OBERSEMINAR ALGEBRAIC GEOMETRY: CUBIC HYPERSURFACES WINTER SEMESTER 2024/25

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Overview: The closed subschemes $X \subset \mathbb{P}^{n+1}$ defined by a homogeneous equation $P(T_0, \ldots, T_{n+1}) = 0$ of degree $d = 3$ over a ground field k are called *cubic hypersurfaces*. They form an astonishingly rich class of schemes, which mark the very beginning of algebraic geometry (elliptic curves, twenty-seven lines, Lüroth problem). To date, they continue to pose massive foundational problems, both of arithmetic and geometric nature. For example, not a single cubic fourfold $X \subset \mathbb{P}^5$ is known *not* to be birational to $\mathbb{P}^4$, although this is expected to hold generically. The goal of this Oberseminar is to learn about the geometry of smooth cubic hypersurfaces, following the recent monograph of Huybrechts [5].

The one-dimensional case apart, cubic hypersurfaces $X \subset \mathbb{P}^{n+1}$ have very ample dualizing sheaf ω_X, hence are what in modern language is called *Fano varieties*. The chief tool to investigate the geometry of such X is another construction bearing Gino Fano's name, the *Fano scheme* $F(X)$ parameterizing the lines $L \subset \mathbb{P}^{n+1}$ contained in X, a closed subscheme of the Hilbert scheme $\mathrm{Hilb}_{\mathbb{P}^{n+1}/k}$. In dimension two, the Fano scheme is zero-dimensional and corresponds to the famous *twenty-seven lines* on cubic surfaces. In higher dimensions, there are two kinds of lines, distinguished by the splitting type of the conormal sheaf $\mathscr{I}/\mathscr{I}^2$ for the embedding $L \subset X$, indicative of further geometric features.

Like curves, which by Torelli are determined by their polarized Jacobian, the cubic hypersurfaces X can be reconstructed from the Fano scheme $F(X)$ of lines. A landmark result of Clemens and Griffiths [3] reveals that every smooth cubic threefold $Y \subset \mathbb{P}^4$ is irrational. This relies on rather particular relations between the *intermediate Jacobian* $J(Y)$ from Hodge theory and the Fano scheme $F(Y)$ of lines. In marked contrast, smooth cubic fourfolds $X \subset \mathbb{P}^5$ arising from *Pfaffians* turn out to be rational.

Time and Place: Monday, 12:30–13:30, seminar room 25.22.03.73.

Schedule: (all dates are tentative and shifts are likely to occur, for example due to internal talks or guests)

Talk 1 (14. October)
Cesar Hilario:
Cohomological invariants.
Elucidate how to compute various cohomological invariants for smooth hypersurfaces $X \subset \mathbb{P}^{n+1}$ in general (Chapter 1, Section 1).

Talk 2 (21. October)
Hugo Zock:
The Fano scheme of lines.
Introduce the Fano scheme $F(X) \subset \mathrm{Hilb}_{\mathbb{P}^{n+1}/k}$ of lines contained in X (Chapter 2, Section 1), and deduce unirationality for cubic hypersurfaces $X \subset \mathbb{P}^{n+1}$ (Corollary 1.21).

Talk 3 (28. October)
Otto Overkamp:
Lines of the first and second type.
Recall Grothendieck's Splitting Theorem for sheaves on $\mathbb{P}^1$, discuss the two types of lines $L \subset X$ (Chapter 2, Section 2), and explain their geometric meaning in the Fano correspondence (Corollary 2.15).

Talk 4 (4. November)
Quentin Posva:
The global Torelli theorem.
Briefly recapitulate the Torelli Theorem for curves, and explain how smooth cubic hypersurfaces $X \subset \mathbb{P}^{n+1}$ can likewise be reconstructed from their Fano scheme $F(X)$ of lines (Chapter 2, Section 3).

Talk 5 (18. November)
Fabian Korthauer:
Moduli and periods.
Explain how the smooth cubic hypersurfaces form a moduli space, a purely algebraic construction in terms of geometric invariant theory or Artin stacks; also discuss their periods, by nature a transcendental concept (Chapter 3, Section 1).

Talk 6 (25. November)
Vicente Monreal:
Cubic surfaces. First give the formula for degree and Euler characteristic of the Fano scheme of lines in general (Chapter 2, Proposition 4.6), then discuss the geometry of cubic surfaces (Chapter 4, Section 1–2).

Talk 7 (2. December)
Bianca Gouthier:
The twenty-seven lines. Examine the twenty-seven lines on a cubic surface from various angles (Chapter 4, Section 3).

Talk 8 (9. December)
Ivo Kroon:
Cubic threefolds: Curves in the Fano scheme. For cubic threefolds $Y \subset \mathbb{P}^4$, the Fano scheme $F(Y)$ of lines is a surface. Discuss various natural curves on it. (Chapter 5, Section 1).

Talk 9 (6. January)
Chen Ping:
Cubic threefolds: The intermediate Jacobian and irrationality. Introduce the intermediate Jacobian $J(Y) = H^{1,2}(Y)/H^3(Y,\mathbb{Z})$, indicate how the Fano correspondence turns this complex torus into a polarized abelian variety, and explain how the singularity of the Theta divisor yields irrationality of $Y \subset \mathbb{P}^4$ (Chapter 5, Section 3 and Corollary 4.11).

Talk 10 (20. January)
Stefan Schröer:
Rationality of pfaffian cubic fourfolds. Recall the Pfaffian of an alternating matrix, introduce the notion of pfaffian cubic fourfolds $X \subset \mathbb{P}^5$ (Chapter 6, Section 2), and elucidate their rationality (Corollary 2.7).

References

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[2] A. Beauville: The Lüroth problem. In: A. Beauville, B. Hassett, A. Kuznetsov, A. Verra (eds.), Rationality problems in algebraic geometry, pp. 1–27. Springer, Cham, 2016.

[3] H. Clemens, P. Griffiths: The intermediate Jacobian of the cubic threefold. Ann. of Math. 95 (1972), 281–356.

[4] G. Fano: Sul sistema ∞^2 di rette contenuto in una varietà cubica generale dello spazio a quattro dimensioni. Torino Atti 39 (1904), 778–792.

[5] D. Huybrechts: The geometry of cubic hypersurfaces. Cambridge University Press, Cambridge, 2023.

[6] J. Kollár: Unirationality of cubic hypersurfaces. J. Inst. Math. Jussieu 1 (2002), 467–476.

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