OBERSEMINAR ALGEBRAIC GEOMETRY: THE NORI FUNDAMENTAL GROUP SCHEME SUMMER SEMESTER 2026

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Overview: The *fundamental group* $\pi_1(X, x_0)$ for a topological space X comprises the homotopy classes of loops with respect to a base point x_0. Under very mild assumptions, it acts via monodromy on the fiber $X'_0 = f^{-1}(x_0)$ of each covering $f : X' \to X$, and in consequence yields a description of the category of all covering spaces.

In order to extend this to the world of schemes, Grothendieck turned the picture upside-down: Using intrinsic properties of the collection $\mathcal{C} = (\mathrm{FinEt}/X)$ of all finite étale schemes over X, abstracted in the notion of *Galois categories*, he defined the *algebraic fundamental group* $\pi_1^{\mathrm{alg}}(X, x_0)$ as the automorphism group of the fiber functor $X' \mapsto X'_0$. In a suitable sense, this pro-finite group unifies the theory of Galois groups from algebra and fundamental groups from topology.

Nori extended this theory even further, to incorporate torsors $X' \to X$ for finite group schemes G, which might be non-reduced. To achieve this, he observed that certain locally free sheaves, the so-called *essentially finite vector bundles*, have the same intrinsic properties as the representations stemming from affine group schemes. This is formalized in the notion of *Tannakian categories*, which are certain k-linear categories $\mathcal{C}$ endowed with internal tensor products and duals. Via fiber functors $\omega : \mathcal{C} \to (\mathrm{Vec}/k)$ one indeed obtains an affine group scheme $G = \underline{\mathrm{Aut}}^\otimes(\omega)$, and $\mathcal{C}$ becomes its representation category. Applied to $\mathcal{C} = (\mathrm{EssFin}/X)$ one obtains the *Nori fundamental group scheme* $\pi_1^N(X, x_0)$, where x_0 is a rational point. In a suitable sense, it classifies triples (G, P, p_0) where G is a finite group scheme, $P \to X$ is a principal homogeneous space, and $p_0 \in P_0$ is a rational point. The construction is particularly useful in positive characteristics, when infinitesimal group schemes enter the picture. Only recently it became clear how to handle issues stemming from imperfectness and base points.

The goal of this Oberseminar is to familiarize with the Nori fundamental group scheme, gain some insight into its structure, and examine examples and applications.

Time and Place: Monday, 12:30–13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative and shifts are likely to occur, for example due to internal talks or guests)

Talk 1 (13. April)
Vicente Monreal (?):
The point of departure: fundamental groups in topology.
Recall that the fundament group $\pi_1(X, x_0)$ of a topological space comprises homotopy classes of loops. Explain how it acts via monodromy on the fibers $X_0' = X' \times_X \{x_0\}$ for covering spaces X', and interpret it as group of deck transformations of the universal covering $\tilde{X}$. References: [18], Section 2 or [8], Chapter 1 or [12] Chapter 8.

Talk 2 (20. April)
Jack Ritschel:
Grothendieck's algebraic fundamental group.
Explain how Grothendieck defined the algebraic fundamental group $\pi_1^{\mathrm{alg}}(X, x_0)$ for schemes via the Galois category (FinEt/X) of finite étale $X' \to X$, together with the fiber functor $X' \mapsto X_0'$. Discuss how this unifies Galois groups from algebra and fundamental groups from topology. References: [7], Exposé V, Sections 4–7 or [18], Section 5.4 or [16].

Talk 3 (11. Mai)
Fabian Rodatz:
Tannakian categories.
Discuss the notion of Tannakian categories $\mathscr{C}$ over a ground field k, the role of fiber functors $\omega : \mathscr{C} \to$ (FinVec/k), the resulting affine group scheme $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and the identification $\mathscr{C}$ with the category of finite-dimensional G-representations. References: [4], §2–3 or [18], Chapter 6 or [9].

Talk 4 (18. Mai)
Stefan Schröer:
Essentially finite vector bundles.
Introduce the notions of finite and essentially finite vector bundles, following [13], Section 3. Discuss that they form, under suitable assumptions on the scheme X, a neutral Tannakian category (EssFin/X).

Talk 5 (1. Juni)
Ismaele Vanni:
Nori's fundamental group scheme.
Introduce the Nori fundamental group scheme $\pi^N(X, x_0)$ for connected reduced schemes X over a ground field k endowed with a rational point, following [14]. Explain how it classifies triples (G, P, p_0) where G is a finite group scheme, $P \to X$ is a principal homogeneous G-space, and $p_0 \in P_0$ is a rational point.

Talk 6 (8. Juni)
Chen Ping:
Computations for abelian varieties.
On of the few examples where a non-trivial computation of the Nori fundamental group is feasible are the abelian varieties. Describe computation in [15].

Talk 7 (15. Juni)
Otto Overkamp:
Decomposing the fundamental group schemes.
Recall that each finite group scheme is an extension of an étale group scheme by an infinitesimal group scheme. Discuss the resulting "Nori-to-Grothendieck" map $\pi_1^N(X, x_0) \to \pi_1^{\mathrm{alg}}(X, x_0)$, and describe its kernel, following [5].

Talk 8 (22. Juni)
Hugo Zock:
Surface singularities.
The local fundamental group $\pi_1^{\mathrm{loc}}(X)$ of the an isolated surface singularity $X = \mathrm{Spec}(R)$ is the algebraic fundamental group of the pointed spectrum $U = X \smallsetminus \{\mathfrak{m}\}$. Mumford [11] showed that is trivial if and only if X is regular, as least in characteristic zero. Artin [1] conjectured that in general, the reason for triviality should be the existence of a branched covering $X' \to X$ with X' regular. Explain the results

of Esnault and Viehweg in this direction [6].

Talk 9 (29. Juni)
NN:
Fundamental group schemes and gerbes.
The role of the base point $x_0 \in X$, reducedness of the scheme X, and perfectness of the ground field k was a constant cause of trouble for the Nori fundamental group scheme. Discuss how Borne and Vistoli [2] used non-abelian cohomology and in particular gerbes to gain control on this.

References

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