Singularities, Monodromy and Zeta Functions
Blatt 3

Exercises for presentation in the exercise class on 8.11.2018

Aufgabe 1:

Suppose $f, g \in \mathbb{Z}[x_1, ..., x_n]$. Prove that

$$\{\bar{x} \in \mathbb{Q}_p^n | v(f(\bar{x})) \geq v(g(\bar{x}))\}$$

is a semi-algebraic set.

Aufgabe 2:

A *valuative ball* in $\mathbb{Q}_p$ is a set of the form

$$B(c; \gamma) := \{x \in \mathbb{Q}_p | v(x - c) \geq \gamma\},$$

where $c \in \mathbb{Q}_p$ is a *centre* of the ball and $\gamma \in \mathbb{Z}$ its *valuative radius*.

(a) Suppose that B is a valuative ball in $\mathbb{Q}_p$ and $x \notin B$. Prove that $y \mapsto \mathrm{ac}_1(y - x)$ is constant on B.

(b) Show that for any $x, y \in \mathbb{Q}_p$ with $v(x - y) > v(x)$, we have $\mathrm{ac}_1(x) = \mathrm{ac}_1(y)$.

(c) Generalising (b), show that for every $l \geq 1$,

$$v(x - y) \geq v(x) + l \Rightarrow \mathrm{ac}_l(x) = \mathrm{ac}_l(y).$$

Aufgabe 3:

Let $f(t) = (t - c_1)(t - c_2)^2 \in \mathbb{Q}_p[t]$ with $c_1 \neq c_2$.

(a) Find $\lambda \in \mathbb{Z}$, $a_1, a_2, a_3 \in \mathbb{Z}$ and $\mu_1, \mu_2, \mu_3 \in \mathbb{Z}$ such that:

- For each $i \in \{1, 2\}$, we have that for all $t \in \mathbb{Q}_p$ with $v(t - c_i) > \lambda$,

$$v(f(t)) = \mu_i + a_i \cdot v(t - c_i);$$

- While for all $t \in \mathbb{Q}_p$ with $v(t - c_1) \leq \lambda$ and $v(t - c_2) \leq \lambda$,

$$v(f(t)) = \mu_3 + a_3 \cdot v(t - c_1).$$

(b) Find $\lambda \in \mathbb{Z}$, $a_1, a_2, a_3 \in \mathbb{Z}$ and $b_1, b_2, b_3 \in \mathbb{F}_p$ such that:

- For each $i \in \{1, 2\}$, we have that for all $t \in \mathbb{Q}_p$ with $v(t - c_i) > \lambda$,

$$\mathrm{ac}_1(f(t)) = b_i \cdot \mathrm{ac}_1(t - c_i)^{a_i};$$

- For $t \in \mathbb{Q}_p$ such that $v(t - c_1) < \lambda$,

$$\mathrm{ac}_1(f(t)) = b_3 \cdot \mathrm{ac}_1(t - c_1)^{a_3};$$

- The remainder of $\mathbb{Q}_p$ is a disjoint union of finitely many valuative balls B_j, each of valuative radius $(\lambda + 1)$, with $\mathrm{ac}_1 \circ f$ constant on each B_j.

(c) Formulate and prove a similar statement for ac_2; i.e. there exists a partition

$$\mathbb{Q}_p = A_1 \cup A_2 \cup A_3 \cup B_1 \cup ... \cup B_m$$

so that there is a simple formula for $\mathrm{ac}_2 \circ f$ on each A_i and that $\mathrm{ac}_2 \circ f$ is constant on each B_j.

Aufgabe 4:

Complete the proof of the Hensel-Rychlik-Newton Lemma.

Course website: http://reh.math.uni-duesseldorf.de/~internet/Zeta_WS18/