Singularities, Monodromy and Zeta Functions
Blatt 4

Exercises for presentation in the exercise class on 15.11.2018

Recall that a function $f : X \to \mathbb{Q}_p^n$ from a semi-algebraic set $X \subset \mathbb{Q}_p^m$ is defined to be a *semi-algebraic (map)* when, for every $r \in \mathbb{N}$ and every semi-algebraic set $S \subset \mathbb{Q}_p^n \times \mathbb{Q}_p^r$, the set $\{(x, z) \in X \times \mathbb{Q}_p^r | (f(x), z) \in S\}$ is semi-algebraic.

Aufgabe 1:

(a) Prove, without using the Projection Theorem, that if $X \subset \mathbb{Q}_p^m$ and $f : X \to \mathbb{Q}_p^n$ are semi-algebraic, then the graph of f is a semi-algebraic subset of $\mathbb{Q}_p^m \times \mathbb{Q}_p^n$.

(b) Assuming the Projection Theorem, prove the converse to (a). That is, prove that

(∗) Every function f whose graph is a semi-algebraic subset of $\mathbb{Q}_p^m \times \mathbb{Q}_p^n$ is a semi-algebraic map.

(c) Now assume (∗) and deduce the Projection Theorem for sets of the form $X \times Y$.

Aufgabe 2:

Let $m, l, k \in \mathbb{Z}$, $X \subset \mathbb{Q}_p^m$ a semi-algebraic set, and $h, a_1, a_2, c : X \to \mathbb{Q}_p$ semi-algebraic maps.

(a) Show that $\{x \in X | v(h(x)) \equiv l \mod (k)\}$ is semi-algebraic.

(b) Show that

$$Y := \{(x, t) \in X \times \mathbb{Q}_p | v(a_1(x)) < v(t - c(x)) < v(a_2(x)) \ \wedge \ v(t - c(x)) \equiv l \mod (k)\}$$

is a semi-algebraic subset of $X \times \mathbb{Q}_p$.

(c) Show that $\pi(Y) \subset \mathbb{Q}_p^m$ is semi-algebraic.

(d) Deduce, for $\nu \in \mathbb{Z}$ and

$$W := \{(x, t) \in X \times \mathbb{Q}_p | v(a_1(x)) < v(t - c(x)) < v(a_2(x)) \ \wedge \ \nu \cdot v(t - c(x)) + v(h(x)) \equiv 0 \mod (k)\},$$

that the set $\pi(W) \subset \mathbb{Q}_p^m$ is semi-algebraic.