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# ALGEBRAIC SPACES THAT BECOME SCHEMATIC AFTER GROUND FIELD EXTENSION

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Abstract. We construct examples of non-schematic algebraic spaces that become schemes after finite ground field extensions.

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## Introduction

Roughly speaking, *algebraic spaces* are generalizations of *schemes* that allow more freely the formation of contractions and quotients, which are somewhat problematic in the category of schemes. For example, if $E$ is a connected curve on a smooth projective surface $Y$ such that the irreducible components form a negative-definite intersection matrix $N = (E_i \cdot E_j)$, the contraction of the curve to a point exists as an algebraic space $X$ that is usually non-schematic if $h^1(\mathscr{O}_E) > 0$. Furthermore, if a finite group $G$ acts freely on a separated scheme $Y$ of finite type, the quotient $X = G\backslash Y$ exists as an algebraic space, but it must be non-schematic if some orbit $G \cdot a$ does not admit an affine open neighborhood.

Despite their importance, it is by no means straightforward to provide geometrically meaningful examples of algebraic spaces that are indeed not schemes. The goal of this note is to highlight the ubiquity of such non-schematic algebraic spaces: We construct algebraic spaces $X$ over non-closed ground fields $k$ that are non-schematic, but where the base-change $X \otimes k'$ to some finite Galois extension become schemes. The construction is easy, and relies on schemes $X_0$ on which some finite quotient $G$ of the absolute Galois group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ acts such that some orbits do not admit affine open neighborhoods. The procedure is formalized in Theorem 1.1, which gives a criterion for Galois twists to be non-schematic.

Our first concrete example comes from a proper normal surface $X_0$ birational to $E \times \mathbb{P}^1$, for some elliptic curve $E$ where the group of rational points $E(k)$ is not torsion. This depends on earlier constructions of myself [19].

The second example is actually a general construction starting from any separated normal scheme $\tilde{Y}$ of finite type that does not admit an ample invertible sheaf. Now $X_0$ is obtained from the disjoint union of two copies of $\tilde{Y}$ via an identification of certain closed points. The idea is extracted from a beautiful observation of Artin ([2], page 286). It also relies on a result of Benoist [4] that there are only finitely many maximal quasiprojective open sets in $\tilde{Y}$, which solves a conjecture by of Białynicki-Birula ([5], page 302), and generalizes earlier results of Kleiman [15] and Włodarczyk [21].

The reviewer pointed out that Huruguen [14] studied schemes such that some base-change aqcuire the structure of a toric or spherical variety, and that in this framework one may also obtain non-schematic algebraic spaces whose base-change become toric or spherical varieties. Let me also mention that there are families of curves $X \to \mathrm{Spec}(R)$ over discrete valuation rings $R$ where $X$ is non-schematic but the base-change to the henselization $R'$ becomes schematic (see [6], Section 6.7 and [20]).

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## 1. Non-schematic twists

Let $S$ be a base scheme, and write $(\mathrm{Aff}/S)$ for the category of affine $S$-schemes. Recall that an *algebraic space* is a contravariant functor

$$F : (\mathrm{Aff}/S) \longrightarrow (\mathrm{Set})$$

satisfying the sheaf axiom for the étale topology, such that the diagonal monomorphism $F \to F \times F$ is relatively representable by schemes, and that there is an étale surjection $U \to F$ from a scheme. Each scheme $X$ can be viewed as an algebraic space, via the Yoneda embedding, and an algebraic space that corresponds to a scheme is called *schematic*. For more details on algebraic spaces, we refer to the monographs of Artin [3], Knutson [16] and Olsson [18].

A tremendous advantage of algebraic spaces is that forming quotient is permissible, in the following very general setting: Let $G$ is an algebraic space endowed with a group structure acting on an algebraic space $F$. Suppose the action is free, and that the structure morphism $G \to S$ is flat and locally of finite type. Then the sheaf quotient $G\backslash F$ is an algebraic space, and the quotient map $F \to G\backslash F$ is flat and locally of finite presentation (for example [17], Lemma 1.1).

For the sake of exposition, we now suppose that $S$ is the spectrum of a ground field $k$. Let $X_0$ be an algebraic space that is separated and of finite type. Then $\mathrm{Aut}_{X_0/k}$ is a group scheme that is locally of finite type. Let $G$ be any group scheme that is locally of finite type, $G \to \mathrm{Aut}_{X_0/k}$ be a homomorphism, and consider the resulting $G$-action on $X_0$. For each principal homogeneous $G$-space $P$, the diagonal action on the product $P \times X_0$ is free, hence the quotient

$$X = {}^P X_0 = P \wedge^G X_0 = G\backslash(P \times X_0)$$

is an algebraic space. Composing the quotient map $q$ for the diagonal action with the inclusion $\Delta_P$ of the diagonal for the principal homogeneous space, we obtain

$$P \times X_0 \stackrel{\Delta_P \times \mathrm{id}_{X_0}}{\longrightarrow} (P \times P) \times X_0 = P \times (P \times X_0) \stackrel{\mathrm{id}_P \times q}{\longrightarrow} P \times X,$$

which is an isomorphism compatible with the projection to $P$. So for each morphism $\operatorname{Spec}(L) \to P$ with some field $L$, we get a canonical identification $X_0 \otimes L = X \otimes L$. In particular, $X$ is a *twisted form* of $X_0$.

We are primarily interested in the case that $G \subset \operatorname{Aut}(X_0)$ is a finite subgroup and $P = \operatorname{Spec}(k')$, where $k'$ is a Galois extension with $G = \operatorname{Gal}(k'/k)^{\mathrm{op}}$. Then $X = {}^P X_0$ is called the *Galois twist* of $X_0$ with respect to $L$. By the above, we have a canonical identification $X \otimes k' = X_0 \otimes k'$. This leads to the following fact:

**Theorem 1.1.** *In the above setting, suppose $X_0$ is schematic, and that there is a rational point $a \in X_0$ whose orbit $G \cdot a \subset X_0$ does not admit an affine open neighborhood. Then the Galois twist $X$ is non-schematic, whereas its base-change $X \otimes k'$ becomes schematic.*

*Proof.* The orbit $Z_0 = G \cdot a$ is finite, hence schematic. Write $A_0 = H^0(Z_0, \mathscr{O}_{Z_0})$ for the coordinate ring. Then the quotient $Z = G\backslash(Z_0 \otimes k')$ is the spectrum for the ring of invariants $L = (A_0 \otimes k')^G$, which coincides with the fixed field $L \subset k'$ for the stabilizer subgroup $H = G_a$. This defines a morphism $\operatorname{Spec}(L) \to X$. One may regard this as a point on the algebraic space, and we now check that $X$ is not schematic near this point.

Seeking a contradiction, we assume that there is an affine open subspace $U \subset X$ over which $\operatorname{Spec}(L) \to X$ factors. Then $U \otimes k' \subset X \otimes k' = X_0 \otimes k'$ is an affine open neighborhood of $Z_0 \otimes k'$. We now use that the projection $X_0 \otimes k' \to X_0$ is the quotient with respect to the Galois action. Clearly

$$V' = \bigcap_{\sigma \in G} (\mathrm{id}_{X_0} \otimes \sigma)(U \otimes k') \subset X_0 \otimes k'$$

is a Galois-invariant affine open neighborhood of $Z_0 \otimes k'$, and its quotient by the Galois action defines an affine open neighborhood for $Z_0$ inside $X_0$, contradiction. Thus the twisted form $X$ is non-schematic. On the other hand, the identification $X \otimes k' = X_0 \otimes k'$ shows that its base-change to $k'$ becomes schematic. □

## 2. Examples

Fix a ground field $k$. We now give some geometrically meaningful examples for which Theorem 1.1 applies.

First consider the smooth projective surface $E \times \mathbb{P}^1$, where $E$ is an elliptic curve. Fix rational points $u, v \in E$ and $\lambda, \mu \in \mathbb{P}^1$. We recall from [19] how this leads to a proper normal surface $X_0$ with two elliptic singularities: Write $\operatorname{Bl}_Z(E \times \mathbb{P}^1)$ for the blowing-up with respect to the center $Z = \{(u, \lambda), (v, \mu)\}$. Let $D_\lambda, D_\mu \subset \operatorname{Bl}_Z(E \times \mathbb{P}^1)$ be the strict transform of the fibers $E \times \{\lambda\}$ and $E \times \{\mu\}$, respectively. These are copies of the elliptic curve $E$, each with self-intersection $s = -1$. In turn, $D = D_\lambda \cup D_\mu$ is a negative-definite curve, so the contraction $f : \operatorname{Bl}_Z(E \times \mathbb{P}^1) \to X_0$ of $D$ exists as an algebraic space ([1], Corollary 6.12), which here is actually a scheme ([19], Section 2.3). This $X_0$ is a proper normal surface with $h^0(\mathscr{O}_{X_0}) = 1$, and its singular locus comprises the two rational points $a = f(D_\lambda)$ and $b = f(D_\mu)$.

**Proposition 2.1.** *There is an involution $\sigma \in \operatorname{Aut}(X_0)$ with $\sigma(a) = b$.*

*Proof.* Without loss of generality we may assume that $\lambda = (1:0)$ and $\mu = (0:1)$. Then the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(k)$ defines an involution $\tau_2 \in \mathrm{Aut}(\mathbb{P}^1)$ with $\tau_2(\lambda) = \mu$. Furthermore, $x \mapsto -x + (u+v)$ is an involution $\tau_1 \in \mathrm{Aut}(E)$ of the underlying genus-one curve with $\tau_1(u) = v$.

The diagonal involution $\tau = \tau_1 \times \tau_2$ on $E \times \mathbb{P}^1$ leaves the center invariant, thus induces an involution on $\mathrm{Bl}_Z(E \times \mathbb{P}^1)$. This induced involution clearly permutes the fibers of the projection to $\mathbb{P}^1$. It follows that the strict transform $D$ of $E \times \{\lambda, \mu\}$ is an invariant subset, and its two connected components are permuted by the induced involution. By the universal property of contractions, we obtain the desired involution $\sigma : X_0 \to X_0$ that permutes the two singularities $a, b \in X_0$. □

Benoist ([4], Theorem 9) showed that each separated normal scheme $Y$ of finite type contains only finitely many maximal quasiprojective open sets $U_1, \ldots, U_r \subset Y$. This solved a conjecture of Białynicki-Birula ([5], page 302), and generalizes earlier results of Kleiman [15] and Włodarczyk [21]. For information on the number $r \geq 0$, see [7], [8] and [9].

**Proposition 2.2.** *If the difference $u - v \in E(k)$ has infinite order, then the proper normal scheme $X_0$ admits no ample invertible sheaf, and there are exactly $r = 2$ maximal quasiprojective open sets, namely*

$$U_1 = X_0 \smallsetminus \{a\} \quad \textit{and} \quad U_2 = X_0 \smallsetminus \{b\}.$$

*In particular, the points $a, b \in X_0$ do not admit a common affine open neighborhood.*

*Proof.* According to [19], Section 2.3 the individual contraction of $D_\lambda \subset \mathrm{Bl}_Z(E \times \mathbb{P}^1)$ yields a projective scheme, which contains a copy of $U_2$ as an open subscheme. Hence $U_2$ is quasiprojective. Suppose it would not be maximal. Then $X_0$ is quasiprojective, so there is a curve $D_0 \subset X_0$ disjoint from $\{a, b\}$. Its strict transform $D \subset E \times \mathbb{P}^1$ is a Cartier divisor with

$$D \cap (E \times \{\lambda\}) = m_1 \cdot (u, \lambda) \quad \text{and} \quad D \cap (E \times \{\mu\}) = m_2 \cdot (v, \mu)$$

for some $m_1, m_2 \geq 1$, as explained in [19], Section 2.5. Interpreting these multiplicities as intersection numbers of the invertible sheaf $\mathscr{L} = \mathscr{O}_{E \times \mathbb{P}^1}(D)$ with linearly equivalent curves, we see that the numbers must coincide. Write $m \geq 1$ for the common value. Since the canonical map $\mathrm{Pic}(E) \oplus \mathrm{Pic}(\mathbb{P}^1) \to \mathrm{Pic}(E \times \mathbb{P}^1)$ is bijective, one gets $\mathscr{O}_E(u)^{\otimes m} \simeq \mathscr{O}_E(v)^{\otimes m}$. Let $\infty \in E$ be the point at infinity, which is the zero element in the group $E(k)$. Using the identification $E(k) = \mathrm{Pic}^0(E)$ given by $w \mapsto \underline{\mathrm{Hom}}(\mathscr{O}_E(\infty), \mathscr{O}_E(w))$, we infer that $u - v \in E(k)$ is annihilated by the integer $m \geq 1$, contradiction. The same argument applies to $U_1$. Summing up, the $U_1, U_2 \subset X_0$ are maximal quasiprojective open sets.

Suppose there is another maximal quasiprojective open set $U \neq U_i$. It must contain both $a, b \in X_0$, hence the two points admit a common affine open neighborhood. With [11], Chapter II, Proposition 3.1 we get a curve $D_0 \subset X_0$ as above and again reach a contradiction. □

We now sketch a completely different construction that gives schemes $X_0$ for which Theorem 1.1 applies. This relies on the following general procedure: Consider any separated scheme $\tilde{Y}$ of finite type. Let $Z \subset \tilde{Y}$ be a finite closed subscheme that admits an affine open neighborhood, and write $A = H^0(Z, \mathscr{O}_Z)$ for the coordinate ring. We can form the cocartesian square

$$\begin{array}{ccc} \operatorname{Spec}(A) & \longrightarrow & \tilde{Y} \\ \big\downarrow & & \big\downarrow q \\ \operatorname{Spec}(k) & \xrightarrow[a]{} & Y \end{array}$$

in the category of sheaves on $(\mathrm{Aff}/k)$. Note that the upper horizontal arrow is the closed embedding of the finite scheme $Z = \operatorname{Spec}(A) = \{z_1, \ldots, z_r\}$, and the vertical arrow to the left is the structure morphism. The push-out $Y$ is an algebraic space ([1], Theorem 6.1), which here is actually a scheme ([10], Theorem 7.1). Such constructions are often called *pinching*. By abuse of notation, we simply say that $q : \tilde{Y} \to Y$ is the *identification that turns the closed points $z_1, \ldots, z_r$ into a rational point $a \in Y$*. Note that $Y$ remains separated and of finite type.

Now suppose that the scheme $\tilde{Y}$ *is normal but does not admit an ample invertible sheaf.* For example, this could be the surface in Proposition 2.2, or Hironaka's proper smooth threefold (see [12], Appendix C, Example 3.4.1). Let $U_1, \ldots, U_s \subset \tilde{Y}$, $s \geq 2$ be the maximal quasiprojective open sets, and choose closed points $u_i \in \tilde{Y} \smallsetminus U_i$. Clearly, the closed set $\{u_1, \ldots, u_s\}$ does not admit an affine open neighborhood. Consequently, there must be certain closed points $z_1, \ldots, z_r \in \tilde{Y}$ that do not admit a common affine open neighborhood, now with $r \geq 2$ minimal. The latter ensures that the $r-1$ points $z_2, \ldots, z_r \in \tilde{Y}$ do admit such neighborhoods. Let $\tilde{Y} \to Y$ be the identification that turns $z_1$ into a rational point $u \in Y$, and also $z_2, \ldots, z_r$ into a rational point $v \in Y$. Note that this is obtained by applying the procedure in the previous paragraph twice. Since $q : \tilde{Y} \to Y$ is finite and hence affine, and $z_1, \ldots, z_r \in \tilde{Y}$ do not admit a common affine open neighborhood, the same holds for $u, v \in Y$. In other words, passing from $\tilde{Y}$ to $Y$ reduces from $r \geq 2$ to the case $r = 2$.

Let $G = \{e, \sigma\}$ by the cyclic group of order two. Regard it as a constant group scheme, and form $G \times Y$. This is the disjoint union of two copies of $Y$, endowed with a permutation action of $G$. Consider the identification $q : G \times Y \to X_0$ that turns $(e, u)$ and $(\sigma, v)$ into a rational point $a \in X_0$, and also turns $(e, v)$ and $(\sigma, u)$ into a rational point $b \in X_0$. For a useful illustration of the geometry of $X_0$, see [2], Picture on page 286.

**Proposition 2.3.** *In the above situation, the rational points $a, b \in X_0$ do not admit a common affine open neighborhood, the $G$-action on $G \times Y$ induces a free action on $X_0$, and we have $G \cdot a = \{a, b\}$.*

*Proof.* Suppose there is an affine open neighborhood $U$ of $a, b \in X_0$. Then $q^{-1}(U)$ is an affine open neighborhood of

$$q^{-1}(\{a, b\}) = \{(e, v), (\sigma, v), (e, u), (\sigma, u)\} = G \times \{u, v\}$$

in $G \times Y$. The intersection with $\{e\} \times Y$ gives an affine open neighborhood of the points $u, v \in Y$, contradiction. The $G$-action leaves $q^{-1}(\{a, b\})$ invariant, while it interchanges the subsets $q^{-1}(a)$ and $q^{-1}(b)$. With the universal property of push-outs, we infer that the permutation action on $G \times Y$ induces a $G$-action on $X_0$, such that $\sigma(a) = b$.

Clearly, the action is free on $X_0 \smallsetminus \{a, b\}$, which can be regarded as an invariant open set of $G \times Y$. Moreover, the stabilizer groups for the rational points $a, b \in X_0$ are trivial. It follows that the $G$-action on $X_0$ is free. □

Note that the composition $Y = \{e\} \times Y \to X_0 \to G\backslash X_0$ is the identification that turns $u, v \in Y$ into a single rational point. In particular, if $\tilde{Y}$ is integral, the same holds for $G\backslash X_0$.

Also note that we may have started the construction with any separated scheme $Y$ of finite type containing two closed points $u, v \in Y$ that do not admit a common affine open neighborhood. For $Y$ normal and $k$ algebraically closed, this are precisely the schemes that do not admit an embedding into any toric variety, according to Włodarczyk's result ([22], Theorem A).

On the other hand, Horrocks [13] showed that the the associated fiber bundle $Y = E \wedge^{\mathbb{G}_m} C$ stemming from a non-trivial principal $\mathbb{G}_m$-bundle $E \to \mathbb{P}^1$ and the rational nodal curve $C$ does not embed into any regular scheme, yet every finite subset admits a common affine open neighborhood. This Cohen-Macaulay proper surface $Y$ can be seen as a Hirzebruch surface with invariant $e > 0$ in which the negative and positive section are identified.

Summing up, we have described above two constructions of separated schemes $X_0$ of finite type, together with an action of the cyclic group $G$ of order two, such that some orbit $G \cdot a = \{a, b\}$ does not admit a common affine open neighborhood. Theorem 1.1 ensures that in both cases the twisted form $X = (X_0 \otimes k')/G$ is a non-schematic algebraic space whose base-change $X \otimes k' = X_0 \otimes k'$ becomes schematic, provided there is a separable extension $k \subset k'$ of degree two. Note that the latter indeed exists if $k$ is a finite field or a number field, or if we replace the ground field $k$ by the transcendental extension $k(T)$.

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