{e_r}),$$

because f has no linear terms and $e_i > 0$. But

$$f(T^{e_1}, \ldots, T^{e_r}) = \lambda_d T^d = (\lambda_d \cdot d!)\gamma^d(T)$$

is nonzero in $W_{m,d+1}$ by the choice of m and d, a contradiction. □

3. The T^1-Lifting criterion for cofibered groupoids

We keep the notation of the preceding section, such that W is a complete discrete valuation ring of mixed characteristic and absolute ramification index $e < p$. Let (Art/W) be the category of formal Artin W-algebras and W-maps. Theorem 2.1 characterizes those formal W-algebras R whose Yoneda functor

$$h_R\colon (\mathrm{Art}/W) \longrightarrow (\mathrm{Set}), \quad A \longmapsto \mathrm{Hom}_W(R, A)$$

is smooth. In Schlessinger's terminology [22], *functors of Artin rings* of the form h_R are called *prorepresentable*. Unfortunately, interesting functors of Artin rings are usually not prorepresentable. Rather, they satisfy a weaker condition, namely they admit a *hull*. It is therefore a good idea to extend results about prorepresentable functors to functors admitting hulls. To avoid the problems discussed in [17], I prefer to work with *cofibered groupoids* instead of functors of Artin rings, which allows us to keep track of automorphisms. Rather than repeating dull definitions, I shall refer to Rim's paper on formal deformation theory [21] and Grothendieck's article on fibered categories [9].

However, we should keep in mind the following example: Each functor of Artin rings $F\colon (\mathrm{Art}/W) \to (\mathrm{Cat})$ into the category of categories defines a cofibered groupoid $\mathcal{G}$ as follows: The objects in $\mathcal{G}$ are pairs (A, X), where A is a formal Artin W-algebra, and $X \in F(A)$ is an object. The morphism between (A, X) and (A', X') are the morphisms $\phi\colon A \to A'$ with $F(\phi)(X) = X'$. The same construction works if F is merely a pseudofunctor.

Throughout, we fix a *semihomogeneous cofibered groupoid* $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ (see [21], Def. 1.2 for definition). Then all fiber categories $\mathcal{G}_A = \pi^{-1}(A)$ are groupoids, and each morphism in $\mathcal{G}$ is *cocartesian*. We shall always assume that the fiber category $\mathcal{G}_k$ is punctual, that is, all homomorphism sets have precisely one element. Let $G = [\mathcal{G}]$ be the induced groupoid whose objects are the fiberwise isomorphism classes of $\mathcal{G}$. Note that $G \to (\mathrm{Art}/W)$ is a semihomogeneous cofibered groupoid, and G_k is a one element set; furthermore, the tangent space $G_{k[\epsilon]}$ is canonically a k-vector space.

For each W-map of formal Artin rings $\phi\colon A \to A'$ and each object $X \in \mathcal{G}$ over A, there is a cocartesian map $X \to X'$ over ϕ. We choose, once and for all, such cocartesian maps, and denote them by $\alpha_\phi(X)\colon X \to \phi_*(X)$. Furthermore, we assume that $(\mathrm{id}_A)_*(X) = X$ and $\alpha_{\mathrm{id}_A} = \mathrm{id}_X$. In other words, we have a *normalized clivage* [9], Section 7. This clivage defines *direct image functors* $\phi_*\colon \mathcal{G}_A \to \mathcal{G}_{A'}$, which are unique up to a unique natural transformation. We also write $X \otimes_A A'$ or $X \otimes A'$ for $\phi_*(X)$.

Note that, given $\phi\colon A \to A'$, we have a canonical bijection between the set of maps $f\colon X \to X'$ over ϕ and the set of pairs (X, g), where $X \in \mathcal{G}$ is an object over A, and $g\colon X \otimes_A A' \to X'$ is a map over A'.

Definition 3.1. Let A be a formal Artin W-algebra, and $X \in \mathcal{G}$ an object over it. Define $T^1(X/A)$ as the set of isomorphism classes of pairs (Y, h), where $Y \in \mathcal{G}$ is an object over $A[\epsilon]$, and $h\colon Y \to X$ is a morphism over $A[\epsilon] \to A$, $\epsilon \mapsto 0$.

Perhaps it goes without saying that a morphism $(Y_1, h_1) \to (Y_2, h_2)$ is a map $f\colon Y_1 \to Y_2$ over $A[\epsilon]$ with $h_2 \circ f = h_1$. A standard argument shows that $T^1(X/A)$ is canonically endowed with a W-module structure ([21], Rmk. 1.3). Furthermore, this construction is functorial: Given a map $f : X_1 \to X_2$, say over $\phi\colon A_1 \to A_2$, the direct image functors define a homomorphism of W-modules

$$f_*\colon T^1(X_1/A_1) \longrightarrow T^1(X_2/A_2), \quad (Y_1, h_1) \mapsto (Y_2, h_2).$$

Here $Y_2 = Y_1 \otimes_{A_1[\epsilon]} A_2[\epsilon]$, and $h_2\colon Y_2 \to X_2$ is the unique map over the projection $A_2[\epsilon] \to A_2$ making the diagram

$$\begin{array}{ccc} Y_1 & \xrightarrow{h_1} & X_1 \\ {\scriptstyle \alpha_{\phi[\epsilon]}(Y_1)}\downarrow & & \downarrow{\scriptstyle f} \\ Y_1 \otimes_{A_1[\epsilon]} A_2[\epsilon] & \xrightarrow[h_2]{} & X_2, \end{array}$$

commutative, see [9], Proposition 6.11.

A first order extension of formal Artin W-algebras is a surjective W-map $A \to A'$ whose kernel annihilates itself. Similarly, a *first order deformation* in $\mathcal{G}$ is a map $X \to X'$ whose image $A \to A'$ is a first order extension of formal Artin W-algebras.

Definition 3.2. We say that a semihomogeneous cofibered groupoid $\pi : \mathcal{G} \to (\mathrm{Art}\,/W)$ has the T^1*-lifting property* if the following two conditions hold:

(i) For each morphism $X \to X'$ over $W_{m+1} \to W_m$ with $m > 0$, the induced map $T^1(X/W_{m+1}) \to T^1(X'/W_m)$ is surjective.

(ii) For each morphism $X \to X'$ over $W_{m,d} \to W_{m,d-1}$ with $m > 0, d > 1$, the induced map $T^1(X/W_{m,d}) \to T^1(X'/W_{m,d-1})$ is surjective.

Concretely, this means that each diagram in $\mathcal{G}$ of solid arrows

$$\begin{array}{ccc} Y & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \qquad \text{over} \qquad \begin{array}{ccc} A[\epsilon] & \overset{\phi[\epsilon]}{\dashrightarrow} & A'[\epsilon] \\ {\scriptstyle \epsilon=0}\downarrow & & \downarrow{\scriptstyle \epsilon=0} \\ A & \xrightarrow[\phi]{} & A', \end{array}$$

where $A \to A'$ is either $W_{m+1} \to W_m$ or $W_{m,d} \to W_{m,d-1}$, can be completed to a diagram in $\mathcal{G}$ including dotted arrows, over the diagram of formal Artin W-algebras to the right. Note that a completion

$$\begin{array}{ccc} Z & \dashrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \qquad \text{over} \qquad \begin{array}{ccc} A[\epsilon]/\epsilon I & \overset{\phi[\epsilon]}{\dashrightarrow} & A'[\epsilon] \\ {\scriptstyle \epsilon=0}\downarrow & & \downarrow{\scriptstyle \epsilon=0} \\ A & \xrightarrow[\phi]{} & A' \end{array}$$

exist by the very definition of semihomogenity, see [21], Remark 1.3. Here $I \subset A$ is the ideal of A' so that $A[\epsilon]/\epsilon I = A \times_{A'} A'[\epsilon]$. Therefore, we may view the T^1-lifting property as a slight strengthening of the semihomogenity property.

Remark 3.3. If $\mathcal{G}$ is the groupoid of deformations of a proper smooth k-scheme X_0, then $T^1(X/A) = \mathrm{Ext}^1(\Omega^1_{X/A}, \mathcal{O}_X)$. We shall see that, under certain smoothness and duality assumptions, the T^1-lifting property is related to cohomological flatness of Kähler differentials $\Omega^n_{X/A}$.

A first order extension $A \to A'$ is called a *small extension* if the ideal $I \subset A$ has length one, that is, $I \simeq k$. We say that $\mathcal{G}$ admits an *obstruction theory* if there is a k-vector space T^2 together with maps as follows: For each small extension $\phi\colon A \to A'$, say with ideal $I \subset A$, there is a map $\mathrm{ob}\colon G_{A'} \to T^2 \otimes_k I$ so that an object $X' \in \mathcal{G}$ over A' admits a small deformation $X \to X'$ over $\phi\colon A \to A'$ if and only if $\mathrm{ob}(X') = 0$. Moreover, these maps must be functorial with respect to the direct image functors, compare [16]. For more on this concept, see [5].

Finally, let $(\mathrm{Art}/W)^\wedge$ be the category of formal W-algebras. Each such W-algebra R can be viewed as pro-object $(R/\mathfrak{m}_R^{n+1})_{n\geq 0}$ for (Art/W), and we define $\mathcal{G}^\wedge$ as the category of pro-objects for $\mathcal{G}$ lying over $(\mathrm{Art}/W)^\wedge$. This yields a semihomogeneous cofibered groupoid $\mathcal{G}^\wedge \to (\mathrm{Art}/W)^\wedge$ extending $\mathcal{G} \to (\mathrm{Art}/W)$.

Theorem 3.4. *Let $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ be a semihomogeneous cofibered groupoid with finite dimensional tangent space $G_{k[\epsilon]}$. Suppose that $\mathcal{G}$ admits an obstruction theory. Assume that there is a formal object $\mathfrak{Y} \in \mathcal{G}^\wedge$ over W, and that $\mathcal{G}$ has the T^1-lifting property. Then $\mathcal{G}$ is smooth.*

Proof. Since $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ is semihomogeneous and has a finite dimensional tangent space, Schlessinger's theorem tells us that there is a *minimally versal formal* object $\mathfrak{X} \in \mathcal{G}^\wedge$, say over the formal W-algebra R (see [21], Thm. 1.1). In other words, the corresponding functor

$$h_R \longrightarrow G, \quad (\phi\colon R \to A) \longmapsto [\mathfrak{X} \otimes_R A]$$

is smooth and induces a bijection on tangent spaces. According to [21], Remark 1.14, our task is to prove that R is a smooth formal W-algebra. To do so, we seek to apply Theorem 2.1 and have to check its three conditions.

Concerning the first condition, note that the isomorphism class of the formal object $\mathfrak{Y}$ over W is induced by a W-map $R \to W$.

Secondly, we have to check that each W-map $R \to W_{m+1}[\epsilon]/(u^m\epsilon)$ lifts to $W_{m+1}[\epsilon]$. Let $Z \in \mathcal{G}$ be an object over $W_{m+1}[\epsilon]/(u^m\epsilon)$ whose isomorphism class is induced by $R \to W_{m+1}[\epsilon]/(u^m\epsilon)$. Applying the restriction functors, we obtain a diagram

$$\begin{array}{ccccccc} Z & \longrightarrow & Y' & & W_{m+1}[\epsilon]/(u^m\epsilon) & \longrightarrow & W_m[\epsilon] \\ \downarrow & & \downarrow & \text{over} & \downarrow & & \downarrow \\ X & \longrightarrow & X' & & W_{m+1} & \longrightarrow & W_m. \end{array}$$

By the T^1-lifting property, we find a commutative diagram

$$\begin{array}{ccccccc} Y & \longrightarrow & Y' & & W_{m+1}[\epsilon] & \longrightarrow & W_m[\epsilon] \\ \downarrow & & \downarrow & \text{over} & \downarrow & & \downarrow \\ X & \longrightarrow & X' & & W_{m+1} & \longrightarrow & W_m. \end{array}$$

Now the isomorphism class of Y is induced by the desired lifting $R \to W_{m+1}[\epsilon]$.

Thirdly, we have to check that each W-map $R \to W_{m,d}$ lifts to $W_{m,d+1}$. This is the most interesting part of the proof, and we shall closely follow Kawamata's arguments [14]. Recall that $W_{m,d} = W\langle T\rangle/(u^m, \gamma^d(T), \gamma^{d+1}(T), \ldots)$. Since there is a W-map $W_{m,1} \to W_{m,2}$, we may assume $d \geq 2$. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & W_{m,d+1}\gamma^d(T) & \to & W_{m,d+1} & \to & W_{m,d} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & W_{m,d}\gamma^{d-1}(T)\epsilon & \to & W_{m,d}[\epsilon] & \to & W_{m,d}[\epsilon]/(\gamma^{d-1}(T)\epsilon) & \to & 0, \end{array}$$

where the vertical maps are given by $T \mapsto T + \epsilon$. Note that the vertical map on the left is bijective, because $\gamma^{d+1}(T+\epsilon) = \gamma^{d+1}(T) + \gamma^d(T)\epsilon$, by the axioms of divided powers ([1], Def. 3.1). The horizontal rows are first order extensions, but not necessarily small extensions. However, we can use the filtration defined by $u^n\gamma^{d+1}(T)$ and $u^n\epsilon\gamma^d(T)$, $n \geq 0$ to obtain small extensions. Using naturality of obstruction maps, we obtain for each $n \geq 0$ a commutative diagram

$$\begin{array}{ccccc} G_{W_{m,d+1}/(u^{n+1}\gamma^d(T))} & \to & G_{W_{m,d+1}/(u^n\gamma^d(T))} & \to & T^2 \otimes ku^{n+1}\gamma^d(T) \\ \downarrow & & \downarrow & & \downarrow \\ G_{W_{m,d}[\epsilon]/(u^{n+1}\epsilon\gamma^{d-1}(T))} & \to & G_{W_{m,d-1}[\epsilon]/(u^n\epsilon\gamma^{d-1}(T))} & \to & T^2 \otimes ku^{n+1}\epsilon\gamma^{d-1}(T). \end{array}$$

The horizontal rows are exact in the sense that an element in the middle lies in the image of the map on the left if and only if it maps to zero on the right. We see that the obstruction for an element in the upper row is zero if and only if the obstruction for its image in the lower row is zero.

Hence, if $Z \in \mathcal{G}$ is an object over $W_{m,d}[\epsilon]/(\gamma^{d-1}(T)\epsilon)$ whose isomorphism class is induced by the composite map $R \to W_{m,d}[\epsilon]/(\gamma^{d-1}(T)\epsilon)$, it suffices to find a first order deformation of the object Z over the first order extension $W_{m,d}[\epsilon] \to W_{m,d}[\epsilon]/(\gamma^{d-1}(T)\epsilon)$. Applying restriction functors, we obtain a commutative diagram

$$\begin{array}{ccc} Z & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \quad \text{over} \quad \begin{array}{ccc} W_{m,d}[\epsilon]/(\gamma^{d-1}(T)\epsilon) & \xrightarrow{\gamma^{d-1}(T)=0} & W_{m,d-1}[\epsilon] \\ {\scriptstyle \epsilon=0}\downarrow & & \downarrow{\scriptstyle \epsilon=0} \\ W_{m,d} & \xrightarrow[\gamma^{d-1}(T)=0]{} & W_{m,d-1}. \end{array}$$

By the T^1-lifting property, we find a commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} \quad \text{over} \quad \begin{array}{ccc} W_{m,d}[\epsilon] & \xrightarrow{\gamma^{d-1}(T)=0} & W_{m,d-1}[\epsilon] \\ {\scriptstyle \epsilon=0}\downarrow & & \downarrow{\scriptstyle \epsilon=0} \\ W_{m,d} & \xrightarrow[\gamma^{d-1}(T)=0]{} & W_{m,d-1}. \end{array}$$

Now the isomorphism class of $Y \in \mathcal{G}$ over $W_{m,d}[\epsilon]$ is induced by the desired lifting $R \to W_{m,d}[\epsilon]$. □

Example 3.5. Consider the functor of Artin W-algebras $A \mapsto \{a \in A \mid a^p = 0\}$ represented by $R = W[T]/(T^p)$, and let $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ be the corresponding semihomogeneous cofibered groupoid. Clearly, R is not a smooth formal W-algebra. However, note that $\Omega^1_{R/W} \otimes k$ is a free $R \otimes k$-module of rank one, generated by dT.

The restriction of $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ to formal Artin k-algebras is Deligne's example discussed in [14], p. 158: It satisfies the T^1-lifting condition in Kawamata's sense without being smooth. Let me check that $\mathcal{G}$ also does not satisfy the T^1-lifting condition in our sense. For simplicity, we assume that k is algebraically closed and that W is absolutely unramified, that is, $e = 1$.

Set $m = 2p+1$, and consider the first order extension $W_{m,p+1} \to W_{m,p}$. Set $\lambda = p^2 \in W_m$, such that $p\lambda^p = 0$ and $p^2\lambda^{p-1} \neq 0$. Then $s = \lambda\gamma^1(T) \in W_{m,p+1}$ satisfies $s^p = 0$. Let $s' \in W_{m,p}$ be its image. Then the deformation

$$r' = (\lambda + \epsilon)\gamma^1(T) \in W_{m,p}[\epsilon]$$

satisfies $(r')^p = 0$. Now suppose that our groupoid $\mathcal{G}$ has the T^1-lifting property. Then there is an element $r \in W_{m,p+1}[\epsilon]$ with $r^p = 0$ restricting to s and r'. We have $r = (\lambda+\epsilon)\gamma^1(T) + x\gamma^p(T)$ for some $x \in W_m$ and calculate

$$0 = r^p = p^2\lambda^{p-1}(p-1)! \cdot \epsilon\gamma^p(T) \neq 0,$$

a contradiction. Hence $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ does not satisfy the T^1-lifting condition.

4. Unobstructedness of Calabi–Yau manifolds

In this section, I shall apply the T^1-lifting criterion to Calabi–Yau manifolds in positive characteristic. Let k be a perfect field of characteristic $p > 0$, and $W = W(k)$ its ring of Witt vectors. Given a proper algebraic k-space X_0, we obtain a semihomogeneous cofibered groupoid $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ as follows. The objects in $\mathcal{G}$ are triples triples (A, X, ϕ), where A is a formal Artin W-algebra, X is a flat proper A-scheme, and $\phi\colon X \otimes k \to X_0$ is an isomorphism. The projection is given by $\pi(A, X, \phi) = A$. A minimally versal formal object $(R, \mathfrak{X}, \phi)$ is nothing but a semiuniversal deformation for X_0. Note that $\mathfrak{X}$ is a flat proper formal algebraic space over the formal scheme $\mathrm{Spf}(R)$. The groupoid $\mathcal{G}$ is smooth if and only if the base R of the minimally versal formal deformation is a smooth formal W-algebra.

Our main result involves the *crystalline topos* and *crystalline cohomology*. Given a formal PD W-algebra A, let $\mathrm{Cris}(X_0/A)$ be the crystalline site described in [1], Section 5. Its objects are pairs $(U \subset T, \delta)$, where $U \subset X_0$ is an open subset, $U \subset T$ is a closed A-embedding, and δ is a compatible PD-structure on the ideal of this embedding. Let X_0/A_{cris} be the associated crystalline topos, and $\mathcal{O}_{X_0/A}$ the corresponding structure sheaf. Note that the crystalline cohomology groups $H^r(X_0/A_{\mathrm{cris}}, \mathcal{O}_{X_0/A})$ are modules over A. Recall that the formal Artin W-algebra

$$W_{m,d} = W_m\langle T\rangle/(\gamma^d(T), \gamma^{d+1}(T), \ldots),$$

is endowed with the canonical compatible PD-structure. According to [13], it is possible to calculate crystalline cohomology on the Zariski site via the de Rham–Witt complex.

Theorem 4.1. *Let X_0 be a smooth proper algebraic k-space with $K_{X_0} = 0$ and $\dim(X_0) \leq p$. Suppose there is a smooth proper formal deformation $\mathfrak{Y} \to \mathrm{Spf}(W)$ of X_0. If the crystalline cohomology groups $H^r(X_0/A_{\mathrm{cris}}, \mathcal{O}_{X_0/A})$, $r \geq 0$ are free A-modules, where A ranges over the* PD *W-algebras $W_{m,d}$, $m, d > 0$, then the semiuniversal deformation $\mathfrak{X} \to \mathrm{Spf}(R)$ of X_0 has a smooth base. In other words, each deformation of X_0 is unobstructed.*

Proof. Let $\pi\colon \mathcal{G} \to (\mathrm{Art}/W)$ be the semihomogeneous cofibered groupoid of deformations (A, X, ϕ) of X_0. We shall apply Theorem 3.4 and have to verify three conditions. First, the formal deformation $\mathfrak{Y} \to \mathrm{Spf}(W)$ is of the form $\mathfrak{Y} = \mathfrak{X} \otimes_R W$ for some W-map $R \to W$.

Secondly, we have to check the T^1-lifting property. To do this, let me recall the concept of cohomological flatness. Let (A, X, ϕ) be a deformation and $\mathcal{F}^\bullet$ a bounded complex of locally free $\mathcal{O}_X$-modules of finite rank. Then $\mathcal{F}^\bullet$ is called *cohomologically flat* if the hypercohomology groups $\mathbb{H}^r(X, \mathcal{F}^\bullet)$ are free A-modules for all $r \in \mathbb{Z}$. By [6], Proposition 7.8.5, this implies that the base change maps

$$\mathbb{H}^r(X, \mathcal{F}^\bullet) \longrightarrow \mathbb{H}^r(X \otimes_A A/I, \mathcal{F}^\bullet \otimes_A A/I)$$

are surjective for all ideals $I \subset A$. We shall apply this to the de Rham complex $\Omega^\bullet_{X/A}$. According to [1], Corollary 7.4, there is a canonical bijection

$$\mathbb{H}^r(X, \Omega^\bullet_{X/A}) = \mathbb{H}^r(X_0/A_{\text{cris}}, \mathcal{O}_{X_0/A}).$$

Now suppose $A = W_{m,d}$. Our assumptions on crystalline cohomology imply that the de Rham complex $\Omega^\bullet_{X/A}$ is cohomologically flat. Next we argue as in [3], proof of Theorem 5.5, that this gives cohomological flatness of the individual sheaves $\Omega^s_{X/A}$. Indeed, by assumption we have $\dim(X_0) \leq p$ and X_0 lifts to W_2, so by [4], Corollary 2.4 the hypercohomology spectral sequence

$$H^r(X_0, \Omega^s_{X_0/k}) \Longrightarrow \mathbb{H}^{r+s}(X_0, \Omega^\bullet_{X_0/k})$$

degenerates. Hence

$$\sum_{r+s=q} h^r(\Omega^s_{X/A}) \geq h^q(\Omega^\bullet_{X/A}) = d \cdot h^q(\Omega^\bullet_{X_0/k}) = d \cdot \sum_{r+s=q} h^r(\Omega^s_{X_0/k}), \tag{1}$$

where $d = \text{length}(A)$. On the other hand, we have $h^r(\Omega^s_{X/A}) \leq d \cdot h^r(\Omega^s_{X_0/k})$ by [3], Corollary 3.4. Together with (1), this implies $h^r(\Omega^s_{X/A}) = d \cdot h^r(\Omega^s_{X_0/k})$. Now each $\Omega^s_{X/A}$ is cohomologically flat by [6], Proposition 7.8.4.

Set $n = \dim(X_0)$. By cohomological flatness of $\Omega^n_{X/A}$, the base change map

$$H^0(X, \Omega^n_{X/A}) \longrightarrow H^0(X_0, \Omega^n_{X_0/k})$$

is surjective. Hence each trivializing section of ω_{X_0} lifts to a trivializing section of $\omega_{X/A}$, and we have $K_{X/A} = 0$. It follows

$$(\Omega^1_{X/A})^\vee = (\Omega^1_{X/A})^\vee \otimes \omega_{X/A} = \Omega^{n-1}_{X/A}, \tag{2}$$

According to [8], Theorem 6.3, we have $T^1(X/A) = \text{Ext}^1(\Omega^1_{X/A}, \mathcal{O}_X)$. Together with (2), this gives

$$T^1(X/A) = H^1(X, \Omega^{n-1}_{X/A}),$$

and you easily check that the restriction map for $T^1(X/A)$ corresponds to the base change map for $H^1(X, \Omega^{n-1}_{X/A})$. Now, using cohomological flatness of $\Omega^{n-1}_{X/A}$, we infer that for each first order extension $(W_{m,d}, X, \phi) \to (W_{m,d-1}, X', \phi')$, the induced map

$$T^1(X/W_{m,d}) \longrightarrow T^1(X/W_{m,d-1})$$

is surjective. The same argument works for first order deformations over the map $W_m \to W_{m-1}$. The upshot is that $\mathcal{G}$ has the T^1-lifting property.

It remains to check that $\mathcal{G}$ admits an obstruction theory. Let $A \to A'$ be a small extension, say with ideal $I \simeq k$. By [8], Theorem 6.3, a given deformation $X' \to \text{Spec}(A')$ of X_0 extends over A if and only if a functorial obstruction

$$\text{ob}(X') \in H^2(X_0, \Theta_{X_0/k} \otimes_k I) = H^2(X_0, \Theta_{X_0/k}) \otimes_k I$$

vanishes. In fact, $\text{ob}(X')$ is nothing but the gerbe of local extension of X over A. So $T^2 = H^2(X_0, \Theta_{X_0/k})$ yields the desired obstruction theory.

We have checked all conditions of Theorem 3.4 and conclude that the semihomogeneous cofibered groupoid $\mathcal{G}$ is smooth. □

Remark 4.2. Hirokado [11] constructed an example of a smooth projective 3-fold X_0 over a field k of characteristic $p = 3$, with $K_{X_0} = 0$ and $h^2(\mathcal{O}_{X_0}) = 0$, so that X_0 does not admit a lifting to characteristic zero. Such a 3-fold does not admit a formal deformation $\mathfrak{X}$ over $\operatorname{Spf}(W)$ as well. Otherwise, the exact sequence

$$\operatorname{Pic}(\mathfrak{X}_{n+1}) \longrightarrow \operatorname{Pic}(\mathfrak{X}_n) \longrightarrow H^2(X_0, \mathcal{O}_{X_0})$$

implies that the formal scheme $\mathfrak{X}$ admits a line bundle whose restriction to X_0 is ample. Then by Grothendieck's Algebraization Theorem, the formal scheme admits an algebraization over $\operatorname{Spec}(W)$, contradiction. We see that X_0 does not satisfy the assumptions of Theorem 4.1, and has obstructed deformations.

Question 4.3. Does Hirokado's example has unobstructed deformations over Artin k-algebras? More generally, do there exist Calabi–Yau manifolds in positive characteristic with unobstructed deformations over Artin k-algebras, but obstructed deformations over Artin W-algebras? Are the sufficient conditions in Theorem 4.1 also necessary?

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