SEMINAR FALL 2025
GEOMETRIC MANIN CONJECTURE

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1. Description

The aim of this seminar is to understand the paper *Geometric Manin's conjecture and rational curves* [LT19] by Lehmann and Tanimoto.

The Geometric Manin's conjecture concerns the dimension and number of components of the space of rational curves on a projective variety. The starting points are Batyrev's heuristics and Manin's conjecture for rational points on Fano varieties over number fields. The authors observe that in order to have a reasonable control on the number of components, one must discount an exceptional set. They thus define the so-called Manin components as those that parametrize families of rational curves that capture the geometry of the variety X, rather than the geometry of a subvariety or of a cover of X. The Geometric Manin's Conjecture predicts an asymptotic formula for a generating function encoding the number and dimension of the Manin components of the space $\mathrm{Mor}(\mathbb{P}^1, X)$ of morphisms of bounded degree. The paper shows that the Geometric Manin's Conjecture holds for Fano threefolds of Picard rank 1 and index 2.

2. The talks

Each talk lasts 2 hours including breaks, proposed schedule Tuesdays 10:00-12:00.

2.1. **Introduction and the cone theorem.**

For the first half introduce the goals and main results of the seminar: cover [LT19, §1], including Batyrev's heuristics [Bou11, §1.2] or [Tan25, §7.3].

For the second half introduce the notation in [LT19, §2] and recall the following notions: numerical equivalence of $\mathbb{R}$-divisors, Néron–Severi group, Theorem of the base, and Picard number from [Laz04, §1.1C]; ample cone, pseudoeffective cone, nef cone, big cone, cone of curves from [Laz04, §1.4, §2.2B]; Cone Theorem and the special case of Fano varieties [Laz04, §1.5F]. Alternative references: [Deb01, §1, §6], [Tan25, §3.1-3.2].

2.2. **The a- and b-invariants.**

Cover [LT19, §3] including a discussion of [HTT15, Propositions 2.7, 2.10], of [LTT18, Proposition 2.10, Corollary 3.9, Lemma 3.10, Proposition 4.1, Theorem 4.8], and of [LT17, Proposition 8.4]. Alternative reference [Tan25, 4.1].

2.3. **Expected dimension.**

In the first half define the space of morphisms $\mathrm{Mor}(\mathbb{P}^1, X)$ and show that it is a scheme (possible references are [Gro95], [Kol96, Theorem I.1.10], [Deb01, §2.2], [Tan25, §7.1.1]), introduce the notion of expected dimension (possible references are [Kol96, Theorems II.1.2, II.1.3], [Deb01, Theorem 2.6, §2.11], [Tan25, Theorem 7.2.2, Corollary 7.2.3]), introduce the notion of free curve including [Kol96, Corollaries II.3.5.4, II.3.10.1] (possible references are: [Kol96, §II.3], [Deb01, §4.2], [Tan25, 7.2.2]).

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In the second half cover [LT19, §4]. Alternative reference [Tan25, §7.4.1].

2.4. **Number of components I.**

Cover [LT19, §5.1-5.2], including a brief reminder of the definition of $\overline{M}_{0,n}(X,\beta)$ for which possible references are [FP97] and [Tan25, §7.1.2].

2.5. **Number of components II.**

Cover [LT19, §5.3-5.4].

2.6. **Geometric Manin Conjecture and examples.**

Cover [LT19, §6-7] until [LT19, Theorem 7.3] included. Alternative reference [Tan25, §7.4.2].

2.7. **Fano threefolds.**

Cover the rest of [LT19, §7]. Alternative reference [Tan25, §8.2].

References

[Bou11] David Bourqui. Asymptotic behaviour of rational curves. arXiv:1107.3824, 2011.

[Deb01] Olivier Debarre. *Higher-dimensional algebraic geometry.* Universitext. Springer-Verlag, New York, 2001.

[FP97] W. Fulton and R. Pandharipande. Notes on stable maps and quantum cohomology. In *Algebraic geometry—Santa Cruz 1995*, volume 62, Part 2 of *Proc. Sympos. Pure Math.*, pages 45–96. Amer. Math. Soc., Providence, RI, 1997.

[Gro95] Alexander Grothendieck. Techniques de construction et théorèmes d'existence en géométrie algébrique. IV. Les schémas de Hilbert. In *Séminaire Bourbaki, Vol. 6*, pages Exp. No. 221, 249–276. Soc. Math. France, Paris, 1995.

[HTT15] Brendan Hassett, Sho Tanimoto, and Yuri Tschinkel. Balanced line bundles and equivariant compactifications of homogeneous spaces. *Int. Math. Res. Not. IMRN*, (15):6375–6410, 2015.

[Kol96] János Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[LT17] Brian Lehmann and Sho Tanimoto. On the geometry of thin exceptional sets in Manin's conjecture. *Duke Math. J.*, 166(15):2815–2869, 2017.

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[Tan25] Sho Tanimoto. Birational geometry and manin's conjecture. in preparation, 2025.

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