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THE INVERSE GALOIS PROBLEM FOR CONNECTED ALGEBRAIC GROUPS

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Abstract. We show that each connected group scheme of finite type over an arbitrary ground field is isomorphic to the component of the identity inside the automorphism group scheme of some projective, geometrically integral scheme. The main ingredients are embeddings into smooth group schemes, equivariant completions, blow-ups of orbit closures, Fitting ideals for Kähler differentials, and Blanchard's Lemma.

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Introduction

Let k be a ground field of characteristic $p \geq 0$, and X a proper scheme. According to a result of Matsumura and Oort ([27], Theorem 3.7), the automorphism group scheme Aut_X exists and the connected component of the identity Aut_X^0 is an *algebraic group*, that is, a group scheme of finite type. Note that the underlying scheme may be non-reduced for $p > 0$. Also note that the group of connected components is not necessarily finitely generated (the first example was constructed by Lesieutre [23]), and understanding its structure attracted a lot of attention in the last decade.

Here we concentrate on the *connected* algebraic group Aut_X^0. Despite a very satisfactory structure theory for algebraic groups (see for example [29]), surprisingly little is known about how the geometry of the scheme X and the geometry of the algebraic group Aut_X^0 are related, in particular over imperfect ground fields and in presence of singularities.

The goal of this paper is to investigate the following problem, which can be seen as a higher-dimensional analog of the classical Inverse Galois Problem for number

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fields: *Given an algebraic group G, does there exist a proper, geometrically integral scheme X whose automorphism group scheme is isomorphic to G?*

Stated in this general form, the answer to the problem is negative: Building on recent work of Lombardo and Maffei [24], Blanc and the first author [2] showed that if X is projective and Aut_X is an abelian variety, the automorphism group of this abelian variety must be finite (see [11] for further developments). In particular, the selfproduct $E \times E$ of any elliptic curve does not occur as the automorphism group scheme of a projective, geometrically integral scheme.

The main result of this paper answers positively the above higher-dimensional analog to the Inverse Galois Problem provided one restricts attention to *connected* algebraic groups:

Theorem. *(See Thm. 2.1) For every connected algebraic group G over a ground field k, there is a projective, geometrically integral scheme X with $\mathrm{Aut}_X^0 = G$.*

One may choose the G-action to be generically free. If G is smooth of dimension $n \geq 1$, one may additionally achieve X normal with $\dim(X) = \max(2n, 3)$. The result generalizes [5], Theorem 1, where the case that G is smooth and k is perfect was treated. As mentioned there, if $p = 0$ then one may additionally take X to be smooth. This follows from the existence of a canonical resolution of singularities (see e.g. [22]), which is not known to exist in positive characteristic and dimension at least 3. So it is an open question whether the above scheme X may be chosen to be smooth. The recent preprint [12] answers this question in the affirmative when G is a *linear* algebraic group, not necessarily connected, and p is arbitrary: then one may even achieve that G is the full automorphism group scheme Aut_X.

As an application of the main theorem, we see that for any finite-dimensional restricted Lie algebra $\mathfrak{g}$ there is a projective, geometrically integral scheme X with $H^0(X, \Theta_X) = \mathfrak{g}$, where Θ_X denotes the tangent sheaf. Recall that the Lie algebra $\mathfrak{g}$ of an algebraic group in characteristic $p > 0$ carries, beside the bracket $[x, y]$, the p-map $x^{[p]}$ as an additional structure, and is called a *restricted Lie algebra*. Note that there are severe restrictions on the restricted Lie algebra $H^0(X, \Theta_X)$ if the proper integral scheme X is a surface of general type, or more generally has *foliation rank* at most 1, according to recent work of Tziolas and the second author [33].

We expect further applications of our result, for example in connection to classifying stacks BG, perhaps analogous to Totaro's approach [35], or with respect to non-abelian cohomology $H^1(k, G)$, as in [20], Section 9.

The idea for the proof of the above theorem is to start with an inclusion $G \subset \mathrm{Aut}_Y$ for some projective, geometrically integral scheme Y, chosen in such a way that G is the stabilizer of some closed subscheme $Z \subset Y$, and pass to the blowing-up $X = \mathrm{Bl}_Z(Y)$. Combining naturality of blowing-ups with Blanchard's Lemma, we get inclusions $G \subset \mathrm{Aut}_X^0$ inside Aut_Y^0, and manage to infer equality by choosing the schemes Y and Z appropriately.

We actually proceed in two steps, treating the case that G is smooth first. In this situation we regard G as a homogeneous space for $G \times G$ via left and right multiplication, and define $Y = V \times V$ as the self-product of some $G \times G$-equivariant completion $G \subset V$. The center $Z \subset Y$ for the blowing-up stems from certain graphs of morphisms $V \to V$. For technical reasons, the case $\dim(G) = 1$ has to be treated

separately. Along the way, we establish several facts on group scheme actions that appear to be of independent interest.

In a second step, we consider algebraic groups G that are non-smooth, which happens only in characteristic $p > 0$. To start with, we establish that G embeds into some smooth connected algebraic group. Such an embedding is in no way canonical, and we loose control of dimensions. But together with the preceding paragraph we get $G \subset \mathrm{Aut}^0_Y$ for some projective, geometrically integral Y. Now the center $Z \subset Y$ is chosen as the closure of some free G-orbit. The local rings on such a free orbit are complete intersections, and acquire a rather simple description after base-changing to the algebraic closure of k. We use this to get information on the *Fitting ideals* of Ω^1_X on the blowing-up $X = \mathrm{Bl}_Z(Y)$, and the corresponding closed subschemes. The latter are intrinsically attached to X, thus are stabilized by all group scheme actions, and from this we infer $G = \mathrm{Aut}^0_X$. To understand these Fitting ideals, we have to make explicit computations for the blowing-up of the polynomial ring $R = k[x_1, \ldots, x_n]$ with respect to ideals of the form $\mathfrak{a} = (x_s^{\nu_s}, \ldots, x_n^{\nu_n})$, where the exponents are p-powers.

Let us also discuss the famous Inverse Galois Problem, which can be stated as follows: Given a number field F and a finite group G, does there exist a finite Galois extension $F \subset E$ with automorphism group $\mathrm{Aut}(E/F)$ isomorphic to G? Hilbert showed this for the symmetric groups ([21], page 124), and Shafarevich established it for solvable groups ([32], see also [30]). Despite many other positive results in this direction, the general case remains open to date. For overviews, see [34] or [36] or [25]. But note that Fried and Kollár [13] showed that the answer to the Inverse Galois Problem becomes affirmative if one drops the assumption that $F \subset E$ is Galois. This was extended to function fields in one variable in the recent preprint [4]: Given a finite étale group scheme G and a (projective, geometrically integral) regular curve Y over k, there exists a regular curve X, finite over Y, such that $G = \mathrm{Aut}_{X/Y} = \mathrm{Aut}_X$. In particular, every finite étale group scheme is the full automorphism group scheme of some regular curve.

The paper is structured as follows: In Section 1 we fix notation and establish several useful general facts on group schemes. Section 2 contains the proof of the main result in the case that G is smooth. In Section 3 we discuss Fitting ideals for Kähler differentials, and the stability of the resulting closed subschemes with respect to group scheme actions. Section 4 contains some explicit computations of such Fitting ideals for certain Rees rings $R[\mathfrak{a}T]$, needed to understand blowing-ups of free G-orbits. This is applied in Section 5, where we prove our main result for non-smooth G.

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1. Preliminaries

In this section we introduce notations and conventions, and establish some general facts on group schemes. Throughout the paper, we work over a ground field k of characteristic $p \geq 0$. Choose an algebraic closure k^{alg}, and denote by k^{sep} the separable closure of k in k^{alg}. For any field extension $k \subset K$ and any scheme X, we write $X_K = X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K)$ for the base-change.

Given a group scheme G, we denote the neutral element by $e \in G(k)$, and the Lie algebra by $\mathrm{Lie}(G)$. We will freely use the fact that every subgroup scheme $H \subset G$ is closed (see [7], Exposé $\mathrm{VI_A}$, Corollaire 0.5.2) An *algebraic group* is a group scheme of finite type. A *locally algebraic group* is a group scheme, locally of finite type. For any locally algebraic group G, the connected component of $e \in G$ is a connected algebraic subgroup, denoted by G^0 and called the *neutral component* (see [8], Theorem II.5.1.1). Note that G may fail to be smooth. In this case we have $p > 0$, and the local ring $\mathscr{O}_{G,e}$ is not regular.

For any proper scheme X, the automorphism group functor $T \mapsto \mathrm{Aut}(X \times T/T)$ on the category (Sch/k) is represented by a locally algebraic group Aut_X, according to [27], Theorem 3.7. Its Lie algebra is $\mathrm{Der}(\mathcal{O}_X) = H^0(X, \Theta_X)$, the space of k-linear derivations $D : \mathscr{O}_X \to \mathscr{O}_X$, or equivalently the space of global sections of the tangent sheaf $\Theta_X = \underline{\mathrm{Hom}}(\Omega^1_X, \mathscr{O}_X)$, see loc. cit., Lemma 3.4. Clearly, the neutral component Aut^0_X has the same Lie algebra.

Let G be a group scheme. A *G-scheme* is a scheme X equipped with a G-action $a : G \times X \to X$. As customary, we write $g \cdot x = a(g, x)$ for $g \in G(T)$, $x \in X(T)$.

We will repeatedly use a result for G-actions called *Blanchard's Lemma*: Let $f : X \to Y$ be a proper morphism of schemes of finite type and let G be a connected algebraic group acting on X. If the canonical map $\mathcal{O}_Y \to f_*(\mathcal{O}_X)$ is bijective, then there is a unique G-action on Y such that f is equivariant (see [3], Proposition 1.1 for the original statement in the setting of complex geometry, and [6], Theorem 7.2.1 for the above scheme-theoretic version). As a consequence, if in addition X is proper then f induces a homomorphism of group schemes $f_* : \mathrm{Aut}^0_X \to \mathrm{Aut}^0_Y$.

Given a group scheme G, a G-scheme X and a closed subscheme $Z \subset X$, the subgroup functor comprising the $g \in G(T)$ such that $g|T' \cdot z \in Z(T')$ for all $T' \to T$ and $z \in Z(T')$ is representable. The resulting subgroup scheme $\mathrm{Stab}_G(Z) \subset G$ is called the *stabilizer*; it acts naturally on Z. Likewise, the subgroup functor given by the condition $g|T' \cdot z = z$ is representable by a normal subgroup scheme of $\mathrm{Stab}_G(Z)$, the *kernel of its action on Z* (see [8], Theorem II.1.3.6 for these facts). In particular, the kernel of the G-action on X is representable by a normal subgroup scheme.

If X is proper and Z is a closed subscheme, we write $\mathrm{Aut}_{X,Z}$ for the stabilizer of Z in $G = \mathrm{Aut}_X$. Its Lie algebra is the space of k-linear derivations $D : \mathscr{O}_X \to \mathscr{O}_X$ that preserve the sheaf of ideals $\mathscr{I} \subset \mathscr{O}_X$ corresponding to Z, see for example [26], Section 2. The blow-up $\mathrm{Bl}_Z(X) = \mathrm{Proj}(\bigoplus_{i=0}^{\infty} \mathscr{I}^i)$ is equipped with an action of $\mathrm{Aut}_{X,Z}$ such that the canonical morphism $f : \mathrm{Bl}_Z(X) \to X$ is equivariant (see loc. cit., Proposition 2.7).

A closed subscheme $Z \subset X$ is called *G-stable* if $\mathrm{Stab}_G(Z) = G$. Equivalently, the morphism $a : G \times Z \to X$ factors through Z. The latter notion of G-stability extends to arbitrary subschemes $Z \subset X$, which are intersections of closed subschemes with

open sets. Note that an open set $U \subset X$ is G-stable if and only if for all field extension $k \subset K$ and all $g \in G(K)$ and $x \in U(K)$, we have $g \cdot x \in U(K)$.

We now collect auxiliary results on G-stability. First recall that the *schematic image* for a morphism $f : X \to Y$ of schemes is the smallest closed subscheme $Z \subset Y$ over which the morphism factors ([15], Section 6.10). It exists without further assumptions, and corresponds to the largest quasicoherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_Y$ with $f^{-1}(\mathscr{I})\mathscr{O}_X = 0$. If f is quasicompact, the underlying topological space of Z is the closure of the set-theoretic image, and $\mathscr{I}$ is the kernel of $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ (see [14], Section 10.8 for these facts). Note that if f is also quasiseparated, the direct image $f_*(\mathscr{O}_X)$ is already quasicoherent. If f is proper, the schematic image coincides with the set-theoretical image, endowed with some scheme structure. The following observation is a version of [26], Lemma 2.5.

Lemma 1.1. *Let G be a group scheme, and $f : X \to Y$ be an equivariant quasi-compact morphism of G-schemes. Then the schematic image $Z \subset Y$ is G-stable.*

Proof. By assumption, we have a commutative square

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id} \times f"] \arrow[d, "a"'] & G \times Y \arrow[d, "b"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where a and b correspond to the G-actions. Also, the schematic image of a is the whole X, since $e \cdot x = x$. In view of the transitivity of schematic images (see [15], Proposition 6.10.3), it suffices to show that the schematic image of $\mathrm{id} \times f$ is $G \times Z$.

Since f is quasi-compact, the formation of its schematic image commutes with flat base change (see [17], Théorème 11.10.5). This yields the desired assertion by using the cartesian square

$$\begin{tikzcd} G \times X \arrow[r, "\mathrm{id} \times f"] \arrow[d, "\mathrm{pr}_X"'] & G \times Y \arrow[d, "\mathrm{pr}_Y"] \\ X \arrow[r, "f"] & Y, \end{tikzcd}$$

where the projections pr_X and pr_Y are flat. □

Lemma 1.2. *Let G be a group scheme, and X a G-scheme of finite type. Then the smooth locus $U \subset X$ is G-stable.*

Proof. Since $U \subset X$ is open, it suffices to check that for all field extensions $k \subset K$ and all $g \in G(K)$, $x \in U(K)$, we have $g \cdot x \in U(K)$. As the formation of the smooth locus commutes with base change by field extensions (see [18], Proposition 17.3.3, Proposition 17.7.1), it suffices to treat the case $K = k$, and that this field is algebraically closed. Then U comprises the points where the local rings are regular, and the statement is immediate. □

Lemma 1.3. *Let G be a connected algebraic group and X a G-scheme of finite type. Then every connected component of X is G-stable. If in addition X is geometrically reduced, then every irreducible component of X is G-stable.*

Proof. Let $Y \subset X$ be a connected component. Then the set-theoretical image $G \cdot Y = a(G \times Y)$ is open in X, since the morphism $a : G \times X \to X$ describing the action is flat, and $G \times Y$ is of finite type. The latter is connected, since G is geometrically connected and Y is connected. Thus, $G \cdot Y$ is connected as well. As Y is contained in $G \cdot Y$, we have $G \cdot Y = Y$ and hence Y is G-stable.

Now assume that X is geometrically reduced. Let $X_1, \ldots, X_n$ be the irreducible components, endowed with reduced scheme structures, and $U \subset X$ the smooth locus. Then $U_i = U \cap X_i$ are the connected components of U, and $U_i \subset X_i$ is schematically dense. In view of Lemma 1.2 and the preceding paragraph, it follows that every U_i is G-stable. Thus, each X_i is G-stable by Lemma 1.1. □

Given a group scheme G and a G-scheme X, we denote by $X^G \subset X$ the *scheme of fixed points*. This is the closed subscheme representing the subfunctor comprising the $x \in X(T)$ such that for all $T' \to T$ and $g \in G(T')$ we have $g \cdot x = x$, see for example [8], Theorem II.1.3.6. The arguments in loc. cit. reveal that for any closed subscheme $F \subset G$, the *scheme of F-fixed points* $X^F \subset X$ is indeed representable by a closed subscheme.

Lemma 1.4. *Let G be a locally algebraic group, and X a G-scheme of finite type. Then there exists a finite closed subscheme $F \subset G$ such that $X^G = X^F$. If G is smooth, then F may be taken étale.*

Proof. Consider the family of closed subschemes X^F of X, where F runs over the finite closed subschemes of G. Since X is noetherian, we may choose such a subscheme F_0 with X^{F_0} minimal. For each F containing F_0 we have $X^F = X^{F_0}$ by minimality. Equivalently, F is a subscheme of the largest subgroup scheme $H \subset G$ that acts trivially on X^{F_0}. Moreover, the family of finite subschemes of G containing F_0 is schematically dense in G, since the latter is locally of finite type. So $H = G$ and hence $X^G = X^{F_0}$ as desired.

If G is smooth, then we argue similarly by considering finite closed subschemes that are étale; these also form a schematically dense family, since $G(k^{\mathrm{sep}})$ is dense in $G_{k^{\mathrm{sep}}}$. □

Remark 1.5. In characteristic zero, every algebraic group G is generated by some finite étale subscheme F (see [5], Lemma 3). Hence $X^G = X^F$ for any G-scheme of finite type X. But this does not hold true in characteristic $p > 0$: For example, the additive group $\mathbb{G}_a$ is not generated by any finite étale subscheme F, because the finite set $F(k^{\mathrm{alg}})$ generates a finite subgroup of $\mathbb{G}_a(k^{\mathrm{alg}}) = k^{\mathrm{alg}}$. Moreover, one may check there exists no finite étale subscheme $F \subset \mathbb{G}_a$ such that $X^{\mathbb{G}_a} = X^F$ for all $\mathbb{G}_a$-schemes X of finite type.

We will also need a version of a classical criterion for a homomorphism of algebraic groups to be an isomorphism.

Lemma 1.6. *Let $f : G \to H$ be a homomorphism of locally algebraic groups. Assume that the induced maps $G(k^{\mathrm{alg}}) \to H(k^{\mathrm{alg}})$ and $\mathrm{Lie}(G) \to \mathrm{Lie}(H)$ are bijective, and that G is smooth. Then f is an isomorphism.*

Proof. By descent, it suffices to treat the case that $k = k^{\mathrm{alg}}$. The kernel N of f is a locally algebraic group that satisfies $N(k) = \{e\}$ and $\mathrm{Lie}(N) = \{0\}$, in view of our assumptions. Thus N is trivial.

Recall that G^0 and H^0 are algebraic groups; moreover, f restricts to a homomorphism $f^0 : G^0 \to H^0$. Applying [8], Corollaire II.5.5.5, we see that f^0 is an open immersion, and hence an isomorphism. It remains to verify that the induced homomorphism $\pi_0(f) : \pi_0(G) = G/G^0 \to H/H^0 = \pi_0(H)$ between étale group schemes is an isomorphism. These group schemes are actually constant, because k is separably closed. Furthermore, $\pi_0(f)$ is surjective, because $G \to H$ and $H \to H/H^0$ are surjective on k-rational points. Suppose the kernel of $\pi_0(f)$ contains some non-trivial $\bar{g} \in G/G^0$. Represent it by some $g \in G(k)$. Then $g \notin G^0(k)$ and $f(g) \in H^0 = f(G^0)$. Thus, there exists $g' \in G^0(k)$ such that $f(g) = f(g')$. Then $g^{-1} \cdot g'$ is non-trivial and belongs to N, contradiction. □

The following observation will be a key ingredient in the proof of our main theorem in positive characteristics:

Proposition 1.7. *Each connected algebraic group G is isomorphic to a subgroup scheme of a smooth connected algebraic group H.*

Proof. There exists a connected affine normal subgroup scheme N such that G/N is an abelian variety (see for example [29], Theorem 8.28). Also, the affinization morphism $G \to G^{\mathrm{aff}} = \operatorname{Spec} H^0(G, \mathscr{O}_G)$ is a faithfully flat homomorphism of algebraic groups. Moreover, its kernel G_{ant} is smooth, connected and contained in the center (see [8], Théorème III.3.8.2 and Corollaire III.3.8.3, or [6], Theorem 1).

We have $G = G_{\mathrm{ant}}N$. Indeed, $G_{\mathrm{ant}}N$ is a normal subgroup scheme of G, and the quotient $G/G_{\mathrm{ant}}N$ is both affine (as a quotient of G/G_{ant}) and an abelian variety (as a quotient of G/N). So we obtain an exact sequence

$$1 \longrightarrow G_{\mathrm{ant}} \cap N \longrightarrow G_{\mathrm{ant}} \times N \longrightarrow G \longrightarrow 1,$$

where $G_{\mathrm{ant}} \cap N$ is identified with a central subgroup scheme of $G_{\mathrm{ant}} \times N$.

We may identify the affine algebraic group N with a subgroup scheme of some general linear group GL_m. By construction, N is contained in the centralizer $C = C_{\mathrm{GL}_m}(G_{\mathrm{ant}} \cap N)$. Moreover, C is smooth (see [19], Lemma 3.5, based on [8], Proposition II.2.1.6). Hence G is isomorphic to a subgroup scheme of the quotient $H = (G_{\mathrm{ant}} \times C)/(G_{\mathrm{ant}} \cap N)$, which is smooth as well. Since G is connected, it is already contained in the neutral component H^0. □

Remark 1.8. Each algebraic group G is an extension of the finite étale group G/G^0 by the connected algebraic group G^0, but the projection $G \to G/G^0$ usually has no section. By the above, we may embed G^0 into some smooth algebraic group, but we do not know how to carry this over to G.

However, it is easy to construct examples of algebraic groups G that cannot be embedded into a connected algebraic group: Let E be an elliptic curve and consider the semidirect product $G = E \rtimes \{\pm 1\}$, where $\{\pm 1\}$ denotes the constant group scheme of order two acting on E via $x \mapsto \pm x$. Then G is a smooth disconnected algebraic group. Suppose G is a subgroup scheme of some connected algebraic group

H, then E is central in H (see [29], Corollary 8.13). But E is not central in H, a contradiction.

2. The main result in the smooth case

We now formulate the main result of this paper:

Theorem 2.1. *For any connected algebraic group G over a field k, there is a projective, geometrically integral G-scheme X with $\mathrm{Aut}_X^0 = G$, and generically free action. If G is smooth of dimension $n \geq 0$, then one may choose X normal with $\dim(X) = \max(2n, 3)$.*

Let us give an immediate application in characteristic $p > 0$, which was actually the starting point for our research: Recall that for any group scheme G and any scheme X the Lie algebras $\mathrm{Lie}(G)$ and $H^0(X, \Theta_X) = \mathrm{Der}(\mathcal{O}_X)$ carry as additional structure the so-called p-map $D \mapsto D^{[p]}$. For the Lie algebra of global vector fields, this is just the p-fold composition in the algebra $\mathrm{Diff}(\mathcal{O}_X)$ of k-linear differential operators. This leads to the notion of *restricted Lie algebra*. By [8], Proposition II.7.4.1, the functor $G \mapsto \mathrm{Lie}(G)$ is an equivalence between the category of algebraic groups annihilated by the relative Frobenius, and the category of finite-dimensional restricted Lie algebras. Such algebraic groups contain but one point. As an immediate consequence of the theorem we get:

Corollary 2.2. *For every finite-dimensional restricted Lie algebra $\mathfrak{g}$ over our ground field k of characteristic $p > 0$, there is a projective, geometrically integral scheme X such that the restricted Lie algebra $H^0(X, \Theta_X)$ is isomorphic to $\mathfrak{g}$.*

This is quite different from the case of characteristic zero, where the Lie algebras of global vector fields on proper schemes are exactly the finite-dimensional linear Lie algebras (see [5], Corollary 1).

Let us now turn to the proof of Theorem 2.1. This section contains the arguments under the additional assumption that G is smooth. The rough idea is as follows: In a first step, we realize G as the group scheme of those automorphisms of a normal, projective, geometrically integral scheme V that commute with finitely many automorphisms $f_1, \ldots, f_m$. In a second step, we replace this commutator condition with a stabilizer condition for a closed subscheme; for this, we replace V with $Y = V \times V$, with closed subscheme the union Z of the graphs Γ_{f_i}. In a third step, we consider the normalized blow-up $X \to Y$ with center Z, and check that $G = \mathrm{Aut}_X^0$ provided that $n \geq 2$. The case $n = 1$ requires some additional attention, and is reduced to $n \geq 2$ via a small trick.

Step 1. The product group scheme $G \times G$ acts from the left on the scheme G via the formula $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$. This action is transitive, and the stabilizer of $e \in G$ is the diagonal $\Delta(G)$; we may thus identify G with the homogeneous space $(G \times G)/\Delta(G)$. According to [29], Theorem 8.44, G admits a $G \times G$-*equivariant projective completion* V. In other words, V is a projective $G \times G$-scheme, containing G as an open set that is schematically dense and $G \times G$-stable. Clearly, V is geometrically integral, since G is smooth and geometrically irreducible ([7], Exposé VI_A,

Proposition 2.4). Also, the $G \times G$-action on V lifts uniquely to the normalization of V, by [6], Proposition 2.5.1. Thus, we may also assume that V is normal.

Denote by $G \times e$ and $e \times G$ the respective left and right copies of G inside $G \times G$. These subgroup schemes commute with each other. This gives a homomorphism of group schemes

$$\psi : G = G \times e \longrightarrow \mathrm{Aut}_V^{e \times G},$$

where $\mathrm{Aut}_V^{e \times G}$ denotes the centralizer of $e \times G \subset \mathrm{Aut}_V$. This is indeed the scheme of fixed points for the conjugacy action of $e \times G$ on Aut_V.

Lemma 2.3. *In the above situation, ψ is an isomorphism.*

Proof. It suffices to treat the case that k is algebraically closed. Observe that the kernel of the $G \times G$-action on G equals the diagonal of the center $Z(G)$. Thus this is also the kernel of the $G \times G$-action on V, i.e., of the corresponding homomorphism $G \times G \to \mathrm{Aut}_V$. Since the intersection of $G \times e$ with the diagonal of the center is trivial, the homomorphism ψ has a trivial kernel, and hence is a closed immersion (as follows from [8], Proposition II.5.5.1). In order to apply Lemma 1.6, we have to show that ψ induces a surjection on rational points and Lie algebras.

Let $h \in \mathrm{Aut}_V^{e \times G}(k)$. Then $h : V \to V$ stabilizes $G \subset V$, which is the open orbit of the action of $e \times G$. Since the induced $h : G \to G$ commutes with the $e \times G$-action by multiplication from the right, it must be the left multiplication by some $g \in G(k)$. Since $G(k) \subset V$ is schematically dense, we obtain $h = \psi(g)$. Thus ψ is surjective on rational points.

It remains to check that the injective map $\mathrm{Lie}(\psi) : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{Aut}_V^{e \times G})$ is bijective. The term on the right is $\mathrm{Der}^{e \times G}(\mathscr{O}_V)$ by [8], Proposition II.4.2.5. Let $i : G \to V$ be the inclusion map of the open set G, which is schematically dense and $G \times G$-equivariant. Then i induces an injection of Lie algebras

$$i^* : \mathrm{Der}^{e \times G}(\mathscr{O}_V) \longrightarrow \mathrm{Der}^{e \times G}(\mathscr{O}_G) = \mathrm{Lie}(G),$$

such that $i^* \circ \mathrm{Lie}(\psi) = \mathrm{id}_{\mathrm{Lie}(G)}$. Having an injective left inverse, the injective linear map $\mathrm{Lie}(\psi)$ must be bijective. $\square$

In particular, the homomorphism ψ gives an identification $G = \mathrm{Aut}_V^{0, e \times G}$, the centralizer of $e \times G$ in Aut_V^0.

Step 2. We now apply Lemma 1.4 to the conjugacy action of $e \times G$ on Aut_V^0, and find some finite étale closed subscheme $F \subset G$ such that $G = \mathrm{Aut}_V^{0, e \times F}$. Clearly, we may assume in addition that $e \in F(k)$. Consider the graph morphism

$$\gamma : F \times V \longrightarrow V \times V, \quad (g, v) \longmapsto (v, (e, g) \cdot v).$$

This morphism is finite: Obviously the morphism $F \times V \to V$ given by $(g, v) \mapsto v$ is finite. The same holds for $(g, v) \mapsto (e, g) \cdot v$, because it is isomorphic to the projection via $(g, v) \mapsto (g, (e, g)^{-1} \cdot v)$. Using that products and compositions of finite morphisms are finite, and that $F \times V$ is separated, we see that the graph morphism is finite.

We set for simplicity $Y = V \times V$, and denote by $Z \subset Y$ the schematic image of γ. We now collect some observations and results on the geometry of Y and Z. Since V is projective and geometrically integral, we see that Y is projective and geometrically

integral as well. Moreover, we have $\dim(Y) = 2\dim(V) = 2\dim(G) = 2n$. Also, Z is geometrically reduced and equidimensional of dimension n.

Note that V is not necessarily geometrically normal (this happens if G is a non-rational form of the additive group, see [31]). Still, we have:

Lemma 2.4. *The scheme Y is normal.*

Proof. We use Serre's Criterion. Since V satisfies Serre's Condition (S_2), the same holds for Y. It remains to verify that Y is regular in codimension one, in other words, satisfies the condition (R_1). Since this holds for V, the closed set $Y \smallsetminus V_{\mathrm{reg}} \times V_{\mathrm{reg}}$ has codimension at least two, where V_{reg} denotes the regular locus. Thus, it suffices in turn to show that $V_{\mathrm{reg}} \times V_{\mathrm{reg}}$ satisfies (R_1). As G is smooth, $G \times V_{\mathrm{reg}}$ is regular and therefore

$$(V_{\mathrm{reg}} \times G) \cup (G \times V_{\mathrm{reg}}) \subset (V \times V)_{\mathrm{reg}}.$$

Moreover, the complementary closed set has codimension at least two, because

$$(V_{\mathrm{reg}} \times V_{\mathrm{reg}}) \smallsetminus ((V_{\mathrm{reg}} \times G) \cup (G \times V_{\mathrm{reg}})) = (V_{\mathrm{reg}} \smallsetminus G) \times (V_{\mathrm{reg}} \smallsetminus G).$$

Thus, $V_{\mathrm{reg}} \times V_{\mathrm{reg}}$ indeed satisfies (R_1). □

By Galois descent, we may identify the finite étale subscheme F of G with the finite subset $F(k^{\mathrm{sep}}) \subset G(k^{\mathrm{sep}})$, equipped with the action of the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$. The scheme $Z_{k^{\mathrm{sep}}}$ is the union of the graphs Γ_f, where $f \in F(k^{\mathrm{sep}})$ and Γ_f denotes the schematic image of the closed immersion

$$V_{k^{\mathrm{sep}}} \longrightarrow Y_{k^{\mathrm{sep}}}, \quad v \longmapsto (v, (e, f) \cdot v).$$

Moreover, each Γ_f is an irreducible component of $Z_{k^{\mathrm{sep}}}$. Also, note that Γ_e is the diagonal $\Delta(V)$. The action of $G_{k^{\mathrm{sep}}} \times e$ on $Y_{k^{\mathrm{sep}}}$ stabilizes each graph Γ_f, and hence Z is $G \times e$-stable.

Let $Y_0 \subset G \times G$ be the open set such that $Y_{0,k^{\mathrm{sep}}}$ consists of the points which lie in at most one graph Γ_f, where $f \in F(k^{\mathrm{sep}})$. Clearly, Y_0 is smooth, dense and $G \times e$-stable; moreover, $G \times e$ acts freely on Y_0. In particular, the $G \times e$-action on Y is generically free. Likewise, $Z_0 = Z \cap Y_0$ is an open set of Z which is smooth, dense and $G \times e$-stable.

As a consequence of Blanchard's Lemma, the canonical homomorphism

$$\mathrm{Aut}_V^0 \times \mathrm{Aut}_V^0 \longrightarrow \mathrm{Aut}_Y^0, \quad (g, h) \longmapsto g \times h$$

of algebraic groups is an isomorphism (see [6], Corollary 7.2.3 for details). Throughout, we regard it as an identification. Also, note that the diagonal homomorphism

$$\Delta : \mathrm{Aut}_V^0 \longrightarrow \mathrm{Aut}_Y^0, \quad g \longmapsto g \times g$$

of algebraic groups is a closed immersion.

Lemma 2.5. *With the above notation, we have* $\mathrm{Stab}_{\mathrm{Aut}_Y^0}(\Gamma_e) = \Delta(\mathrm{Aut}_V^0)$. *Moreover,*

$$\mathrm{Stab}_{\mathrm{Aut}^0_{Y_{k^{\mathrm{sep}}}}}(\Gamma_e) \cap \mathrm{Stab}_{\mathrm{Aut}^0_{Y_{k^{\mathrm{sep}}}}}(\Gamma_f) = \Delta(\mathrm{Aut}^{0,e\times f}_{V_{k^{\mathrm{sep}}}})$$

for every $f \in F(k^{\mathrm{sep}})$.

Proof. Let T be a scheme and $g, h \in \mathrm{Aut}(V \times T/T)$. Then the resulting $g \times h \in \mathrm{Aut}(Y \times T/T)$ stabilizes $\Delta(V) \times T$ if and only if for any $T' \to T$ and $v \in V(T')$, we have $(g \times h)(v, v) \in \Delta(V)(T')$, in other words $g(v) = h(v)$. This yields the first equality. The second equality is checked similarly. □

Next, recall the identification $G \times e = \mathrm{Aut}_V^{0,e\times G}$; we denote by $\eta : G \to \mathrm{Aut}_V^0$ the corresponding closed immersion.

Lemma 2.6. *The homomorphism of algebraic groups*

$$\eta \times \eta : G \longrightarrow \mathrm{Aut}_V^0 \times \mathrm{Aut}_V^0 = \mathrm{Aut}_Y^0, \quad g \longmapsto ((v, w) \mapsto ((g, e) \cdot v, (g, e) \cdot w)))$$

yields an identification $G = \mathrm{Aut}_{Y,Z}^0$.

Proof. We may assume that k is algebraically closed. Since the kernel of η is trivial, the same holds for $\eta \times \eta$, so the latter is a closed immersion. Let $f \in F$. Then Γ_f is stable by the action of G on Y via $\psi \times \psi$, since $G \times e$ centralizes $e \times f$. Thus, the G-action on Y stabilizes Z, and $\psi \times \psi$ factors over $\mathrm{Aut}_{Y,Z}^0$.

It remains to check that the resulting closed immersion $G \to \mathrm{Aut}_{Y,Z}^0$ is an isomorphism. The group scheme $\mathrm{Aut}_{Y,Z}^0$ stabilizes the irreducible component $\Gamma_f \subset Z$, $f \in F$ in view of Lemma 1.3. In other words, $\mathrm{Aut}_{Y,Z}^0$ normalizes the $\Gamma_f \subset Y$. By Lemma 2.5, it is contained in the intersection of the $\Delta(\mathrm{Aut}_V^{0,e\times f})$. By construction, we have $G = \mathrm{Aut}_V^{0,e\times F} = \bigcap_{f\in F} \mathrm{Aut}_V^{0,e\times f}$, and the assertion follows. □

Step 3. Let X be the normalization of the blow-up $\mathrm{Bl}_Z(Y)$ with respect to the center $Z \subset Y$, and write $f : X \to Y$ for the resulting birational morphism. Then X is again normal, projective, and geometrically integral. Moreover, the G-action on Y via $\eta \times \eta$ lifts uniquely to an action on X via a homomorphism

$$f^* : G \longrightarrow \mathrm{Aut}_X^0 .$$

Lemma 2.7. *If $n \geq 2$ then the above map f^* is an isomorphism.*

Proof. We begin with some observations. We have $\mathcal{O}_Y \xrightarrow{\sim} f_*(\mathcal{O}_X)$ by the normality of Y (Lemma 2.4) and Zariski's Main Theorem. Thus, Blanchard's Lemma yields a homomorphism of algebraic groups

$$f_* : \mathrm{Aut}_X^0 \longrightarrow \mathrm{Aut}_Y^0 .$$

Since f is birational, f^* and f_* have trivial kernels, and hence are closed immersions. Moreover, the composition $f_* \circ f^* : G \to \mathrm{Aut}_Y^0$ equals $\eta \times \eta$, since this holds over a schematically dense open set of Y. Using Lemma 2.6, we may thus identify G with $\mathrm{Aut}_{Y,Z}^0$, and $f_* \circ f^*$ with id. Also, note that the pull-back morphism

$$f_0 : X_0 = f^{-1}(Y_0) \longrightarrow Y_0$$

is the blow-up of the smooth variety Y_0 along the smooth subvariety Z_0 of pure codimension n. In particular, X_0 and the exceptional divisor E_0 of f_0 are smooth. Thus, the schematic closure $E \subset X$ of E_0 is geometrically reduced. Moreover, the schematic image of E in Y is Z, since $f_0(E_0) = Z_0$ is schematically dense in Z (but E may be strictly contained in the exceptional locus of f).

We now show that the closed immersion f^* is an isomorphism, by checking that it is surjective on k^{alg}-points and on Lie algebras (Lemma 1.6). We may thus assume k

algebraically closed. Then $\mathrm{Aut}^0_X(k)$ stabilizes the exceptional locus of f, and hence its irreducible components (Lemma 1.3). Thus, $\mathrm{Aut}^0_X(k)$ stabilizes E, and hence Z. This shows that f^* is surjective on k-points.

Next, we show that $\mathrm{Lie}(f^*) : \mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{Aut}^0_X)$ is surjective. For this, we use a rigidity property of the exceptional divisor E_0. We have an exact sequence of coherent sheaves on X_0:

$$0 \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_0}(E_0) \longrightarrow \mathcal{O}_{E_0}(E_0) \longrightarrow 0,$$

and hence an exact sequence on Y_0

$$0 \longrightarrow f_{0,*}(\mathcal{O}_{X_0}) \longrightarrow f_{0,*}(\mathcal{O}_{X_0}(E_0)) \longrightarrow f_{0,*}(\mathcal{O}_{E_0}(E_0)) \longrightarrow R^1 f_{0,*}(\mathcal{O}_{X_0}).$$

Moreover, $f_{0,*}(\mathcal{O}_{X_0}) = \mathcal{O}_{Y_0}$ by Zariski's Main Theorem again; $f_{0,*}(\mathcal{O}_{X_0}(E_0)) = \mathcal{O}_{Y_0}$ as $\mathrm{codim}_{Y_0} f_0(E_0) = n \geq 2$; and $R^1 f_{0,*}(\mathcal{O}_{X_0}) = 0$ by [1], Exposé VII, Lemme 3.5. Thus, we have $f_{0,*}(\mathcal{O}_{E_0}(E_0)) = 0$. In particular, $H^0(E_0, \mathcal{O}_{E_0}(E_0)) = 0$.

Now let $D \in \mathrm{Lie}(\mathrm{Aut}^0_X) = \mathrm{Der}(\mathcal{O}_X)$. Then D induces $D_0 \in \mathrm{Der}(\mathcal{O}_{X_0}) = H^0(X_0, \Theta_{X_0})$. Let $\mathcal{N}_{E_0} = \mathcal{O}_{E_0}(E_0)$, then the image of D_0 under the composition $\Theta_{X_0} \to \Theta_{X_0}|_{E_0} \to \mathcal{N}_{E_0}$ is 0. As $H^0(E_0, \mathcal{N}_{E_0}) = \mathrm{Hom}_{X_0}(\mathcal{I}_{E_0}, \mathcal{O}_{X_0}/\mathcal{I}_{E_0})$ where $\mathcal{I}_{E_0} = \mathcal{O}_{E_0}(-E_0)$, this means that $D_0 \in \mathrm{Der}(\mathcal{O}_{X_0}; \mathcal{I}_{E_0})$ (the derivations of $\mathcal{O}_{X_0}$ which preserve the sheaf of ideals of E_0). In view of Lemma 1.1, it follows that $D \in \mathrm{Der}(\mathcal{O}_X; \mathcal{I}_E)$. So the image of D in $\mathrm{Der}(\mathcal{O}_Y)$ lies in $\mathrm{Der}(\mathcal{O}_Y; \mathcal{I}_Z)$ (as follows e.g. from Lemma 1.1 again). Since $\mathrm{Der}(\mathcal{O}_Y; \mathcal{I}_Z) = \mathrm{Lie}(G)$ by Lemma 2.6, this completes the proof. □

It remains to treat the case where $n = 1$. Then Y is a surface, and Z a curve. Choose an elliptic curve E with origin 0. Then $\mathrm{Aut}^0_{E,0}$ is trivial, and hence we obtain $\mathrm{Aut}^0_{Y,Z} \xrightarrow{\sim} \mathrm{Aut}^0_{Y\times E, Z\times 0}$ by Blanchard's Lemma. In view of Lemma 2.6, we thus have $G \xrightarrow{\sim} \mathrm{Aut}^0_{Y\times E, Z\times 0}$, where $Y \times E$ is a threefold, and $Z \times 0$ a curve. We now consider the normalized blow-up X of $Y \times E$ along $Z \times 0$, and argue as in the proof of Lemma 2.7 to get $G \simeq \mathrm{Aut}^0_X$.

Remark 2.8. Let G be a smooth connected algebraic group of dimension 1. If the ground field k is perfect, then G is either an elliptic curve, or the additive group $\mathbb{G}_a$, or a k-form of the multiplicative group $\mathbb{G}_m$. In each of these cases, one may easily construct a smooth projective surface X such that $G \simeq \mathrm{Aut}^0_X$ (see the end of Section 2.1 in [5] for details). As a consequence, Theorem 2.1 holds with $\dim(X) = 2n$.

If k is imperfect, then we get in addition the non-trivial k-forms of $\mathbb{G}_a$; these are described in [31]. Given such a k-form G, is there a normal projective surface X such that $G \simeq \mathrm{Aut}^0_X$? If the answer is affirmative, then Theorem 2.1 holds again with $\dim(X) = 2n$. We have just seen that there is a normal projective threefold X with $G \simeq \mathrm{Aut}^0_X$, but one may check that there is no normal projective curve satisfying this assertion.

3. Fitting ideals for Kähler differentials

One crucial idea to prove Theorem 2.1 for non-smooth algebraic groups is to control actions via Fitting ideals for Kähler differentials. In this section we collect some more or less obvious facts, which turn out to be very useful. Let us start by recalling the theory of Fitting ideals (compare [10] and [9], Section 20.2): Suppose

M is a module of finite presentation over some commutative ring R, and choose a particular presentation

$$R^{\oplus n} \xrightarrow{P} R^{\oplus m} \longrightarrow M \longrightarrow 0.$$

Let $\mathfrak{a}_i \subset R$ be the ideal generated by the $(m-i)$-minors of the matrix $P \in \mathrm{Mat}_{m\times n}(R)$, for any integer $i \leq m$. Then $\mathfrak{a}_m$ is the unit ideal, $\mathfrak{a}_{m-1}$ is generated by the matrix entries, and we have $\mathfrak{a}_i \supset \mathfrak{a}_{i-1}$ according to Laplace expansion for determinants. It turns out that these ideals depend only on the module rather than the presentation, and one calls them *Fitting ideals* $\mathrm{Fitt}_i(M) = \mathfrak{a}_i$. Note that $\mathrm{Fitt}_m(M) = R$, and our definition extends to all integers by setting $\mathrm{Fitt}_i(M) = R$ for $i > m$. One immediately sees that $\mathrm{Fitt}_{i+1}(M \oplus R) = \mathrm{Fitt}_i(M)$. Also note that the *annihilator ideal* $\mathfrak{b} = \mathrm{Ann}_R(M)$ is related to Fitting ideals by the formulas $\mathfrak{b} \cdot \mathrm{Fitt}_{i+1}(M) \subset \mathrm{Fitt}_i(M)$ and $\mathfrak{b}^m \subset \mathrm{Fitt}_0(M) \subset \mathfrak{b}$.

Obviously Fitting ideals commute with arbitrary base change, in particular with localizations. This leads to the following generalization: Let $\mathscr{F}$ be a quasicoherent sheaf of finite presentation on some scheme X. Then the Fitting ideals for $M = \Gamma(U, \mathscr{F})$ for the various affine open sets $U = \mathrm{Spec}(R)$ are compatible, thus define a quasicoherent *sheaf of Fitting ideals* $\mathrm{Fitt}_i(\mathscr{F}) \subset \mathscr{O}_X$. Let us write $X_i \subset X$ for the corresponding closed subscheme. By the Nakayama Lemma, the complementary open set $U_i \subset X$ is the locus of all points a having an open neighborhood U such that $\mathscr{F}|U$ can be written as a quotient of $\mathscr{O}_U^{\oplus i}$. The crucial point here is that the Fitting ideal endows the closed set $X_i \subset X$ with an intrinsic scheme structure, compatible with base change along any $X' \to X$. The inclusion of ideals $\mathrm{Fitt}_i(\mathscr{F}) \supset \mathrm{Fitt}_{i-1}(\mathscr{F})$ correspond to inclusions of subschemes $X_i \subset X_{i-1}$.

Now let k be a ground field, and X a k-scheme of finite type. Then the sheaf of Kähler differentials $\Omega^1_X = \Omega^1_{X/k}$ is coherent, and hence quasicoherent and of finite presentation. Consider the Fitting ideals $\mathrm{Fitt}_i(\Omega^1_X)$ and the resulting closed subschemes $X_i \subset X$.

Proposition 3.1. *Assume that X is a G-scheme for some group scheme G. Then the closed subschemes $X_i \subset X$ are G-stable.*

Proof. We have to check that for any scheme T, each T-valued point $g \in G(T)$ normalizes the base change $X_i \times T$. Since Kähler differentials and Fitting ideals commute with base change, it suffices to show that X_i is stable with respect to any T-automorphism $g : X \times T \to X \times T$. Such an automorphism induces a bijection $\varphi : g^*(\Omega^1_{X\times T/T}) \to \Omega^1_{X\times T/T}$, and thus $\mathrm{Fitt}_i(g^*\Omega^1_{X\times T/T}) = \mathrm{Fitt}_i(\Omega^1_{X\times T/T})$. On the other hand, we have $g^{-1}\,\mathrm{Fitt}_i(\Omega^1_{X\times T/T}) \cdot \mathscr{O}_{X\times T} = \mathrm{Fitt}_i(g^*\Omega^1_{X\times T/T})$, again because Fitting ideals commute with base change. So we see that $g : X \times T \to X \times T$ sends the ideal corresponding to $X_i \times T \subset X \times T$ to itself, and thus normalizes $X_i \times T$. $\square$

Let X and Y be schemes that are separated and of finite type, and $f : X \to Y$ be a proper morphism with $f_*(\mathscr{O}_X) = \mathscr{O}_Y$, and suppose that a connected algebraic group G acts on X. According to Blanchard's Lemma, there is a unique G-action on Y making the morphism f equivariant. Let $X_i \subset X$ be the closed subscheme defined by $\mathrm{Fitt}_i(\Omega^1_X)$, and denote its schematic image by $Z_i \subset Y$.

Corollary 3.2. *In the above setting, the closed subschemes $Z_i \subset Y$ are G-stable.*

Proof. This follows by combining Lemma 1.1 and Proposition 3.1. □

4. Computations with Rees rings

Let k be a field, and $R = k[x_1, \ldots, x_n]$ be the polynomial ring in $n \geq 1$ indeterminates x_i. Fix some integers $1 \leq s \leq n$ and $v_s, \ldots, v_n \geq 1$, and consider the ideal

$$\mathfrak{a} = (x_s^{v_s}, \ldots, x_n^{v_n}).$$

We then form the *Rees ring* $R[\mathfrak{a}T]$, which is the R-subalgebra of the polynomial ring $R[T]$ generated by the homogeneous elements $x_s^{v_s}T, \ldots, x_n^{v_n}T$. We now seek to understand, from a purely algebraic view, a certain Fitting ideal for the Kähler differentials $\Omega^1_{R[\mathfrak{a}T]} = \Omega^1_{R[\mathfrak{a}T]/k}$. This bears geometric consequences, because the homogeneous spectrum of the Rees ring is the blow-up of $\mathbb{A}^n = \mathrm{Spec}(R)$ with respect to the center $Z = \mathrm{Spec}(R/\mathfrak{a})$.

The Rees ring $R[\mathfrak{a}T]$ is generated as a graded k-algebra by the homogeneous elements

$$x_1, \ldots, x_n \quad \text{and} \quad x_s^{v_s}T, \ldots, x_n^{v_n}T.$$

Between these generators we have the obvious relations $x_i^{v_i} \cdot x_j^{v_j}T = x_j^{v_j} \cdot x_i^{v_i}T$ for all $s \leq i < j \leq n$, and these actually generate the ideal of all relations. Indeed, the canonical morphism $\mathrm{Sym}^\bullet(\mathfrak{a}) \to R[\mathfrak{a}T]$ is bijective, and the obvious relations generate the ideal of all relations for the symmetric algebra (see [28], Theorem 1 and Lemma 2). In turn, $\Omega^1_{R[\mathfrak{a}T]}$ is generated as module over the Rees ring by $2n - s + 1$ differentials $dx_1, \ldots, dx_n, d(x_s^{v_s}T), \ldots, d(x_n^{v_n}T)$. These are only subject to the $\binom{n-s+1}{2}$ relations $d(x_i^{v_i} \cdot x_j^{v_j}T) = d(x_j^{v_j} \cdot x_i^{v_i}T)$.

We now assume that the ground field k has characteristic $p > 0$, and that there is an integer $s \leq l < n$ such that p divides the the exponents $v_s, \ldots, v_l$ whereas $v_{l+1} = \ldots = v_n = 1$. This ensures

$$d(x_i^{v_i}) = v_i x_i^{v_i - 1} dx_i = \begin{cases} 0 & \text{if } i \leq l; \\ dx_i & \text{else.} \end{cases}$$

The following result computes one Fitting ideal for the Kähler differentials outside the locus defined by the irrelevant ideal $R[\mathfrak{a}T]_+ = (x_s^{v_s}T, \ldots, x_n^{v_n}T)$:

Theorem 4.1. *Assumptions as above. Then the two ideals*

$$\mathrm{Fitt}_{n+l-s+1}(\Omega^1_{R[\mathfrak{a}T]}) \quad \textit{and} \quad (x_s^{v_s}, \ldots, x_n^{v_n}, x_s^{v_s}T, \ldots, x_l^{v_l}T)$$

in the Rees ring $R[\mathfrak{a}T]$ induce the same ideals in the localization $R[\mathfrak{a}T]_g$, for any $g = x_r^{v_r}T$, $s \leq r \leq n$.

Proof. We first reduce the problem to the case $s = 1$. Consider the subrings $R_0 = k[x_1, \ldots, x_{s-1}]$ and $R' = k[x_s, \ldots, x_n]$ inside the Rees ring, and the ideal $\mathfrak{a}' = (x_s^{v_s}, \ldots, x_n^{v_n})$ inside R'. We then have a tensor product decomposition

$$R[\mathfrak{a}T] = k[x_1, \ldots, x_{s-1}, x_s, \ldots, x_n, x_s^{\nu_s}T, \ldots, x_n^{\nu_n}T] = R_0 \otimes_k R'[\mathfrak{a}'T]$$

of the Rees ring, and thus a direct sum decomposition

$$\Omega^1_{R[\mathfrak{a}T]} = \left(\Omega^1_{R_0} \otimes_{R_0} R[\mathfrak{a}T]\right) \oplus \left(\Omega^1_{R'[\mathfrak{a}'T]/k} \otimes_{R'[\mathfrak{a}'T]} R[\mathfrak{a}T]\right)$$

for the Kähler differentials (see [9], Proposition 16.5). Using that the summand on the left is free of rank $s-1$, we infer

$$\mathrm{Fitt}_{n+l-s+1}(\Omega^1_{R[\mathfrak{a}T]}) = \mathrm{Fitt}_{(n-s+1)+(l-s+1)}(\Omega^1_{R'[\mathfrak{a}'T]}) \otimes_{R'[\mathfrak{a}'T]} R[\mathfrak{a}T].$$

This reduces our problem to the case $s = 1$.

In this situation, we have $2n$ generators, which come in three kinds, namely

$$dx_1, \ldots, dx_l \quad \text{and} \quad dx_{l+1}, \ldots, dx_n \quad \text{and} \quad d(x_1^{v_1}T), \ldots, d(x_n^{v_n}T).$$

Likewise, we group the relations into three types, using the product rule and our assumptions on the exponents: First we have the $\binom{l}{2}$ relations

$$x_i^{v_i} \cdot d(x_j^{v_j}T) - x_j^{v_j} \cdot d(x_i^{v_i}T), \quad 1 \le i < j \le l. \tag{1}$$

Then there are the $l(n-l)$ relations

$$x_i^{v_i} \cdot d(x_jT) - x_j \cdot d(x_i^{v_i}T) - x_i^{v_i}T \cdot d(x_j), \quad 1 \le i \le l \text{ and } l+1 \le j \le n. \tag{2}$$

Finally, the remaining $\binom{n-l}{2}$ relations take the form

$$x_i \cdot d(x_jT) + x_jT \cdot dx_i - x_j \cdot d(x_iT) - x_iT \cdot dx_j, \quad l+1 \le i < j \le n. \tag{3}$$

Recall that we assume $p \mid v_i$ for $s \le i \le l$, and $v_{l+1} = \ldots = v_n = 1$, which indeed enters in the above formulas. Using these generators and relations, we obtain a presentation of $\Omega^1_{R[\mathfrak{a}T]}$ by a matrix P of size $2n \times \binom{n}{2}$. Since we have three kinds of generators and three types of relations, one can view P as a block matrix with $3 \cdot 3 = 9$ blocks, roughly indicated as follows:

$$P = \left(\begin{array}{c|c|c} & & \\ \hline & & x_jT \\ & -x_i^{v_i}T & -x_iT \\ \hline -x_j^{v_j} & -x_j & \\ x_i^{v_i} & & \\ & x_i^{v_i} & -x_j \\ & & x_i \end{array}\right) \tag{4}$$

Note that the columns of P are indexed by the 2-element subsets $\{i,j\} \subset \{1,\ldots,n\}$, which we regard as $1 \le i < j \le n$, equipped with the lexicographic order. Also note that each column contains either two, three or four non-zero entries, and that all three upper blocks are zero.

We have to gain control on the minors of size $2n-(n+l) = n-l$, after localization of $g = x_r^{v_r}T$ with fixed $1 \le r \le n$. The case $l = n-1$ can be dealt with immediately: Then the minors in question are just the matrix entries, and the assertion becomes obvious. Note that in this case, the three blocks to the right are empty. From now on, we assume that $l \le n-2$, such that the blocks on the right are non-empty.

Suppose $r \le l$. After renumbering, we may assume $r = 1$. Now look at the central block: The $n-l$ rows indexed by $\{i,j\}$ with $i = 1$ and $l+1 \le j \le n$ yield a scalar submatrix of the form $x_1^{v_1}T \cdot E_{n-l}$, whose scalar $x_1^{v_1}T = g$ is invertible in the

localization $R[\mathfrak{a}T]_g$. Here E_{n-l} denotes the identity matrix of size $(n-l)\times(n-l)$. Consequently, the Fitting ideal becomes the unit ideal upon localization, and the same obviously holds for the ideal $\mathfrak{b} = (x_1^{v_1}, \dots, x_n^{v_n}, x_1^{v_1}T, \dots, x_l^{v_l}T)$.

We now suppose $r \geq l+1$. Our task is to verify the equality

$$\mathrm{Fitt}_{n+l}(\Omega^1_{R[\mathfrak{a}T]})_g = \mathfrak{b}_g$$

of ideals in $R[\mathfrak{a}T]_g$. After renumbering, we may assume $r = l+1$, so that $g = x_{l+1}T$. We first check that every generator of $\mathfrak{b}$ is contained in the localized Fitting ideal. The trick is to examine the right middle block: Removing its first row and keeping the $n-l-1$ columns indexed by $\{i,j\}$ with $i = l+1$ and $l+2 \leq j \leq n$ yields the scalar submatrix $-gE_{n-l-1}$. Now fix some $i \leq l$ and look at the central block: Its column corresponding to $\{i, l+1\}$ has as single entry $-x_i^{v_i}T$. We infer that P contains an upper triangular submatrix of the form

$$\begin{pmatrix} -x_i^{v_i}T & * \\ 0 & -gE_{n-l-1} \end{pmatrix} \in \mathrm{Mat}_{n-d}(R[\mathfrak{a}T])$$

and conclude that $x_i^{v_i}T$ is contained in the localized Fitting ideal, for all $1 \leq i \leq l$. But then also $x_i^{v_i} = x_{l+1}/g \cdot x_i^{v_i}T$ belongs to it. For $j \geq l+1$, we use the lower middle block to produce a submatrix of the form

$$\begin{pmatrix} * & -gE_{n-l-1} \\ -x_j & 0 \end{pmatrix} \in \mathrm{Mat}_{n-d}(R[\mathfrak{a}T]).$$

Now Laplace expansion reveals that x_j belongs to the localized Fitting ideal.

It remains to verify that the localized Fitting ideal is contained in $\mathfrak{b}_g$, which is the most difficult part of the argument. Note that all but the entries from the right middle block B already belong to $\mathfrak{b}$. We seek to remove the critical entries from the matrix by using elementary row and column operations. For this we have to inspect B in more detail. In light of the lexicographic order on the indices $\{i,j\}$, it takes the form

$$B = \left(\begin{array}{ccc|ccc} x_{l+2}T & \cdots & x_nT & 0 & \cdots & 0 \\ \hline -g & & & & & \\ & \ddots & & & * & \\ & & -g & & & \end{array}\right),$$

with $n-l$ rows and $\binom{n-l}{2}$ columns. Note that the columns to the left have indices $\{i,j\}$ with $i = l+1$ and $l+2 \leq j \leq n$. To proceed, we perform elementary row operations over the localization $R[\mathfrak{a}T]_g$ to remove the entries $x_{l+2}T, \dots, x_nT$ from the top row. The crucial observation is that this process does not afflict the remaining zeros in the top row: For any index $\{j, j'\}$ with $l+2 \leq j < j' \leq n$ the corresponding column in B contains two non-zero entries, namely $x_{j'}T$ and $-x_jT$. The combined row operations add

$$\frac{x_jT}{-g} \cdot (x_{j'}T) + \frac{x_{j'}T}{-g} \cdot (-x_jT) = 0$$

into the $\{j, j'\}$-position of the top row. Summing up, the top row becomes trivial. Performing additionally elementary column operations, we make the lower right block zero as well.

Now examine the effect of the above elementary row and column operations on the whole matrix P. The result is a matrix P' of the form

$$P' = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline * & * & B' \\ \hline * & * & * \end{array}\right),$$

where $B' = \left(\begin{smallmatrix} 0 & 0 \\ -gE_{n-l-1} & 0 \end{smallmatrix}\right)$, and all entries from the $*$-blocks belong to the ideal $\mathfrak{b}_g$. Performing further elementary row and column operations, we may assume that in P' also the neighbors to the left and below of the subblock $-gE_{n-l-1}$ become zero. Clearly, the entries in the $*$-blocks still belong to the ideal $\mathfrak{b}_g$. As explained in [9], page 493 our localized ideal $\mathrm{Fitt}_{n+l}(\Omega^1_{R[\mathfrak{a}T]})_g$, which is generated by the $(n-l)$-minors of the matrix P', is also generated by the entries of the submatrix Q' obtained by removing the rows and columns passing through the subblock $-gE_{n-l-1}$. It follows that the localized Fitting ideal must belong to $\mathfrak{b}_g$. □

We now translate the above result into more geometric form: Set $Y = \mathbb{A}^n = \mathrm{Spec}(R)$ and $Z = \mathrm{Spec}(\bar{R})$, where $\bar{R} = R/\mathfrak{a}$. Form the blow-up $X = \mathrm{Bl}_Z(Y)$ and let $f : X \to Y$ be the ensuing morphism. The exceptional divisor $E = f^{-1}(Z)$ is the homogeneous spectrum of

$$R[\mathfrak{a}T] \otimes_R \bar{R} = \bar{R}[x_s^{v_s}T, \ldots, x_n^{v_n}T].$$

This gives an identification $E = \mathbb{P}^{n-s}_{\bar{R}}$, and we may regard the generators $x_i^{v_i}T$ as global sections for the invertible sheaf $\mathscr{O}_X(1) = \mathscr{O}_X(-E)$.

Corollary 4.2. *In the above setting, the closed subscheme $X' \subset X$ corresponding to the Fitting ideal $\mathrm{Fitt}_{n+l-s}(\Omega^1_X)$ is the linear subscheme inside the exceptional divisor $E = \mathbb{P}^{n-s}_{\bar{R}}$ given by the equations $x_i^{v_i}T = 0$ for $s \le i \le l$. If furthermore $n \ge l+1$, then the schematic image of X' coincides with the center $Z \subset Y$.*

Proof. Write $S = R[\mathfrak{a}T]$ for the Rees ring, and fix one of the homogeneous generators $g = x_i^{v_i}T$ of degree one. Consider the localization S_g and its degree-zero part $S_{(g)}$. The latter defines the basic open set $D_+(g) = \mathrm{Spec}(S_{(g)})$ inside the blow-up $X = \mathrm{Proj}(S)$. According to [16], paragraph (2.2.1) the canonical map $S_{(g)} \otimes_k k[T^{\pm 1}] = S_{(g)}[T^{\pm}] \to S_g$ given by the assignment $T \mapsto g/1$ is bijective. In turn, we get a decomposition

$$\Omega^1_{S_g} = (\Omega^1_{S_{(g)}} \otimes_{S_{(g)}} S_g) \oplus (\Omega^1_{k[T^{\pm 1}]} \otimes_{k[T^{\pm 1}]} S_g),$$

by [9], Proposition 16.5. The summand on the right is free of rank one, and we conclude that the j-th Fitting ideal for $\Omega^1_{S_g}$ inside S_g is induced from the $(j-1)$-th Fitting ideal for $\Omega^1_{S_{(g)}}$ inside the homogeneous localization $S_{(g)}$.

We now apply Theorem 4.1 with $j = n+l-s+1$, and see that X' is contained in the exceptional divisor $E = \mathbb{P}^{n-s}_{\bar{R}}$. In fact, it is a linear subscheme of codimension $l-s+1$. We thus have $X' = \mathbb{P}^d_{\bar{R}}$ with $d = (n-s)-(l-s+1) = n-l-1$. The schematic image is given by the spectrum of $H^0(X', \mathscr{O}_{X'})$. Under the additional condition $n \ge l+1$ we have $d \ge 0$, and this ring of global sections is indeed $\bar{R}$. □

5. The non-smooth case

This section contains the proof of Theorem 2.1 in the case that G is non-smooth. Then our ground field k has characteristic $p > 0$. The proof requires some preparation and appears at the end of the section.

Let us first consider the following situation: Suppose that G acts freely on some geometrically integral scheme of finite type V. Then there exists a dense open G-stable set $V_0 \subset V$ such that the quotient V_0/G exists as a scheme, V_0/G is of finite type, and the projection $V_0 \to V_0/G$ is a G-torsor (see [7], Exposé V, Théorème 8.1). We may also assume that V_0 is smooth by using Lemma 1.2; then V_0/G is smooth as well.

Replacing V with V_0, we may therefore assume that V is smooth and the total space of some G-torsor $V \to V/G$, where V/G is a smooth scheme of finite type. We thus find a finite étale subscheme $F \subset V/G$. The cartesian square

$$\begin{array}{ccc} W & \longrightarrow & F \\ \downarrow & & \downarrow \\ V & \longrightarrow & V/G \end{array}$$

defines a G-stable closed subscheme $W \subset V$. We now consider the blow-up $U = \mathrm{Bl}_W(V)$, and write $f : U \to V$ for the ensuing morphism. Define integers by the equations

$$n = \dim(V) \quad \text{and} \quad s - 1 = \dim(G) \quad \text{and} \quad l = \dim_k(\mathfrak{g}), \tag{5}$$

where $\mathfrak{g} = \mathrm{Lie}(G)$.

Proposition 5.1. *In the above setting, let $U' \subset U$ be the closed subscheme defined by the sheaf of ideals $\mathrm{Fitt}_{n+l-s}(\Omega^1_U)$, and $V' \subset V$ the schematic image of U'. If $n \geq l + 1$ then V' coincides with the center $W \subset V$.*

Proof. To verify this equality of closed subschemes, it suffices to treat the case that k is algebraically closed. Then F is a finite sum of copies of $\mathrm{Spec}(k)$. By passing to suitable open sets, it suffices to treat the case that $F = \mathrm{Spec}(k)$. Fix a closed point $a \in V$. Our task is to check that the two closed subschemes $V', W \subset V$ give the same ideal in the formal completion $\mathscr{O}^\wedge_{V,a}$. To proceed, note that $s = \dim(G) + 1 \leq \dim_k(\mathfrak{g}) = l$ because G is non-smooth. Thus $n + l - s \geq n$. Since Ω^1_V is locally free of rank n we see that $\mathrm{Fitt}_{n+l-s}(\Omega^1_V) = \mathcal{O}_V$. It follows that $V' \subset W$, at least as closed sets, and this already settles the case $a \notin W$.

Suppose now that $a \in W$. Then the orbit map $G \to U$, $g \mapsto g \cdot a$ gives an identification $G = W$. Set $R = \mathscr{O}^\wedge_{V,a}$ and $A = \mathscr{O}^\wedge_{W,a} = \mathscr{O}^\wedge_{G,e}$. According to [8], Chapter III, Corollary 6.4, we have $A = k[[u_1, \ldots, u_l]]/(u_s^{v_s}, \ldots, u_l^{v_l})$ for some $1 \leq s \leq l$ and some p-powers $v_s, \ldots, v_l \neq 1$. Moreover, R is isomorphic to a formal power series ring in n indeterminates over the field k. By Lemma 5.2 below, there is a regular system of parameters $x_1, \ldots, x_n \in R$ such that the kernel of the surjection $R \to A$ is generated by $x_s^{v_s}, \ldots, x_l^{v_l}, x_{l+1}, \ldots, x_n$. Setting $v_{l+1} = \ldots = v_n = 1$ we get

$$\mathscr{O}^\wedge_{V,a} = k[[x_1, \ldots, x_n]] \quad \text{and} \quad \mathscr{O}^\wedge_{W,a} = k[[x_1, \ldots, x_n]]/(x_s^{v_s}, \ldots, x_n^{v_n}).$$

This is exactly the situation in Corollary 4.2, after base change from the polynomial ring $k[x_1, \ldots, x_n]$ to the formal power series ring. Since the formation of blow-ups and schematic images commute with flat base change, and formation of Fitting ideals commutes with arbitrary base change, the corollary ensures that the center and schematic image coincide in $\mathrm{Spec}(\mathscr{O}_{V,a}^{\wedge})$. $\square$

In the preceding proof we have used some facts on formal power series rings: Suppose A is a local noetherian ring of the form $A = k[[u_1, \ldots, u_l]]/(f_s, \ldots, f_l)$, where $u_1, \ldots, u_l$ are indeterminates and $f_s, \ldots, f_l$ are formal power series that form a regular sequence contained in $\mathfrak{m}_R^2$. Then $\dim(A) = l - (l - s + 1) = s - 1$, whereas the *embedding dimension* equals $\mathrm{edim}(A) = \dim_k(\mathfrak{m}_A/\mathfrak{m}_A^2) = l$. Now let $\varphi : R \to A$ be any surjection from a formal power series ring R in $n \geq l$ indeterminates over the field k. Then $\mathfrak{a} = \mathrm{Ker}(\varphi)$ is generated by a regular sequence contained in $\mathfrak{m}_R$, according to [18], Proposition 19.3.2. The following observation gives additional information:

Lemma 5.2. *In the above situation, there exists a regular system of parameters $x_1, \ldots, x_n \in R$ such that $\mathfrak{a}$ is generated by $f_i(x_1, \ldots, x_l)$ with $s \leq i \leq l$, together with the x_j with $l + 1 \leq j \leq n$.*

Proof. Consider the short exact sequence of vector spaces

$$0 \longrightarrow \mathfrak{a}/(\mathfrak{a} \cap \mathfrak{m}_R^2) \longrightarrow \mathfrak{m}_R/\mathfrak{m}_R^2 \longrightarrow \mathfrak{m}_A/\mathfrak{m}_A^2 \longrightarrow 0.$$

Write $\bar{u}_i \in A$ for the classes of the indeterminates u_i. Choose $x_1, \ldots, x_l \in \mathfrak{m}_R$ mapping to $\bar{u}_1, \ldots, \bar{u}_l \in \mathfrak{m}_A$. The latter form a basis of the cotangent space $\mathfrak{m}_A/\mathfrak{m}_A^2$, so the former stay linearly independent in $\mathfrak{m}_R/\mathfrak{m}_R^2$. Choose $x_{l+1}, \ldots, x_n \in R$ that belong to $\mathfrak{a}$ and that yield a basis of $\mathfrak{a}/(\mathfrak{a} \cap \mathfrak{m}_R^2)$. Then $x_1, \ldots, x_n$ form a basis in $\mathfrak{m}_R/\mathfrak{m}_R^2$, hence constitute a regular system of parameters in R. Set $g_i = f_i(x_1, \ldots, x_l)$ for $s \leq i \leq l$, and form the ideal $\mathfrak{b} = (g_s, \ldots, g_l, x_{l+1}, \ldots, x_n)$. Clearly $\mathfrak{b} \subset \mathfrak{a}$, so the surjection $\varphi : R \to A$ induces a map

$$k[[x_1, \ldots, x_r]]/(g_s, \ldots, g_l) = R/\mathfrak{b} \longrightarrow A = k[[u_1, \ldots, u_l]]/(f_s, \ldots, f_l)$$

This sends generators to generators, and relations to relations, whence is bijective. Hence the kernel $\mathfrak{a}/\mathfrak{b}$ vanishes, and thus $\mathfrak{b} = \mathfrak{a}$. $\square$

Proof of Theorem 2.1 for non-smooth G. According to Proposition 1.7, the algebraic group G embeds into a smooth connected algebraic group H. By Section 2, there is a projective scheme Y with $\mathrm{Aut}_Y^0 = H$, and we may furthermore assume that Y is normal and geometrically integral, and that there is a dense open set $V \subset Y$ that is H-stable, with free H-action. We may also assume that $\dim(Y) = \max(2\dim(H), 3)$. Choose $W \subset V$ as described before Proposition 5.1, write $Z \subset Y$ for its schematic closure, let $X = \mathrm{Bl}_Z(Y)$ be the blow-up, and $f : X \to Y$ be the resulting birational morphism. Since Y is normal, the map $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is bijective, according to Zariski's Main Theorem.

By Blanchard's Lemma, there is a unique action of Aut_X^0 on Y making the morphism $f : X \to Y$ equivariant. As f is birational, we thus get an inclusion $\mathrm{Aut}_X^0 \subset \mathrm{Aut}_Y^0 = H$. On the other hand, since the G-action on V stabilizes W, the G-action on Y stabilizes the schematic closure Z (Lemma 1.1). So this action lifts uniquely to an action on X (see [26], Proposition 2.7). This yields an inclusion

$G \subset \mathrm{Aut}^0_X$, and our task is to show that this is an equality. For this we may assume that k is algebraically closed.

As in (5), we define integers $n = \dim(X)$ and $s-1 = \dim(G)$ and $l = \dim(\mathfrak{g})$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Consider the open set $U = f^{-1}(V)$, and let $U' \subset U$ be the closed subscheme corresponding to the sheaf of Fitting ideals $\mathrm{Fitt}_{n+l-s}(\Omega^1_U)$. We have $n = \dim(X) \geq 2\dim(H) \geq 2l \geq l+1$ by our assumptions. According to Proposition 5.1, the schematic image of U' in V coincides with W. Since Aut^0_X stabilizes the open set $U = f^{-1}(V)$, it must also stabilize U'. In turn, its action on V stabilizes W. Fix a closed point $a \in W$, and set $G' = \mathrm{Aut}^0_X$. Recall that $H = \mathrm{Aut}^0_Y$ acts freely on V. So the orbit maps give inclusions of schemes $G \subset G' = G' \cdot a \subset G \cdot a = G$, thus $G = G'$. □

Remark 5.3. In general, the scheme X constructed in the above proof is non-normal. Consider indeed the finite group scheme $G = \alpha_p \times \alpha_{p^2}$. Then G is a subgroup scheme of the smooth connected algebraic group $H = \mathbb{G}_a \times \mathbb{G}_a$. So the above construction yields a G-scheme X which contains a G-stable dense open set, isomorphic to the blow-up of a dense open set of $H \times H = \mathbb{A}^4$ along the G-orbit of a k-rational point. To show that X is non-normal, it suffices to check that the blow-up of $\mathbb{A}^4$ along the G-orbit of the origin is non-normal along the exceptional divisor. We then have $R = k[x_1, x_2, x_3, x_4]$, $\mathfrak{a} = (x_3^p, x_4^{p^2})$ and $S = R[\mathfrak{a}T] = k[x_1, x_2, x_3, x_4, x_3^pT, x_4^{p^2}T]$. Let $g = x_3^pT$, then the degree-0 localization $S_{(g)}$ satisfies

$$S_{(g)} = k[x_1, x_2, x_3, x_4, x_4^{p^2}/x_3^p].$$

The normalization of $S_{(g)}$ in its fraction field contains x_4^p/x_3, which is not in $S_{(g)}$. So $S_{(g)}$ is indeed non-normal along the exceptional divisor (the zero subscheme of x_3).

6. Conflicts of interest

The authors have no conflicts of interest to declare that are relevant to this article.

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