Dynamical zeta functions, Nielsen-Reidemeister fixed point theory and Reidemeister torsion

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Zeta functions of groups and dynamical systems
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Introduction

Inspired by the Hasse-Weil zeta function of an algebraic variety over a finite field, Artin and Mazur defined the Artin - Mazur zeta function for an arbitrary map $f : X \to X$ of a topological space X:

$$F_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{F(f^n)}{n} z^n\right)$$

where $F(f^n)$ is the number of isolated fixed points of f^n. Artin and Mazur showed that for a dense set of the space of smooth maps of a compact smooth manifold into itself the Artin-Mazur zeta function $F_f(z)$ has a positive radius of convergence.

Later Manning proved the rationality of the Artin - Mazur zeta function for diffeomorphisms of a smooth compact manifold satisfying Smale axiom A. On the other hand there exist maps for which Artin-Mazur zeta function is transcendental. The Artin-Mazur zeta function was historically the first dynamical zeta function for *discrete* dynamical system. The next dynamical zeta function was defined by Smale and Milnor . This is the Lefschetz zeta function of discrete dynamical system:

$$L_f(z) := \exp\left(\sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n\right),$$

where

$$L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr} \left[f_{*k}^n : H_k(X;\mathbb{Q}) \to H_k(X;\mathbb{Q}) \right]$$

is the Lefschetz number of the iterate f^n of f.
The Lefschetz zeta function is always a rational function of z and is given by the formula:

$$L_f(z) = \prod_{k=0}^{\dim X} \det\left(I - f_{*k}.z \right)^{(-1)^{k+1}}.$$

We apply a simple linear algebra calculation connecting a trace and a determinant:

$$\begin{aligned}
\exp\left(\sum_{n=1}^{\infty}\frac{\mathrm{tr}\,B^n}{n}z^n\right) &= \\
&= \exp\left(\mathrm{tr}\sum_{n=1}^{\infty}\frac{B^n}{n}z^n\right) \\
&= \exp\left(\mathrm{tr}(-\log(1-Bz))\right) \\
&= \frac{1}{\det(1-Bz)}.
\end{aligned}$$

This implies a formula above connecting a graded trace and a graded determinant.

Nielsen - Reidemeister fixed point theory

There is another way of counting the fixed points of f^n - according to Nielsen and Reidemeister. This is counting of fixed points of a map in the presence of the foundamental group of a space.

I would like to start from the definitions.

We assume everywhere X to be a connected, compact polyhedron and $f: X \to X$ to be a continuous map. Let $p: \tilde{X} \to X$ be the universal cover of X and $\tilde{f}: \tilde{X} \to \tilde{X}$ a lifting of f, i.e., $p \circ \tilde{f} = f \circ p$.

Definition

Two liftings $\tilde{f}'$ and $\tilde{f}$ of f are said to be *conjugate* if there exists covering translation $\gamma \in \Gamma \cong \pi_1(X)$, such that $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$. Lifting classes are equivalence classes by conjugacy. Notation:

$$[\tilde{f}] = \{\gamma \circ \tilde{f} \circ \gamma^{-1} | \gamma \in \Gamma\}$$

The subset $p(\mathrm{Fix}(\tilde{f})) \subset \mathrm{Fix}(f)$ is called the *fixed point class* of f

determined by the lifting class $[\tilde{f}]$.

Lemma

(1) $\mathrm{Fix}(f) = \cup_{\tilde{f}} p(\mathrm{Fix}(\tilde{f}))$.
(2) $p(\mathrm{Fix}(\tilde{f})) = p(\mathrm{Fix}(\tilde{f}'))$ *if* $[\tilde{f}] = [\tilde{f}']$.
(3) $p(\mathrm{Fix}(\tilde{f})) \cap p(\mathrm{Fix}(\tilde{f}')) = \emptyset$ *if* $[\tilde{f}] \neq [\tilde{f}']$.

Our definition of a fixed point class is via the universal covering space. It essentially says: two fixed point of f are in the same class iff there is a lifting $\tilde{f}$ of f having fixed points above both of them. There is another way of saying this, which does not use covering space explicitly, hence is very useful in identifying fixed point classes. Two fixed points x_0 and x_1 of f belong to the same fixed point class iff there is a path c from x_0 to x_1 such that $c \cong f \circ c$ (

homotopy relative endpoints - "Nielsen disk"). This fact can be considered as an equivalent definition(geometrical) of a non-empty fixed point class. Every map f has only finitely many non-empty fixed point classes, each a compact subset of X.
Examples: a map of degree d of the circle S^1, involution on torus T^2.
A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of f (and hence the number of all fixed point classes) is called the *Reidemeister number* of f, denoted by $R(f)$. This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of f, denoted by $N(f)$.
The Nielsen number is always finite. $R(f)$ and $N(f)$ are homotopy

invariants. In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as f.
Let G be a group and $\phi : G \to G$ an endomorphism. Two elements $\alpha, \beta \in G$ are said to be ϕ-*conjugate* iff there exists $\gamma \in G$ such that $\beta = \gamma\alpha\phi(\gamma)^{-1}$. Reidemeister classes (twisted conjugacy classes, ϕ-conjugacy classes) of an automorphism (endomorphism) ϕ of a (countable discrete) group G are the classes $\{g\}_\phi$ of the equivalence relation

$$g \sim xg\phi(x^{-1}), \qquad g, x \in G.$$

The number of them is the Reidemeister number $R(\phi)$. For $\phi = \mathrm{Id}$

we have the usual conjugacy classes.
Reidemeister bijection: Reidemeister number of continuous map f coincides with Reidemeister number (the number of twisted conjugacy classes) of induced endomorphism f_* of the fundamental group of X: $R(f) = R(f_*)$.

Taking a dynamical point of view, we consider the iterates of f and ϕ, and define following zeta functions connected with the Nielsen-Reidemeister fixed point theory. The Reidemeister zeta functions of f and ϕ and the Nielsen zeta function of f are defined as power series:

$$R_\phi(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right),$$

$$R_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n\right),$$

$$N_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n\right).$$

The investigation and computation of the Reidemeister zeta function $R_\phi(z)$of a group endomorphism ϕ is an algebraic ground of the computation and investigation of zeta functions $R_f(z)$ and $N_f(z)$.

The Reidemeister bijection implies that Reidemeister zeta function of continuous map f coincides with Reidemeister zeta function of induced endomorphism f_* of the fundamental group of X:
$R_f(z) = R_{f_*}(z)$.
Whenever we mention the Reidemeister zeta function, we shall assume that it is well-defined and so $R(f^n) < \infty$ and $R(\phi^n) < \infty$ for all $n > 0$. There are spaces and maps for which $R_f(z)$ is not defined. The zeta functions $R_f(z)$ and $N_f(z)$ are homotopy invariants. The function $N_f(z)$ has a positive radius of convergence for any continuous map f. The above zeta functions are directly analogous to the Lefschetz zeta function of a map. We start with an example that shows how different can be the Nielsen, the Reidemeister and the Lefschetz zeta functions.

Example

Let $f: S^2 \vee S^4 \to S^2 \vee S^4$ to be a continuous map of the bouquet of spheres such that the restriction $f|_{S^4} = id_{S^4}$ and the degree of the restriction $f|_{S^2}: S^2 \to S^2$ equal to -2. Then $L(f) = 0$, hence $N(f) = 0$ since $S^2 \vee S^4$ is simply connected. For $k > 1$ we have $L(f^k) = 2 + (-2)^k \neq 0$, therefore $N(f^k) = 1$. $R(f^k) = 1$ for all $k \geq 1$ since $S^2 \vee S^4$ is simply connected. From this we have by direct calculation that

$$N_f(z) = \exp(-z) \cdot \frac{1}{1-z}; \; R_f(z) = \frac{1}{1-z}; \; L_f(z) = \frac{1}{(1-z)^2(1+2z)}.$$

Hence $N_f(z)$ is a meromorphic function, and $R_f(z)$ and $L_f(z)$ are rational and different.
The following main problem was investigated: for which spaces and maps and for which groups and endomorphisms are the Nielsen and Reidemeister zeta functions rational functions? Are these functions algebraic functions?
The knowledge that a zeta function is a rational function is important because it shows that the infinite sequence of coefficients of the corresponding power series is closely interconnected, and is given by the finite set of zeros and poles of the zeta function.

Topological entropy the radius of convergence of the Nielsen zeta function

The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition.

Let $f: X \to X$ be a self-map of a compact metric space. For given $\epsilon > 0$ and $n \in \mathbb{N}$, a subset $E \subset X$ is said to be (n, ϵ)-separated under f if for each pair $x \neq y$ in E there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \epsilon$. Let $s_n(\epsilon, f)$ denote the largest cardinality of any (n, ϵ)-separated subset E under f. Thus $s_n(\epsilon, f)$ is the greatest number of orbit segments $x, f(x), \cdots, f^{n-1}(x)$ of length n that can be distinguished one from another provided we can only distinguish between points of X that are at least ϵ apart. Now let

$$h(f, \epsilon) := \limsup_n \frac{1}{n} \log\ s_n(\epsilon, f)$$

$$h(f) := \limsup_{\epsilon \to 0} h(f, \epsilon).$$

The number $0 \leq h(f) \leq \infty$, which to be independent of the metric d used, is called the topological entropy of f. If $h(f,\epsilon) > 0$ then, up to resolution $\epsilon > 0$, the number $s_n(\epsilon, f)$ of distinguishable orbit segments of length n grows exponentially with n. So $h(f)$ measures the growth rate in n of the number of orbit segments of length n with arbitrarily fine resolution.

For a "hyperbolic"(Axiom A) diffeomorphism of a manifold topological entropy

$$h(f) = \limsup_n \frac{1}{n} \cdot \log \# Fix(f^n),$$

so $h(f)$ measures the growth rate of the number of periodic points.

A basic relation between topological entropy $h(f)$ and Nielsen numbers was found by N. Ivanov-1982. We present here a very short proof of the Ivanov's inequality.

Lemma

Let f be a continuous map on a compact connected polyhedron X. Then

$$h(f) \geq \limsup_n \frac{1}{n} \cdot \log N(f^n) := \log N^\infty(f)$$

Proof: Let δ be such that every loop in X of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f(x), f(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of f^n. Thus $|E_n| = N(f^n)$. By the definition of $h(f)$, it suffices to show that E_n is (n,ϵ)-separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $o \leq i < n$ hence for all $i \geq 0$. Pick a path c_i from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $0 \leq i < n$ and let $c_n = c_0$. By the choice of δ and ϵ, $f \circ c_i \simeq c_{i+1}$ for all i, so $f^n \circ c_0 \simeq c_n = c_0$. This means x, y in the same fixed point class of f^n, contradicting the construction of E_n.

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

We denote by R the radius of convergence of the Nielsen zeta function $N_f(z)$. Let $h = \inf h(g)$ over all maps g of the same homotopy type as f.

Theorem

For a continuous map of a compact polyhedron X into itself,

$$R \geq \exp(-h) > 0. \quad (1)$$

Proof.

The inequality $R \geq \exp(-h)$ follows from the previous lemma, the Cauchy-Hadamard formula, and the homotopy invariance of the radius R of the Nielsen zeta function $N_f(z)$. We consider a smooth compact manifold M which is a regular neighborhood of X and a smooth map $g : M \to M$ of the same homotopy type as f. As is known, the entropy $h(g)$ is finite. Thus $\exp(-h) \geq \exp(-h(g)) > 0$. □

Nielsen zeta function of a periodic map

Sometimes one can answer above questions without directly calculating the Nielsen numbers $N(f^n)$, but using the connection between Nielsen numbers of iterates. We denote $N(f^n)$ by N_n.We shall say that $f: X \to X$ is a periodic map of period m, if f^m is the identity map $id_X : X \to X$. Let $\mu(d), d \in N$, be the Möbius function of number theory. As is known, it is given by the following equations: $\mu(d) = 0$ if d is divisible by a square different from one ; $\mu(d) = (-1)^k$ if d is not divisible by a square different from one , where k denotes the number of prime divisors of d; $\mu(1) = 1$.

Theorem (F. - V. Pilyugina, 1985)

Let f be a periodic map of least period m of the connected compact polyhedron X . Then the Nielsen zeta function is equal to

$$N_f(z) = \prod_{d|m} \sqrt[d]{(1-z^d)^{-P(d)}},$$

where the product is taken over all divisors d of the period m, and $P(d)$ is the integer $P(d) = \sum_{d_1|d} \mu(d_1) N_{d|d_1}$.

Proof. Since $f^m = id$, for each $j, N_j = N_{m+j}$. Since $(k, m) = 1$, there exist positive integers t and q such that $kt = mq + 1$. So $(f^k)^t = f^{kt} = f^{mq+1} = f^{mq}f = (f^m)^q f = f$. Consequently, $N(f) = N(f^k)$. For an arbitrary period m one can prove completely analogously that $N_d = N_{di}$, if $(i, m/d) = 1$, where d is a divisor of m. Using these series of equal Nielsen numbers, one can regroup the terms of the series in the exponential of the Nielsen zeta function so as to get logarithmic functions by adding and subtracting missing terms with necessary coefficient. The same argument shows that $N(f^d) = N(f^{di})$ if $(i, m/d) = 1$ where d divisor m. Using these series of equal numbers we obtain the result by direct calculation

Nielsen zeta function of a periodic map

$$
\begin{aligned}
N_f(z) &= \exp\left(\sum_{i=1}^{\infty}\frac{N(f^i)}{i}z^i\right) \\
&= \exp\left(\sum_{d|m}\sum_{i=1}^{\infty}\frac{P(d)}{d}\cdot\frac{z^{d^i}}{i}\right) \\
&= \exp\left(\sum_{d|m}\frac{P(d)}{d}\cdot\log(1-z^d)\right) \\
&= \prod_{d|m}\sqrt[d]{(1-z^d)^{-P(d)}}
\end{aligned}
$$

where the integers $P(d)$ are calculated recursively by the formula

$$P(d) = N_d - \sum_{d_1|d;d_1\neq d} P(d_1).$$

Moreover, if the last formula is rewritten in the form

$$N_d = \sum_{d_1|d} P(d_1)$$

and one uses the Möbius Inversion law for real function in number theory, then

$$P(d) = \sum_{d_1|d} \mu(d_1) \cdot N_{d/d_1}.$$

Reidemeister zeta function

In this section we study the problem of rationality of the Reidemeister zeta function. The first group of related problems includes a study of validity of the TBFT (twisted Burnside-Frobenius theorem (or theory)) for different classes of groups and a proof of the Gauss congruences for the Reidemeister numbers of iterations. TBFT says that the Reidemeister number $R(\phi)$ of automorphism ϕ of a group G is equal to the number of finite-dimensional fixed points of the induced map $\widehat{\phi}$ on the unitary dual space $\widehat{G}$ if $R(\phi) < \infty$. TBFT was proved for automorphisms of abelian, finite, compact, abelian-by-finite and polycyclic-by-finite

Definition (Unitary Dual)

Denote by $\widehat{G}$ the set of equivalence classes of unitary irreducible representations of G and by $\widehat{G}_f$ its part corresponding to finite-dimensional representations. The class of ρ in $\widehat{G}$ we will denote by $[\rho]$. An automorphism ϕ of G induces a bijection $\widehat{\phi}: \widehat{G} \to \widehat{G}$ by the formula $[\widehat{\phi}(\rho)] := [\rho \circ \phi]$.

If G is a finite group and $\phi = \mathrm{Id}$ then TBFT becomes the classical Burnside-Frobenius theorem: the number of classes of irreducible representations of finite group G is equal to the number of conjugacy classes of elements of G.

TBFT, first example

For the simplest infinite group $G = \mathbb{Z}$ and its unique non-trivial isomorphism $\phi = -\mathrm{Id}$ we have

$$m \sim k + m - (-k) = m + 2k, \qquad \forall\, k$$

Thus even and odd numbers form 2 Reidemeister classes.

The dual object can be identified with the unit circle $S^1 \subset \mathbb{C}$ as follows: each (one dimensional) irreducible representation is defined at $m \in \mathbb{Z}$ as multiplication by $(e^{i\alpha})^m = e^{i\alpha m}$. In this way, $e^{i\alpha} \in S^1$ corresponds to this representation (denote by ρ_α). Then $\widehat{\phi}(\rho_\alpha)(m) = (\rho_\alpha)(-m) = (e^{i\alpha})^{-m} = (e^{-i\alpha})^m$. Thus, $\widehat{\phi}$ coincides with the complex conjugation and has on S^1 exactly two fixed points: ± 1. Hence, $F(\widehat{\phi}) = 2 = R(\phi)$. This example shows, in particular, that even for "simple" infinite groups the number of twisted conjugacy classes can be finite (in contrast with the ordinary classes).

Finite groups

Let ϕ-class functions be functions, which are constant on Reidemeister classes of ϕ, i.e. twisted invariant functions: $gf\phi(g^{-1}) = f$. Evidently, $R(\phi)$ is equal to the dimension of the space of such functions. On the other hand, for the $L^2(G)$ we have the Peter-Weyl decomposition:

$$L^2(G) \cong \bigoplus_{\rho\in\widehat{G}} \text{End } V_\rho, \qquad \text{End } V_\rho \cong \text{Mat}(\dim\rho, \mathbb{C}),$$

which respects the left and right G-actions (we have written $L^2(G)$, but this is $C[G] = C^*(G)$ because the group is finite).

Thus

$$R(\phi) = \dim\{\text{space of twisted invariant elements of } L^2(G)\} =$$

$$\sum_{[\rho]\in\widehat{G}} \dim T_\rho, \qquad T_\rho := \{F \in \mathrm{End}\, V_\rho | F = \rho(g) F \rho(\phi(g^{-1}) \text{ for all } g \in G\}$$

$$= \sum_{[\rho]\in\widehat{G}} \dim\{\text{space of intertwinning operators of } \rho \to \rho\circ\phi\} =$$

$$= \sum_{[\rho]\in\widehat{G}} \begin{cases} 1, & \text{if } \rho \sim \rho\circ\varphi \\ 0, & \text{if } \rho \not\sim \rho\circ\varphi \end{cases} = \#Fix(\widehat{\phi}).$$

We therefore have [F. - R.Hill, 1994]:

$$R(\phi) = \#\,\mathrm{Fix}\left(\hat{\phi} : \hat{G} \to \hat{G}\right) \tag{2}$$

The following nice way of calculation is known (Brauer ?). Consider the natural action ϕ^* on class-functions (for usual conjugacy classes). Then $\phi^* = \widehat{\phi}$ under the identification of Burnside-Frobenius. The trace of this operator should be the same in the basis of class-functions and in the basis of characters. In both cases the operator acts by transpositions of basic elements, thus, its trace is equal to the number of fixed element. Hence $\#Fix(\widehat{\phi})=$ the number of ϕ-invariant usual conjugacy classes. The TBFT in finite group case implies $R(\phi) = \#Fix((\widehat{\phi})$. Hence,

$R(\phi) =$ the number of ϕ-invariant usual classes. The above example with $\mathbb{Z}$ shows that this is not correct for infinite groups.

Abelian groups

Let G be a locally compact abelian topological group. We write $\hat{G}$ for the set of continuous homomorphisms from G to the circle $U(1) = \{z \in C : |z| = 1\}$. This is a group with pointwise multiplication. We call $\hat{G}$ the *Pontryagin dual* of G. When we equip $\hat{G}$ with the compact-open topology it becomes a locally compact abelian topological group. The dual of the dual of G is canonically isomorphic to G.

A continuous endomorphism $f : G \to G$ gives rise to a continuous endomorphism $\hat{f} : \hat{G} \to \hat{G}$ defined by

$$\hat{f}(\chi) := \chi \circ f.$$

If G is a finitely generated free abelian group then a homomorphism $\chi : G \to U(1)$ is completely determined by its values on a basis of G, and these values may be chosen arbitrarily. The dual of G is thus a torus whose dimension is equal to the rank of G.
If $G = Z/nZ$ then the elements of $\hat{G}$ are of the form

$$x \to e^{\frac{2\pi iyx}{n}}$$

with $y \in \{1, 2, \ldots, n\}$. A cyclic group is therefore (uncanonically) isomorphic to itself.
The dual of $G_1 \oplus G_2$ is canonically isomorphic to $\hat{G}_1 \oplus \hat{G}_2$. From this we see that any finite abelian group is (non-canonically) isomorphic to its own Pontryagin dual group, and that the dual of any finitely generated discrete abelian group is the direct sum of a

Torus and a finite group.
We shall require the following statement:

Proposition

Let $\phi : G \to G$ be an endomorphism of an abelian group G. Then the kernel $\ker \left[\hat{\phi} : \hat{G} \to \hat{G}\right]$ is canonically isomorphic to the Pontryagin dual of $\mathrm{Coker}\ \phi$.

PROOF. We construct the isomorphism explicitly. Let χ be in the dual of $\mathrm{Coker}\ (\phi : G \to G)$. In that case χ is a homomorphism

$$\chi : G/\operatorname{im}(\phi) \longrightarrow U(1).$$

There is therefore an induced map

$$\overline{\chi} : G \longrightarrow U(1)$$

which is trivial on $\mathrm{im}(\phi)$.
This means that $\overline{\chi} \circ \phi$ is trivial, or in other words $\hat{\phi}(\overline{\chi})$ is the identity element of $\hat{G}$. We therefore have $\overline{\chi} \in \ker(\hat{\phi})$.
If on the other hand we begin with $\overline{\chi} \in \ker(\hat{\phi})$, then it follows that χ is trivial on $\mathrm{im}\,\phi$, and so $\overline{\chi}$ induces a homomorphism

$$\chi : G/\mathrm{im}(\phi) \longrightarrow U(1)$$

and χ is then in the dual of $\mathrm{Coker}\ \phi$. The correspondence $\chi \leftrightarrow \overline{\chi}$ is clearly a bijection.

TBFT for Abelian groups

By Proposition above, the Pontrjagin dual of the cokernel of $(1-\phi):G\to G$ is canoically isomorphic to the kernel of the dual map $\widehat{(1-\phi)}:\hat{G}\to\hat{G}$. Since $\mathrm{Coker}\ (1-\phi)$ is finite, we have

$$\#\mathrm{Coker}\ (1-\phi)=\#\widehat{\mathrm{Coker}\ (1-\phi)}=\#\ker\widehat{(1-\phi)}.$$

The map $\widehat{1-\phi}$ is equal to $\hat{1}-\hat{\phi}$. Its kernel is thus the set of fixed points of the map $\hat{\phi}:\hat{G}\to\hat{G}$. We therefore have [F. - R.Hill, 1994]:

$$R(\phi)=\#\,\mathrm{Fix}\left(\hat{\phi}:\hat{G}\to\hat{G}\right) \tag{3}$$

Polycyclic groups

Let $G' = [G, G]$ be the *commutator subgroup* or *derived group* of G, i.e. the subgroup generated by commutators. G' is invariant under any homomorphism, in particular it is normal. It is the smallest normal subgroup of G with an abelian factor group. Denoting $G^{(0)} := G$, $G^{(1)} := G'$, $G^{(n)} := (G^{(n-1)})'$, $n \geq 2$, one obtains *derived series* of G:

$$G = G^{(0)} \supset G' \supset G^{(2)} \supset \cdots \supset G^{(n)} \supset \ldots \quad (4)$$

If $G^{(n)} = e$ for some value n, i.e. the series (4) stabilizes by trivial group, then the group G is *solvable*.

A solvable group is a *polycyclic group*, if it has a derived series with all $G^{(n)}$ finitely generated and all factors $G^{(n)}/G^{(n+1)}$ are cyclic. A group is said to have max if every its subgroup is finitely generated. It is known that a solvable group has max if and only if it is polycyclic. A polycyclic group is virtually poly-$\mathbb{Z}$.
In fact, the TBFT for group is closely related to a generalization of the following well-known notion.

Definition

A group G is *conjugacy separable* if any pair g, h of non-conjugate elements of G are non-conjugate in some finite quotient of G.

It was proved by Remeslenikov and Formanek that polycyclic-by-finite(or almost polycyclic) groups are conjugacy separable (see Daniel Segal book).

We can introduce the following notion, which coincides with the previous definition in the case $\phi = \mathrm{Id}$.

Definition

A group G is ϕ-*conjugacy separable* with respect to an automorphism $\phi : G \to G$ if any pair g, h of non-ϕ-conjugate elements of G are non-$\overline{\phi}$-conjugate in some finite quotient of G respecting ϕ.

Lemma

Let ρ be a finite dimensional irreducible representation of G on V_ρ, and $\phi : G \to G$ is an automorphism.
1). There exists a twisted invariant function $\omega : G \to C$ being a matrix coefficient of ρ if and only if $\widehat{\phi}[\rho] = [\rho]$.
2). In this case such ω is unique up to scaling.
3). If we have several distinct $\widehat{\phi}$-fixed representations, then the correspondent twisted invariant functions are linearly independent.

Function ω is defined by the formula

$$\omega : g \mapsto \mathrm{Tr}(S \circ \rho(g)), \quad (5)$$

where S is an intertwining operator between ρ and $\rho \circ \varphi$:

$$\rho(\varphi(x))S = S\rho(x) \quad \text{for any } x \in G.$$

In particular, TBFT is true for ϕ if and only if these matrix coefficients form a base of the space of ϕ-class functions. One gets the following statement.

Theorem (F. - E. Troitsky, 2007)

Suppose, $R(\phi) < \infty$. If a group G is ϕ-conjugacy separable then TBFT is true for G.

Proof: Indeed, let $F_{ij} : G \to K_{ij}$ distinguish ith and jth ϕ-conjugacy classes, where K_{ij} are finite groups, $i, j = 1, \ldots, R(\phi)$. Let

$F : G \to \oplus_{i,j} K_{ij}$, $F(g) = \sum_{i,j} F_{ij}(g)$, be the diagonal mapping and K its image. Then the map $F : G \to K$ gives TBFT. Indeed, each ϕ-class function f on G is a linear combination of functionals coming from some finite collection $\{\rho_i\}$ of fixed by the map $\widehat{\phi}$ elements of $\widehat{G}_f$. These representations $\rho_1, \ldots, \rho_s$ are in fact representations of the form $\pi_i \circ F$, where π_i are irreducible representations of the finite group K and $F : G \to K$, as above.
The following construction relates ϕ-conjugacy classes and some conjugacy classes of another group.
Consider the action of $\mathbb{Z}$ on G, i.e. a homomorphism $\mathbb{Z} \to \mathrm{Aut}(G)$, $n \mapsto \phi^n$. Let Γ be a corresponding semi-direct product $\Gamma = G \rtimes \mathbb{Z}$:

$$\Gamma := < G, t \mid tgt^{-1} = \phi(g) > \qquad (6)$$

in terms of generators and relations, where t is a generator of Z. The group G is a normal subgroup of Γ. As a set, Γ has the form

$$\Gamma = \sqcup_{n \in Z} G \cdot t^n, \qquad (7)$$

where $G \cdot t^n$ is the coset by G containing t^n.

Lemma

Two elements x, y of G are ϕ-conjugate iff xt and yt are conjugate in the usual sense in Γ. Therefore $g \mapsto g \cdot t$ is a bijection from the set of ϕ-conjugacy classes of G onto the set of conjugacy classes of Γ contained in $G \cdot t$.

Proof. If x and y are ϕ-conjugate then there is a $g \in G$ such that $gx = y\phi(g)$. This implies $gx = ytgt^{-1}$ and therefore $g(xt) = (yt)g$ so xt and yt are conjugate in the usual sense in Γ. Conversely, suppose xt and yt are conjugate in Γ. Then there is a $gt^n \in \Gamma$ with $gt^nxt = ytgt^n$. From the relation $txt^{-1} = \phi(x)$ we obtain $g\phi^n(x)t^{n+1} = y\phi(g)t^{n+1}$ and therefore $g\phi^n(x) = y\phi(g)$. Hence, y and $\phi^n(x)$ are ϕ-conjugate. Thus, y and x are ϕ-conjugate, because x and $\phi(x)$ are always

ϕ-conjugate: $\phi(x) = x^{-1}x\phi(x)$.

Theorem

Let $F:\Gamma\to K$ be a morphism onto a finite group K which separates two conjugacy classes of Γ in $G\cdot t$. Then the restriction $F_G := F|_G : G \to \mathrm{Im}(F|_G)$ separates the corresponding ϕ-conjugacy classes in G.

Proof: First let us remark that $\mathrm{Ker}(F_G)$ is ϕ-invariant. Indeed, suppose $F_G(g) = F(g) = e$. Then

$$F_G(\phi(g)) = F(\phi(g)) = F(tgt^{-1}) = F(t)F(t)^{-1} = e$$

(the kernel of F is a normal subgroup).

Let gt and $\widetilde{g}t$ be some representatives of the mentioned conjugacy classes. Then

$$F((ht^n)gt(ht^n)^{-1}) \neq F(\widetilde{g}t), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(ht^ngt) \neq F(\widetilde{g}tht^n), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(h\phi^n(g)t^{n+1}) \neq F(\widetilde{g}\phi(h)t^{n+1}), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

$$F(h\phi^n(g)) \neq F(\widetilde{g}\phi(h)), \qquad \forall h \in G,\ n \in \mathbb{Z},$$

in particular, $F(hg\phi(h^{-1})) \neq F(\widetilde{g})\ \forall h \in G$.

Theorem (F. - E. Troitsky, 2007)

Let G be a polycyclic-by-finite group. Suppose, $R(\phi) < \infty$. Then TBFT is true for G i.e.

$$R(\phi) = \# \operatorname{Fix}\left(\hat{\phi} : \hat{G}_f \to \hat{G}_f\right) \tag{8}$$

Proof: The class of polycyclic-by-finite groups is closed under taking semidirect products by Z. Indeed, let G be an polycyclic-by-finite group. Then there exists a characteristic (polycyclic) subgroup P of finite index in G. Hence, $P \rtimes Z$ is a polycyclic normal group of $G \rtimes Z$ of the same finite index.

Polycyclic-by-finite groups are conjugacy separable . It remains to apply theorem for ϕ-conjugacy separable group above.

Gauss congruences: an application of TBFT

In dynamical context we study Reidemeister numbers of iterations. Nice analytical properties of the Reidemeister zeta function indicate that the Reidemeister numbers $R(\phi^n)$ are closely interconnected. The manifestation of this are Gauss congruences for Reidemeister numbers.

More precisely, let μ be the Möbius function:

$$\mu(d) = \begin{cases} 1, & \text{if } d = 1; \\ (-1)^k, & \text{if } d \text{ — is a product of} \\ & k \text{ distinct prime numbers;} \\ 0, & \text{if } d \text{ is not square free.} \end{cases}$$

In number theory, the following Gauss congruence for integers holds:

$$\sum_{d|n} \mu(d)\ a^{n/d} \equiv 0 \mod n$$

for any integer a and any natural number n.
In the case of a prime power $n = p^r$, the Gauss congruences turn into the Euler congruences. Indeed, for $n = p^r$ the Möbius function

$\mu(n/d) = \mu(p^r/d)$ is different from zero only in two cases: when $d = p^r$ and when $d = p^{r-1}$. Therefore, from the Gauss congruence we obtain the Euler congruence

$$a^{p^r} \equiv a^{p^{r-1}} \mod p^r$$

This congruence is equivalent to the following classical Euler's theorem:

$$a^{\varphi(n)} \equiv 1 \mod n$$

where $(a, n) = 1$.

These congruences have been generalized from integers a to some other mathematical invariants such as the traces of all integer matrices A and the Lefschetz numbers of iterations of a map:

$$\sum_{d|n} \mu(d)\ \mathrm{tr}(A^{n/d}) \equiv 0 \mod n, \tag{9}$$

$$\mathrm{tr}(A^{p^r}) \equiv \mathrm{tr}(A^{p^{r-1}}) \mod p^r. \tag{10}$$

$$\sum_{d|n} \mu(d)\ L(f^{n/d}) \equiv 0 \mod n \tag{DL}$$

These congruences are now also called the Dold congruences. It is shown that the above congruences (9), (10) and (DL) are equivalent.

Suppose, the TBFT holds for a group. Then, using congruences for fixed points of iterations of the dual map one can obtain the Gauss congruences for Reidemeister numbers of iterations

Theorem

Let G be a polycyclic-by-finite group. Suppose, $R(\phi^n) < \infty$ for all n. Then we have Gauss congruences for Reidemeister numbers of iterations:

$$\sum_{d|n} \mu(d) \cdot R(\phi^{n/d}) \equiv 0 \mod n, \tag{11}$$

Proof. we have following evident consequence of TBFT: $R(\phi^n) = \#\, Fix(\widehat{\phi}^n|_{\widehat{G}_f}) = \sum_{d|n} P_d$, where P_d denote the number of periodic points of $\widehat{\phi}$ on $\widehat{G}_f$ of least period d. Applying the Möbius' inversion formula we obtain $P_n = \sum_{d|n} \mu(d) \cdot R(\phi^{n/d})$. But number P_n is always divisible by n, because P_n is exactly n times the number of orbits of $\widehat{\phi}$ of length n.

Rationality of the Reidemeister zeta function of endomorphisms of finitely generated Abelian groups

For a finitely generated Abelian group G we define the finite subgroup G^{finite} to be the subgroup of torsion elements of G. We denote the quotient $G^\infty := G/G^{finite}$. The group G^∞ is torsion free. Since the image of any torsion element by a homomorphism must be a torsion element, the function $\phi : G \to G$ induces maps

$$\phi^{finite} : G^{finite} \longrightarrow G^{finite}, \quad \phi^\infty : G^\infty \longrightarrow G^\infty.$$

The dual group of G^∞ is a torus whose dimension is the rank of G. This is canonically a closed subgroup of $\hat{G}$. We shall denote it $\hat{G}_0$. The quotient $\hat{G}/\hat{G}_0$ is canonically isomorphic to the dual of G^{finite}.

We shall call a torus $\hat{G}_i$ periodic if there is an iteration $\hat{\phi}^s$ such that $\hat{\phi}^s(\hat{G}_i) \subset \hat{G}_i$. If this is the case, then the map $\hat{\phi}^s : \hat{G}_i \to \hat{G}_i$ is a translation of the map $\hat{\phi}^s : \hat{G}_0 \to \hat{G}_0$ and has the same number of fixed points as this map. If $\hat{\phi}^s(\hat{G}_i) \not\subset \hat{G}_i$ then $\hat{\phi}^s$ has no fixed points in $\hat{G}_i$. The map on the torus

$$\hat{\phi}_0 : \hat{G}_0 \to \hat{G}_0$$

lifts to a linear map F of the universal cover, which is in this case the Lie algebra of $\hat{G}$. It is well known that Lefschetz number of $\hat{\phi}_0$ equals $\det(F - Id)$.

Theorem (F. - R.Hill, 1994)

Let $\phi : G \to G$ be an endomorphism of a finitely generated Abelian group. Then we have the following $R(\phi^n) = \mid L(\hat{\phi}^n) \mid$, where $L(\hat{\phi}^n)$ is the Lefschetz number of $\hat{\phi}$ thought of as a self-map of the topological space $\hat{G}$. From this it follows that zeta functon $R_\phi(z)$ is rational function and is equal to:

$$R_\phi(z) = L_{\hat{\phi}}(\sigma z)^{(-1)^r}, \tag{12}$$

where $\sigma = (-1)^p$ where p is the number of real eingevalues $\lambda \in Spec(F)$ such that $\lambda < -1$ and r is the number of real eingevalues $\lambda \in Spec(F)$ such that $\mid \lambda \mid > 1$. If G is finite abelian group then this reduces to $R(\phi^n) = L(\hat{\phi}^n)$ and $R_\phi(z) = L_{\hat{\phi}}(z)$.

Proof. If G is finite abelian then $\hat{G}$ is a discrete finite set, so the number of fixed points is equal to the Lefschetz number. This finishes the proof in the case that G is finite.

If G a finitely generated Abelian group it is only necessary to check that the number of fixed points of $\hat{\phi}^n$ is equal to the absolute value of its Lefschetz number. We assume without loss of generality that $n=1$. We are assuming that $R(\phi)$ is finite, so the fixed points of $\hat{\phi}$ form a discrete set. We therefore have $L(\hat{\phi}) = \sum_{x\in \mathrm{Fix}\,\hat{\phi}} \mathrm{ind}(\hat{\phi}, x)$.

Since ϕ is a group endomorphism, the zero element $0 \in \hat{G}$ is always fixed. Let x be any fixed point of $\hat{\phi}$. We then have a commutative diagram

$$\begin{array}{ccccccc} g & & \hat{G} & \xrightarrow{\hat{\phi}} & \hat{G} & & g \\ \updownarrow & & \updownarrow & & \updownarrow & & \updownarrow \\ g+x & & \hat{G} & \xrightarrow{\hat{\phi}} & \hat{G} & & g+x \end{array}$$

in which the vertical functions are translations on $\hat{G}$ by x. Since the vertical maps map 0 to x, we deduce that

$$\mathrm{ind}(\hat{\phi}, x) = \mathrm{ind}(\hat{\phi}, 0)$$

and so all fixed points have the same index. It is now sufficient to show that $\mathrm{ind}(\hat{\phi}, 0) = \pm 1$. This follows because the map on the torus $\hat{\phi}_0 : \hat{G}_0 \to \hat{G}_0$ lifts to a linear map F of the universal cover, which is the Lie algebra of $\hat{G}$. The index is then the sign of the determinant of the identity map minus this lifted map F. This determinant cannot be zero, because $1-\hat{\phi}$ must have finite kernel by our assumption that the $R(\phi)$ is finite (if $\det(1-F)=0$ then the kernel of $1-\hat{\phi}$ is a positive dimensional subgroup of $\hat{G}$, and therefore infinite). So we have $R(\phi^n)=\#\operatorname{Fix}\left(\hat{\phi}^n:\hat{G}\to\hat{G}\right)=\mid L(\hat{\phi}^n)\mid=(-1)^{r+pn}L(\hat{\phi}^n)$ for all n. Then zeta function $R_\phi(z)=L_{\hat{\phi}}(\sigma z)^{(-1)^r}$ is rational function.

Rationality of the Reidemeister zeta function of endomorphisms of finitely generated Abelian groups - second proof

A Convolution Product

When $R_\phi(z)$ is a rational function the infinite sequence $\{R(\phi^n)\}_{n=1}^\infty$ of Reidemeister numbers is determined by a finite set of complex numbers - the zeros and poles of $R_\phi(z)$.

Lemma

$R_\phi(z)$ is a rational function if and only if there exists a finite set of complex numbers α_i and β_j such that $R(\phi^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n$ for every $n > 0$.

PROOF Suppose $R_\phi(z)$ is a rational function. Then

$$R_\phi(z) = \frac{\prod_i (1-\alpha_i z)}{\prod_j (1-\beta_j z)},$$

where $\alpha_i, \beta_j \in C$. Taking the logarithmic derivative of both sides and then using the geometric series expansion we see that $R(\phi^n) = \sum_j \beta_j^n - \sum_i \alpha_i^n$. The converse is proved by a direct calculation.

For two sequences (x_n) and (y_n) we may define the corresponding zeta functions:

$$X(z) := \exp\left(\sum_{n=1}^{\infty} \frac{x_n}{n} z^n\right),$$

$$Y(z) := \exp\left(\sum_{n=1}^{\infty} \frac{y_n}{n} z^n\right).$$

Alternately, given complex functions X and Y (defined in a neighbourhood of 0) we may define sequences

$$x_n := \frac{d^n}{dz^n} \log\left(X(z)\right) |_{z=0},$$

$$y_n := \frac{d^n}{dz^n} \log\left(Y(z)\right) |_{z=0}.$$

Taking the componentwise product of the two sequences gives another sequence, from which we obtain another complex function. We call this new function the *additive convolution* of X and Y, and we write it

$$(X * Y)(z) := \exp\left(\sum_{n=1}^{\infty} \frac{x_n . y_n}{n} z^n\right).$$

It follows immediately from lemma 1 that if X and Y are rational functions then $X * Y$ is a rational function. In fact we may show using the same method the following

Lemma (Convolution of rational functions)

Let

$$X(z) = \prod_i (1-\alpha_i z)^{m(i)}, \quad Y(z) = \prod_j (1-\beta_j z)^{l(j)}$$

be rational functions in z. Then $X * Y$ is the following rational function

$$(X * Y)(z) = \prod_{i,j} (1-\alpha_i \beta_j z)^{-m(i).l(j)}. \tag{13}$$

Lemma

Let $\phi : Z^k \rightarrow Z^k$ be a group endomorphism. Then we have

$$R_\phi(z) = \left(\prod_{i=0}^{k} \det(1 - \Lambda^i \phi.\sigma z)^{(-1)^{i+1}} \right)^{(-1)^r} \qquad (14)$$

where $\sigma = (-1)^p$ with p the number of $\mu \in \operatorname{spec} \phi$ such that $\mu < -1$, and r the number of real eigenvalues of ϕ whose absolute value is > 1. Λ^i denotes the exterior power.

PROOF Since Z^k is abelian, we have as before,

$$R(\phi^n) = \#\mathrm{Coker}\ (1 - \phi^n).$$

On the other hand we have

$$\#\mathrm{Coker}\ (1-\phi^n) = \mid \det(1-\phi^n) \mid,$$

and hence $R(\phi^n) = (-1)^{r+pn} \det(1-\phi^n)$. It is well known from linear algebra that $\det(1-\phi^n) = \sum_{i=0}^{k} (-1)^i \operatorname{tr}(\Lambda^i \phi^n)$.
From this we have the following "trace formula" for Reidemeister numbers:

$$R(\phi^n) = (-1)^{r+pn} \sum_{i=0}^{k} (-1)^i \operatorname{tr}(\Lambda^i \phi^n). \qquad (15)$$

We now calculate directly

$$\begin{aligned}
R_\phi(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{R(\phi^n)}{n}z^n\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^r\sum_{i=0}^{k}(-1)^i\,\mathrm{tr}(\Lambda^i\phi^n)}{n}(\sigma z)^n\right)\\
&= \left(\prod_{i=0}^{k}\left(\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{tr}(\Lambda^i\phi^n).(\sigma z)^n\right)\right)^{(-1)^i}\right)^{(-1)^r}\\
&= \left(\prod_{i=0}^{k}\det\left(1-\Lambda^i\phi.\sigma z\right)^{(-1)^{i+1}}\right)^{(-1)^r}.
\end{aligned}$$

Lemma

Let $\phi : G \to G$ be an endomorphism of a finite abelian group G. Then we have the following "Euler product" expression

$$R_\phi(z) = \prod_{[\gamma]} \frac{1}{1 - z^{\#[\gamma]}} \tag{16}$$

where the product is taken over the periodic orbits of ϕ in G.

We give two proofs of this lemma. The first proof is given here and the second proof is a special case of the proof for nonabelian finite group.

PROOF Since G is abelian, we again have,

$$\begin{aligned} R(\phi^n) &= \#\mathrm{Coker}\ (1-\phi^n) \\ &= \# G / \#\,\mathrm{im}(1-\phi^n) \\ &= \# G / \#(G/\ker(1-\phi^n)) \\ &= \# G / (\# G / \#\ker(1-\phi^n)) \\ &= \#\ker(1-\phi^n) \\ &= \#\,\mathrm{Fix}(\phi^n) \end{aligned}$$

We shall call an element of G periodic if it is fixed by some iteration of ϕ. A periodic element γ is fixed by ϕ^n iff n is divisible by the cardinality the orbit of γ. We therefore have

$$\begin{aligned} R(\phi^n) &= \sum_{\substack{\gamma \;\; periodic \\ \#[\gamma] | n}} 1 \\ &= \sum_{\substack{[\gamma] \;\; such \;\; that, \\ \#[\gamma] | n}} \#[\gamma]. \end{aligned}$$

From this follows

Rationality of the Reidemeister zeta function of endomorphisms of finitely generated Abelian groups

$$
\begin{aligned}
R_\phi(z) &= \exp\left(\sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n\right) \\
&= \exp\left(\sum_{[\gamma]} \sum_{\substack{n=1 \\ \#[\gamma] \mid n}}^{\infty} \frac{\#[\gamma]}{n} z^n\right) \\
&= \prod_{[\gamma]} \exp\left(\sum_{n=1}^{\infty} \frac{\#[\gamma]}{\#[\gamma] n} z^{\#[\gamma] n}\right) =
\end{aligned}
$$

$$= \prod_{[\gamma]} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} z^{\#[\gamma]n}\right)$$

$$= \prod_{[\gamma]} \exp\left(-\log\left(1 - z^{\#[\gamma]}\right)\right)$$

$$= \prod_{[\gamma]} \frac{1}{1 - z^{\#[\gamma]}}.$$

For a finitely generated abelian group G we define the finite subgroup G^{finite} to be the subgroup of torsion elements of G. We denote the quotient $G^{\infty} := G/G^{finite}$. The group G^{∞} is torsion free. Since the image of any torsion element by a homomorphism must be a torsion element, the function $\phi : G \to G$ induces maps

$$\phi^{finite} : G^{finite} \longrightarrow G^{finite}, \quad \phi^{\infty} : G^{\infty} \longrightarrow G^{\infty}.$$

Theorem (F. - R.Hill, 1994)

If G is a finitely generated abelian group and ϕ an endomorphism of G then $R_\phi(z)$ is a rational function and is equal to the following additive convolution:

$$R_\phi(z) = R_\phi^\infty(z) * R_\phi^{finite}(z). \tag{17}$$

where $R_\phi^\infty(z)$ is the Reidemeister zeta function of the endomorphism $\phi^\infty : G^\infty \to G^\infty$, and $R_\phi^{finite}(z)$ is the Reidemeister zeta function of the endomorphism $\phi^{finite} : G^{finite} \to G^{finite}$. The rational functions $R_\phi^\infty(z)$ and $R_\phi^{finite}(z)$ are given by the formulae above

PROOF By Pontrjagin duality we have as above

$$R(\phi^n) = \#\,\mathrm{Fix}\left(\hat{\phi}^n : \hat{G} \to \hat{G}\right).^1 \qquad (18)$$

The dual group of G^∞ is a torus whose dimension is the rank of G. This is canonically a closed subgroup of $\hat{G}$. We shall denote it $\hat{G}_0$. The quotient $\hat{G}/\hat{G}_0$ is canonically isomorphic to the dual of G^{finite}. It is therefore finite. From this we know that $\hat{G}$ is a union of finitely many disjoint tori. We shall call these tori $\hat{G}_0, \dots, \hat{G}_r$.
We shall call a torus $\hat{G}_i$ periodic if there is an iteration $\hat{\phi}^s$ such that $\hat{\phi}^s(\hat{G}_i) \subset \hat{G}_i$. If this is the case, then the map $\hat{\phi}^s : \hat{G}_i \to \hat{G}_i$ is a translation of the map $\hat{\phi}^s : \hat{G}_0 \to \hat{G}_0$ and has the same number of fixed points as this map. If $\hat{\phi}^s(\hat{G}_i) \not\subset \hat{G}_i$ then $\hat{\phi}^s$ has no fixed points

in $\hat{G}_i$. From this we see

$$\#\operatorname{Fix}\left(\hat{\phi}^n : \hat{G} \to \hat{G}\right) = \#\operatorname{Fix}\left(\hat{\phi}^n : \hat{G}_0 \to \hat{G}_0\right) \times \#\{\hat{G}_i \mid \hat{\phi}^n(\hat{G}_i) \subset \hat{G}_i\}.$$

We now rephrase this

$$\#\operatorname{Fix}\left(\hat{\phi}^n : \hat{G} \to \hat{G}\right)$$
$$= \#\operatorname{Fix}\left(\widehat{\phi^{\infty}}^n : \hat{G}_0 \to \hat{G}_0\right) \times \#\operatorname{Fix}\left(\widehat{\phi^{finite}}^n : \hat{G}/(\hat{G}_0) \to \hat{G}/(\hat{G}_0)\right).$$

From this we have

$$R_\phi(z) = R_{(\phi^\infty)}(z) * R_{(\phi^{finite})}(z).$$

The rationality of $R_\phi(z)$ and the formulae for $R_\phi^\infty(z)$ and $R_\phi^{finite}(z)$ follow from the previous lemmas.

Rationality of the Reidemeister zeta function of endomorphisms of finite groups

In this section we consider finite non-abelian groups. We shall write the group law multiplicatively. We generalize our results on endomorphisms of finite abelian groups to endomorphisms of finite non-abelian groups. We shall write $\{g\}$ for the ϕ-conjugacy class of an element $g \in G$. We shall write $< g >$ for the ordinary conjugacy class of g in G. We continue to write $[g]$ for the ϕ-orbit of $g \in G$, and we also write now $[< g >]$ for the ϕ-orbit of the ordinary conjugacy class of $g \in G$. We first note that if ϕ is an endomorphism of a group G then ϕ maps conjugate elements to conjugate elements. It therefore induces an endomorphism of the

If G is abelian then a conjugacy class consists of a single element. The following is thus an extension of the result in abelian case:

Theorem (F. - R.Hill, 1994)

Let G be a finite group and let $\phi : G \to G$ be an endomorphism. Then $R(\phi)$ is the number of ordinary conjugacy classes $< x >$ in G such that $< \phi(x) >=< x >$.

Proof From the definition of the Reidemeister number we have,

$$R(\phi) = \sum_{\{g\}} 1$$

where $\{g\}$ runs through

the set of ϕ-conjugacy classes in G. This gives us immediately

$$\begin{aligned} R(\phi) &= \sum_{\{g\}} \sum_{x\in\{g\}} \frac{1}{\#\{g\}} \\ &= \sum_{\{g\}} \sum_{x\in\{g\}} \frac{1}{\#\{x\}} \\ &= \sum_{x\in G} \frac{1}{\#\{x\}}. \end{aligned}$$

We now calculate for any $x \in G$ the order of $\{x\}$. The class $\{x\}$ is the orbit of x under the G-action

$$(g, x) \longmapsto gx\phi(g)^{-1}.$$

We verify that this is actually a G-action:

$$\begin{aligned}
(id, x) &\longmapsto id.x.\phi(id)^{-1} \\
&= x, \\
(g_1g_2, x) &\longmapsto g_1g_2.x.\phi(g_1g_2)^{-1} \\
&= g_1g_2.x.(\phi(g_1)\phi(g_2))^{-1} \\
&= g_1g_2.x.\phi(g_2)^{-1}\phi(g_1)^{-1} \\
&= g_1(g_2.x.\phi(g_2)^{-1})\phi(g_1)^{-1}.
\end{aligned}$$

We therefore have from the orbit-stabilizer theorem,

$$\#\{x\} = \frac{\#G}{\#\{g \in G \mid gx\phi(g)^{-1} = x\}}.$$

The condition $gx\phi(g)^{-1} = x$ is equivalent to

$$x^{-1}gx\phi(g)^{-1} = 1 \quad \Leftrightarrow \quad x^{-1}gx = \phi(g)$$

We therefore have

$$R(\phi) = \frac{1}{\#G}\sum_{x\in G}\#\{g\in G \mid x^{-1}gx = \phi(g)\}.$$

Changing the summation over x to summation over g, we have:

$$R(\phi) = \frac{1}{\#G}\sum_{g\in G}\#\{x\in G \mid x^{-1}gx = \phi(g)\}.$$

If $<\phi(g)>\neq<g>$ then there are no elements x such that

$x^{-1}gx = \phi(g)$. We therefore have:

$$R(\phi) = \frac{1}{\#G} \sum_{\substack{g\in G \text{ such that} \\ <\phi(g)>=<g>}} \#\{x \in G \mid x^{-1}gx = \phi(g)\}.$$

The elements x such that $x^{-1}gx = \phi(g)$ form a coset of the subgroup satisfying $x^{-1}gx = g$. This subgroup is the centralizer of g in G which we write $C(g)$. With this notation we have,

$$\begin{aligned} R(\phi) &= \frac{1}{\#G} \sum_{\substack{g\in G \text{ such that} \\ <\phi(g)>=<g>}} \#C(g) \\ &= \frac{1}{\#G} \sum_{\substack{<g>\subset G \text{ such that} \\ <\phi(g)>=<g>}} \# < g > .\#C(g). \end{aligned}$$

The last identity follows because $C(h^{-1}gh) = h^{-1}C(g)h$. From the orbit stabilizer theorem, we know that $\# < g > .\#C(g) = \#G$. We therefore have $R(\phi) = \#\{< g > \subset G \mid < \phi(g) > = < g >\}$. From this theorem we have immediately,

Theorem (F. - R.Hill, 1994)

Let ϕ be an endomorphism of a finite group G. Then $R_\phi(z)$ is a rational function with a functional equation. In particular we have,

$$R_\phi(z) = \prod_{[<g>]} \frac{1}{1 - z^{\#[<g>]}}, R_\phi\left(\frac{1}{z}\right) = (-1)^a z^b R_\phi(z). \qquad (19)$$

The product here is over all periodic ϕ-orbits of ordinary conjugacy classes of elements of G. The number $\#[< g >]$ is the number of conjugacy classes in the ϕ-orbit of the conjugacy class $< g >$. In the functional equation the numbers a and b are respectively the number of periodic ϕ-orbits of conjugacy classes of elements of G and the number of periodic conjugacy classes of elements of G.

PROOF From the previous theorem we know that $R(\phi^n)$ is the number of conjugacy classes $< g > \subset G$ such that $\phi^n(< g >) \subset < g >$. We can rewrite this

$$R(\phi^n) = \sum_{\substack{[< g >] \text{ such that} \\ \#[< g >] \mid n}} \#[< g >].$$

From this we have,

$$R_\phi(z) = \prod_{[<g>]} \exp\left(\sum_{\substack{n=1 \text{ such that} \\ \#[<g>] \mid n}}^{\infty} \frac{\#[<g>]}{n} z^n \right).$$

The first formula now follows by using the power series expansion for $\log(1-z)$. The functional equation follows from the previous theorem by direct computation.

Rationality of the Reidemeister zeta function of endomorphisms of finitely generated torsion free nilpotent groups

Consider the lower central series of a finitely generated group G: $G = G_1 \supset G_2 \supset \cdots$, where $G_j = [G, G_{j-1}]$ is the j-fold commutator subgroup $\gamma_j(G)$ of G. The group G is called **nilpotent** if $G_j = 1$ for some j. When $G_c \neq 1$ but $G_{c+1} = 1$, we say that it is c-**step** nilpotent.

In this section we consider finitely generated torsion free nilpotent group Γ. It is well known(Malcev) that such group Γ is a uniform discrete subgroup of a simply connected nilpotent Lie group G (uniform means that the coset space G/Γ is compact).The coset space $M = G/\Gamma$ is called a nilmanifold.Since $\Gamma = \pi_1(M)$ and M is a $K(\Gamma, 1)$, every endomorphism $\phi : \Gamma \to \Gamma$ can be realized by a selfmap $f : M \to M$ such that $f_* = \phi$ and thus $R(f) = R(\phi)$. Any endomorphism $\phi : \Gamma \to \Gamma$ can be uniquely extended to an endomorphism $F : G \to G$. Let $\tilde{F} : \tilde{G} \to \tilde{G}$ be the corresponding Lie algebra endomorphism induced from F.

Lemma

If Γ is a finitely generated torsion free nilpotent group and ϕ an endomorphism of Γ .Then

$$R(\phi) = (-1)^{r+p} \sum_{i=0}^{m} (-1)^i \operatorname{tr} \Lambda^i \tilde{F}, \tag{20}$$

where m is $rg\Gamma = \dim M$, p the number of $\mu \in \operatorname{spec} \tilde{F}$ such that $\mu < -1$, and r the number of real eigenvalues of $\tilde{F}$ whose absolute value is > 1.

PROOF: Let $f : M \to M$ be a map realizing ϕ on a compact nilmanifold M of dimension m.We suppose that the Reidemeister number $R(f) = R(\phi)$ is finite.The finiteness of $R(f)$ implies the nonvanishing of the Lefschetz number $L(f)$.A strengthened version of Anosov's theorem states, in particular, that if $L(f)\neq 0$ than $N(f)=|L(f)|=R(f)$. It is well known that $L(f)=\det(\tilde{F}-1)$. From this we have

$$R(\phi)=R(f)=|L(f)|=|\det(1-\tilde{F})|=(-1)^{r+p}\det(1-\tilde{F})=$$

$$=(-1)^{r+p}\sum_{i=0}^{m}(-1)^i\,\mathrm{tr}\,\Lambda^i\tilde{F}.$$

Theorem

If Γ is a finitely generated torsion free nilpotent group and ϕ an endomorphism of Γ .Then $R_\phi(z)$ is a rational function and is equal to

$$R_\phi(z) = \left(\prod_{i=0}^{m} \det(1 - \Lambda^i \tilde{F} \cdot \sigma \cdot z)^{(-1)^{i+1}}\right)^{(-1)^r} \qquad (21)$$

where $\sigma = (-1)^p$,p , r, m and $\tilde{F}$ is defined in Lemma above.

PROOF Lemma above implies the trace formula for $R(\phi^n)$:

$$R(\phi^n) = (-1)^{r+pn} \sum_{i=0}^{m} (-1)^i \operatorname{tr}(\Lambda^i \tilde{F})^n$$

We now calculate directly

$$
\begin{aligned}
R_\phi(z) &= \exp\left(\sum_{n=1}^{\infty}\frac{R(\phi^n)}{n}z^n\right)\\
&= \exp\left(\sum_{n=1}^{\infty}\frac{(-1)^r\sum_{i=0}^{k}(-1)^i\,\mathrm{tr}(\Lambda^i\tilde{F}^n)}{n}(\sigma z)^n\right)\\
&= \left(\prod_{i=0}^{k}\left(\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}\,\mathrm{tr}(\Lambda^i\tilde{F}^n)\cdot(\sigma z)^n\right)\right)^{(-1)^i}\right)^{(-1)^r}\\
&= \left(\prod_{i=0}^{k}\det\left(1-\Lambda^i\tilde{F}\cdot\sigma z\right)^{(-1)^{i+1}}\right)^{(-1)^r}.
\end{aligned}
$$

Reidemeister and Nielsen zeta functions for maps of infra-nilmanifolds and infra-solvmanifolds

I will try to explain a essential progress in the problem of the rationality and properties of the Nielsen and the Reidemeister zeta function for maps of infra-nilmanifolds and infra-solvmanifolds of type $\mathbb{R}$ and their connection to the topological entropy of maps and the Reidemeister torsion of the corresponding mapping tori and also the progress in the problem of Gauss congruences for Nielsen and Reidemeister numbers. Zeta functions $R_f(z)$ and $N_f(z)$ coincide and are rational on on infra-nilmanifolds and on infra-solvmanifolds of type $\mathbb{R}$. This allows to give a linear bound for the number of

We consider almost Bieberbach groups $\Pi \subset G \rtimes \mathrm{Aut}(G)$, where G is a connected, simply connected nilpotent Lie group, and infra-nilmanifolds $M = \Pi \backslash G$. It is known that these are exactly the class of almost flat Riemannian manifolds . It is L. Auslander's result that $\Gamma := \Pi \cap G$ is a lattice of G, and is the unique maximal normal nilpotent subgroup of Π. The group $\Phi = \Pi/\Gamma$ is the *holonomy group* of Π or M. Thus we have the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & G \rtimes \mathrm{Aut}(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & 1 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \xrightarrow{p} & \Phi & \longrightarrow & 1
\end{array}$$

Thus Φ sits naturally in $\mathrm{Aut}(G)$. Denote $\rho : \Phi \to \mathrm{Aut}(\mathfrak{G})$, $A \mapsto A_* =$ the differential of A.
Let $M = \Pi \backslash G$ be an infra-nilmanifold. Any continuous map $f : M \to M$ induces a homomorphism $\phi : \Pi \to \Pi$. We can choose an affine element $(d, D) \in G \rtimes \mathrm{Endo}(G)$ such that

$$\phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \forall \alpha \in \Pi. \tag{22}$$

This implies that the affine map $(d, D) : G \to G$ induces a continuous map on the infra-nilmanifold $M = \Pi \backslash G$, which is homotopic to f. That is, f has an affine homotopy lift (d, D). We can choose a fully invariant subgroup $\Lambda \subset \Gamma$ of Π which is of finite index. Therefore $\phi(\Lambda) \subset \Lambda$ and so ϕ induces the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda & \longrightarrow & \Pi & \longrightarrow & \Psi & \longrightarrow & 1 \\ & & \downarrow \phi' & & \downarrow \phi & & \downarrow \bar{\phi} & & \\ 1 & \longrightarrow & \Lambda & \longrightarrow & \Pi & \longrightarrow & \Psi & \longrightarrow & 1 \end{array}$$

where $\Psi = \Pi/\Lambda$ is finite. Applying (22) for $\lambda \in \Lambda \subset \Pi$, we see that

$$\phi(\lambda) = dD(\lambda)d^{-1} = (\tau_d D)(\lambda)$$

where τ_d is the conjugation by d. The homomorphism $\phi' : \Lambda \to \Lambda$ induces a unique Lie group homomorphism $F = \tau_d D : G \to G$, and hence a Lie algebra homomorphism $F_* : \mathfrak{G} \to \mathfrak{G}$. On the other hand, since $\phi(\Lambda) \subset \Lambda$, f has a lift $\bar{f} : N \to N$ on the nilmanifold $N := \Lambda \backslash G$ which finitely and regularly covers M and has Ψ as its group of covering transformations.

Theorem (Averaging Formula (JB Lee - KB Lee, 2006))

Let f be a continuous map on an infra-nilmanifold $\Pi\backslash G$ with holonomy group Φ. Let f have an affine homotopy lift (d, D) and let $\phi : \Pi \to \Pi$ be the homomorphism induced by f. Then we have

$$L(f) = \frac{1}{|\Phi|}\sum_{A\in\Phi}\det(I - A_*F_*) = \frac{1}{|\Phi|}\sum_{A\in\Phi}\det(I - A_*D_*),$$

$$N(f) = \frac{1}{|\Phi|}\sum_{A\in\Phi}|\det(I - A_*F_*)| = \frac{1}{|\Phi|}\sum_{A\in\Phi}|\det(I - A_*D_*)|,$$

$$R(f) = R(\phi) = \frac{1}{|\Phi|}\sum_{A\in\Phi}\sigma\left(\det(A_* - F_*)\right) = \frac{1}{|\Phi|}\sum_{A\in\Phi}\sigma\left(\det(A_* - D_*)\right)$$

where $\sigma : \mathbb{R} \to \mathbb{R}\cup\{\infty\}$ is defined by $\sigma(0) = \infty$ and $\sigma(x) = |x|$ for all $x \neq 0$.

We can choose a linear basis of $\mathfrak{G}$ so that $\rho(\Phi) = \Phi_* \subset \mathrm{Aut}(\mathfrak{G})$ can be expressed as diagonal block matrices

$$\begin{bmatrix} \Phi_1 & 0 \\ 0 & \Phi_2 \end{bmatrix} \subset \mathrm{GL}(n_1, \mathbb{R}) \chi \mathrm{GL}(n_2, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$$

and D_* can be written in block triangular form

$$\begin{bmatrix} D_1 & * \\ 0 & D_2 \end{bmatrix}$$

where D_1 and D_2 have eigenvalues of modulus ≤ 1 and > 1, respectively. We can assume $\Phi = \Phi_1 \chi \Phi_2$. Every element $\alpha \in \Pi$ is of the form $(a, A) \in G \rtimes \mathrm{Aut}(G)$ and α is mapped to $A = (A_1, A_2)$.

We define

$$\Pi_+ = \{\alpha \in \Pi \mid \det A_2 = 1\}.$$

Then Π_+ is a subgroup of Π of index at most 2. If $[\Pi : \Pi_+] = 2$, then Π_+ is also an almost Bieberbach group and the corresponding infra-nilmanifold $M_+ = \Pi_+ \backslash G$ is a double covering of $M = \Pi \backslash G$; the map f lifts to a map $f_+ : M_+ \to M_+$ which has the same affine homotopy lift (d, D) as f. If D_* has no eigenvalues of modulus > 1, then for any $A \in \Phi$, $A = A_1$ and in this case we take $\Pi_+ = \Pi$.

Theorem (F.- Jong Bum Lee, 2013)

Let f be a continuous map on an infra-nilmanifold with an affine homotopy lift (d, D). Assume $N(f^n) = |L(f^n)|$ for all $n > 0$ and none of the eigenvalues of D_* is a root of unity. Then the Nielsen zeta function $N_f(z)$ is a rational function and is equal to

$$N_f(z) = L_f((-1)^q z)^{(-1)^r}$$

where q is the number of real eigenvalues of D_* which are < -1 and r is the number of real eigenvalues of D_* of modulus > 1. When the Reidemeister zeta function $R_f(z)$ is defined, we have

$$R_f(z) = R_\phi(z) = N_f(z).$$

The class of infra-solvmanifolds of type $\mathbb{R}$ contains and shares a lot of properties of the class of infra-nilmanifolds such as the averaging formula for Nielsen numbers. Therefore, the statement about $N_f(z)$ can be generalized directly to the class of infra-solvmanifolds of type $\mathbb{R}$.

Let S be a connected and simply connected solvable Lie group. A discrete subgroup Γ of S is a *lattice* of S if $\Gamma\backslash S$ is compact, and in this case, we say that the quotient space $\Gamma\backslash S$ is a *special* solvmanifold. Let $\Pi \subset \mathrm{Aff}(S)$ be a torsion-free finite extension of the lattice $\Gamma = \Pi \cap S$ of S. That is, Π fits the short exact sequence

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S & \longrightarrow & \mathrm{Aff}(S) & \longrightarrow & \mathrm{Aut}(S) & \longrightarrow & 1 \\ & & \uparrow & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \Gamma & \longrightarrow & \Pi & \longrightarrow & \Pi/\Gamma & \longrightarrow & 1 \end{array}$$

Then Π acts freely on S and the manifold $\Pi\backslash S$ is called an *infra-solvmanifold*. The finite group $\Phi = \Pi/\Gamma$ is the *holonomy group* of Π or $\Pi\backslash S$. It sits naturally in $\mathrm{Aut}(S)$. Thus every infra-solvmanifold $\Pi\backslash S$ is finitely covered by the special solvmanifold $\Gamma\backslash S$. An infra-solvmanifold $M = \Pi\backslash S$ is of type $\mathbb{R}$ if S is of *type* $\mathbb{R}$ or *completely solvable*, i.e., if $\mathrm{ad}\, X : \mathfrak{S} \to \mathfrak{S}$ has only real eigenvalues for all X in the Lie algebra $\mathfrak{S}$ of S.

Recall that a connected solvable Lie group S contains a sequence of closed subgroups

$$1 = N_1 \subset \cdots \subset N_k = S$$

such that N_i is normal in N_{i+1} and $N_{i+1}/N_i \cong \mathbb{R}$ or $N_{i+1}/N_i \cong S^1$. If the groups $N_1, \cdots, N_k$ are normal in S, the group S is called *supersolvable*. The supersolvable Lie groups are the Lie groups of type $\mathbb{R}$.

Lemma (Wilking-2000)

For a connected Lie group S, the following are equivalent:

(1) *S is supersolvable.*

(2) *All elements of $\mathrm{Ad}(S)$ have only positive eigenvalues.*

(3) *S is of type $\mathbb{R}$.*

We shall assume that $f: M \to M$ is a continuous map on an infra-solvmanifold $M = \Pi \backslash S$ of type $\mathbb{R}$ with holonomy group Φ. Then f has an affine homotopy lift $(d, D): S \to S$, and so f^k has an affine homotopy lift $(d, D)^k = (d', D^k)$ where $d' = dD(d) \cdots D^{k-1}(d)$.

Theorem (K. Dekimpe - G.J. Dugardein; F. -J.B. Lee, 2013)

Let f be a continuous map on an infra-solvmanifold $\Pi\backslash S$ of type $\mathbb{R}$ with an affine homotopy lift (d, D). Then the Reidemeister zeta function, whenever it is defined, is a rational function and is equal to

$$R_f(z) = N_f(z) = \begin{cases} L_f((-1)^n z)^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ \left(\frac{L_{f_+}((-1)^n z)}{L_f((-1)^n z)}\right)^{(-1)^{p+n}} & \text{when } \Pi \neq \Pi_+, \end{cases}$$

Recall the following

Theorem (Bourbaki, Algebra)

Let σ be a Lie algebra automorphism. If none of the eigenvalues of σ is a root of unity, then the Lie algebra must be nilpotent.

Theorem

If the Reidemeister zeta function $R_f(z)$ is defined for a homeomorphism f on an infra-solvmanifold M of type $\mathbb{R}$, then M is an infra-nilmanifold.

Proof. Let f be a homeomorphism on an infra-solvmanifold $M=\Pi\backslash S$ of type $\mathbb{R}$. We may assume that f has an affine map as a homotopy lift. There is a special solvmanifold $N = \Lambda\backslash S$ which covers M finitely and on which f has a lift $\bar{f}$, which is induced by a Lie group automorphism D on the solvable Lie group S. We have an averaging formula for Reidemeister numbers:

$$R(f^n) = \frac{1}{[\Pi : \Lambda]} \sum_{\bar{\alpha}\in\Pi/\Lambda} R(\bar{\alpha}\bar{f}^n).$$

Assume now that f defines the Reidemeister zeta function. Then $R(f^n) < \infty$ for all $n > 0$. The above averaging formula implies that $R(\bar{f}^n) < \infty$ for all n. We have

$$R(\bar{f}^n) = N(\bar{f}^n) = |L(\bar{f}^n)| > 0.$$

Since $L(\bar{f}^n) = \det(I - D_*^n) \neq 0$ for all $n > 0$, this would imply that the differential D_* of D has no roots of unity. By Borel theorem, S must be nilpotent.

Results obtained for continuous maps motivate the following

Conjecture

Reidemeister zeta function is a rational function for endomorphisms of polycyclic-by-finite groups.

Functional Equation

To write down a functional equation for the Reidemeister and the Nielsen zeta function, we recall the following functional equation for the Lefschetz zeta function:

Lemma (Deligne,Fried)

Let M be a closed orientable manifold of dimension m and let $f : M \to M$ be a continuous map of degree d. Then

$$L_f\left(\frac{\alpha}{dz}\right) = \epsilon\,(-\alpha dz)^{(-1)^m \chi(M)}\, L_f(\alpha z)^{(-1)^m}$$

where $\alpha = \pm 1$ and $\epsilon \in \mathbb{C}$ is a non-zero constant such that if $|d| = 1$ then $\epsilon = \pm 1$.

Proof.

In the Lefschetz zeta function formula, we may replace f_* by $f^*: H^*(M;\mathbb{Q}) \to H^*(M;\mathbb{Q})$. Let $\beta_k = \dim H_k(M;\mathbb{Q})$ be the kth Betti number of M. Let $\lambda_{k,j}$ be the (complex and distinct) eigenvalues of $f_{*k}: H_k(M;\mathbb{Q}) \to H_k(M;\mathbb{Q})$. Via the natural non-singular pairing in the cohomology $H^k(M;\mathbb{Q}) \otimes H^{m-k}(M;\mathbb{Q}) \to \mathbb{Q}$, the operators f^*_{m-k} and $d(f^*_k)$ are adjoint to each other. Hence since $\lambda_{k,j}$ is an eigenvalue of f^*_k, $\mu_{\ell,j} = d/\lambda_{k,j}$ is an eigenvalue of $f^*_{m-k} = f^*_\ell$. Furthermore, $\beta_k = \beta_{m-k} = \beta_\ell$.

Functional Equation

Consequently, we have

$$L_f\left(\frac{\alpha}{dz}\right) = \prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(1-\lambda_{k,j}\frac{\alpha}{dz}\right)^{(-1)^{k+1}}$$

$$= \prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(1-\frac{d}{\lambda_{k,j}}\alpha z\right)^{(-1)^{k+1}}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{k}}$$

$$= \prod_{\ell=0}^{m}\prod_{j=1}^{\beta_{m-\ell}}(1-\mu_{\ell,j}\alpha z)^{(-1)^{m-\ell+1}}\prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{m-\ell}}$$

$$= \left(\prod_{\ell=0}^{m}\prod_{j=1}^{\beta_\ell}(1-\mu_{\ell,j}\alpha z)^{(-1)^{\ell+1}}\prod_{k=0}^{m}\prod_{j=1}^{\beta_k}\left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{\ell}}\right)^{(-1)^m} =$$

$$= L_f(\alpha z)^{(-1)^m} \cdot (-\alpha dz)^{\sum_{\ell=0}^{m}(-1)^\ell \beta_\ell} \cdot \prod_{k=0}^{m} \prod_{j=1}^{\beta_k} \lambda_{k,j}^{(-1)^{k+1}} =$$

$$= L_f(\alpha z)^{(-1)^m} \epsilon(-\alpha dz)^{(-1)^m \chi(M)}.$$

Here,

$$\epsilon = \prod_{k=0}^{m} \prod_{j=1}^{\beta_k} \lambda_{k,j}^{(-1)^{k+1}} = \pm \prod_{k=0}^{m} \det(f_k^*).$$

Functional equation for endomorphisms of finitely generated Abelian groups

Functional equation of a convolution Let $X(z)$ and $Y(z)$ be rational functions satisfying the following functional equations

$$X\left(\frac{1}{d_1.z}\right) = K_1 z^{-e_1} X(z)^{f_1}, \quad Y\left(\frac{1}{d_2.z}\right) = K_2 z^{-e_2} Y(z)^{f_2},$$

with $d_i \in C^{\times}$, $e_i \in Z$,$K_i \in C^{\times}$ and $f_i \in \{1, -1\}$. Suppose also that $X(0) = Y(0) = 1$.

Then the rational function $X * Y$ has the following functional equation:

$$(X * Y)\left(\frac{1}{d_1 d_2 z}\right) = K_3 z^{-e_1 e_2}(X * Y)(z)^{f_1 f_2} \tag{23}$$

for some $K_3 \in C^{\times}$.

Lemma (Functional equation for the torsion free part)

Let $\phi : Z^k \to Z^k$ be an endomorphism. The Reidemeister zeta function $R_\phi(z)$ has the following functional equation:

$$R_\phi\left(\frac{1}{dz}\right) = \epsilon_1 . R_\phi(z)^{(-1)^k}. \tag{24}$$

where $d = \det\phi$ and ϵ_1 is a constant in $C^\times$.

Proof Via the natural nonsingular pairing $(\Lambda^i Z^k) \otimes (\Lambda^{k-i} Z^k) \to C$ the operators $\Lambda^{k-i}\phi$ and $d.(\Lambda^i\phi)^{-1}$ are adjoint to each other.
We consider an eigenvalue λ of $\Lambda^i\phi$. By lemma above , this contributes a term $\left((1 - \frac{\lambda\sigma}{dz})^{(-1)^{i+1}}\right)^{(-1)^r}$ to $R_\phi\left(\frac{1}{dz}\right)$. We rewrite

this term as

$$\left(\left(1-\frac{d\sigma z}{\lambda}\right)^{(-1)^{i+1}}\left(\frac{-dz}{\lambda\sigma}\right)^{(-1)^i}\right)^{(-1)^r}$$

and note that $\frac{d}{\lambda}$ is an eigenvalue of $\Lambda^{k-i}\phi$. Multiplying these terms together we obtain,

$$R_\phi\left(\frac{1}{dz}\right)=\left(\prod_{i=1}^{k}\prod_{\lambda^{(i)}\in\text{spec}\,\Lambda^i\phi}\left(\frac{1}{\lambda^{(i)}\sigma}\right)^{(-1)^i}\right)^{(-1)^r}\times R_\phi(z)^{(-1)^k}.$$

The variable z has disappeared because

$$\sum_{i=0}^{k}(-1)^i \dim \Lambda^i Z^k = \sum_{i=0}^{k}(-1)^i \; {}^iC_k = 0.$$

Lemma (Functional equation for the finite part)

Let $\phi : G \to G$ be an endomorphism of a finite, abelian group G. The Reidemeister zeta function $R_\phi(z)$ has the following functional equation:

$$R_\phi\left(\frac{1}{z}\right) = (-1)^p z^q R_\phi(z), \tag{25}$$

where q is the number of periodic elements of ϕ in G and p is the number of periodic orbits of ϕ in G.

Proof This is a simple calculation.

We begin with formula for Reidemeister zeta of finite abelian group.

$$
\begin{aligned}
R_\phi\left(\frac{1}{z}\right) &= \prod_{[\gamma]} \frac{1}{1-z^{-\#[\gamma]}} \\
&= \prod_{[\gamma]} \frac{z^{\#[\gamma]}}{z^{\#[\gamma]}-1} \\
&= \prod_{[\gamma]} \frac{-z^{\#[\gamma]}}{1-z^{\#[\gamma]}} \\
&= \prod_{[\gamma]} -z^{\#[\gamma]} \times \prod_{[\gamma]} \frac{1}{1-z^{\#[\gamma]}} \\
&= \prod -z^{\#[\gamma]} \times R_\phi(z).
\end{aligned}
$$

The statement now follows because $\sum_{[\gamma]} \#[\gamma] = q$.

Theorem (Functional equation, F.- R. Hill, 1994)

Let $\phi : G \to G$ be an endomorphism of a finitely generated abelian group G. If G is finite the functional equation of R_ϕ is described in lemma above. If G is infinite then R_ϕ has the following functional equation:

$$R_\phi\left(\frac{1}{dz}\right) = \epsilon_2 . R_\phi(z)^{(-1)^{\mathrm{rank}\, G}}. \tag{26}$$

where $d = \det\left(\phi^\infty : G^\infty \to G^\infty\right)$ and ϵ_2 is a constant in $C^\times$.

Proof We have $R_\phi(z) = R_\phi^\infty(z) * R_\phi^{finite}(z)$. In the previous two lemmas we have obtained functional equations for the functions

$R_\phi^\infty(z)$ and $R_\phi^{finite}(z)$. Convolution lemma now gives the functional equation for $R_\phi(z)$.

Theorem (F.- Jong Bum Lee, 2013)

Let f be a continuous map on an orientable infra-solvmanifold $M = \Pi \backslash S$ of type $\mathbb{R}$ with an affine homotopy lift (d, D). Then the Reidemeister zeta function, whenever it is defined, and the Nielsen zeta function have the following functional equations:

$$R_f\left(\frac{1}{dz}\right) = \begin{cases} R_f(z)^{(-1)^m} \epsilon^{(-1)^{p+n}} & \text{when } \Pi = \Pi_+; \\ R_f(z)^{(-1)^m} \epsilon^{-1} & \text{when } \Pi \neq \Pi_+ \end{cases}$$

where d is a degree f, $m = \dim M$, ϵ is a constant in $\mathbb{C}^\times$, $\sigma = (-1)^n$, p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1. If $|d| = 1$ then $\epsilon = \pm 1$.

Theorem

Let f be a continuous map on an infra-solvmanifold of type $\mathbb{R}$ induced by an affine map. Then $AM_f(z) = N_f(z)$, i.e., $AM_f(z)$ is a rational function with functional equation.

Reidemeister torsion

Like the Euler characteristic, the Reidemeister torsion is algebraically defined.
Roughly speaking, the Euler characteristic is a graded version of the dimension, extending the dimension from a single vector space to a complex of vector spaces. In a similar way, the Reidemeister torsion is a graded version of the absolute value of the determinant of an isomorphism of vector spaces.

Let $d^i : C^i \to C^{i+1}$ be a cochain complex C^* of finite dimensional vector spaces over $\mathbb{C}$ with $C^i = 0$ for $i < 0$ and large i. If the cohomology $H^i = 0$ for all i we say that C^* is *acyclic*. If one is given positive densities Δ_i on C^i then the Reidemeister torsion $\tau(C^*, \Delta_i) \in (0, \infty)$ for acyclic C^* is defined as follows:

Definition

Consider a chain contraction $\delta^i : C^i \to C^{i-1}$, i.e., a linear map such that $d \circ \delta + \delta \circ d = \mathrm{id}$. Then $d + \delta$ determines a map $(d+\delta)_+ : C^+ := \oplus C^{2i} \to C^- := \oplus C^{2i+1}$ and a map $(d+\delta)_- : C^- \to C^+$. Since the map $(d+\delta)^2 = id + \delta^2$ is unipotent, $(d+\delta)_+$ must be an isomorphism. One defines $\tau(C^*, \Delta_i) := |\det(d+\delta)_+|$.

Reidemeister torsion is defined in the following geometric setting. Suppose K is a finite complex and E is a flat, finite dimensional, complex vector bundle with base K. We recall that a flat vector bundle over K is essentially the same thing as a representation of $\pi_1(K)$ when K is connected. If $p \in K$ is a base point then one may move the fibre at p in a locally constant way around a loop in K. This defines an action of $\pi_1(K)$ on the fibre E_p of E above p. We call this action the holonomy representation $\rho : \pi \to GL(E_p)$.

Conversely, given a representation $\rho : \pi \to GL(V)$ of π on a finite dimensional complex vector space V, one may define a bundle $E = E_\rho = (\tilde{K} \times V)/\pi$. Here $\tilde{K}$ is the universal cover of K, and π acts on $\tilde{K}$ by covering transformations and on V by ρ. The holonomy of E_ρ is ρ, so the two constructions give an equivalence of flat bundles and representations of π.

If K is not connected then it is simpler to work with flat bundles. One then defines the holonomy as a representation of the direct sum of π_1 of the components of K. In this way, the equivalence of flat bundles and representations is recovered.

Suppose now that one has on each fibre of E a positive density which is locally constant on K. In terms of ρ_E this assumption just means $|\det \rho_E| = 1$. Let V denote the fibre of E. Then the cochain complex $C^i(K;E)$ with coefficients in E can be identified with the direct sum of copies of V associated to each i-cell σ of K. The identification is achieved by choosing a basepoint in each component of K and a basepoint from each i-cell. By choosing a flat density on E we obtain a preferred density Δ_i on $C^i(K,E)$. A case of particular interest is when E is an acyclic bundle, meaning that the twisted cohomology of E is zero ($H^i(K;E)=0$). In this case one defines the R-torsion of (K,E) to be $\tau(K;E)=\tau(C^*(K;E),\Delta_i)\in(0,\infty)$. It does not depend on the choice of flat density on E.

The Reidemeister torsion of an acyclic bundle E on K has many nice properties. Suppose that A and B are subcomplexes of K.

Then we have a multiplicative law:

$$\tau(A\cup B;E)\cdot\tau(A\cap B;E)=\tau(A;E)\cdot\tau(B;E) \qquad (27)$$

that is interpreted as follows. If three of the bundles $E|A\cup B,\rangle E|A\cap B,\rangle E|A,\rangle E|B$ are acyclic then so is the fourth and the equation (27) holds.
Another property is the simple homotopy invariance of the Reidemeister torsion. In particular τ is invariant under subdivision. This implies that for a smooth manifold, one can unambiguously define $\tau(K;E)$ to be the torsion of any smooth triangulation of K.
In the case $K=S^1$ is a circle, let A be the holonomy of a generator of the fundamental group $\pi_1(S^1)$. One has that E is acyclic if and

only if $I - A$ is invertible and then

$$\tau(S^1; E) = |\det(I - A)|$$

Note that the choice of generator is irrelevant as $I - A^{-1} = (-A^{-1})(I - A)$ and $|\det(-A^{-1})| = 1$.
These three properties of the Reidemeister torsion are the analogues of the properties of Euler characteristic (cardinality law, homotopy invariance and normalization on a point), but there are differences. Since a point has no acyclic representations ($H^0 \neq 0$) one cannot normalize τ on a point as we do for the Euler characteristic, and so one must use S^1 instead. The multiplicative cardinality law for the Reidemeister torsion can be made additive just by using $\log \tau$, so the difference here is inessential.

More important for some purposes is that the Reidemeister torsion is not an invariant under a general homotopy equivalence: as mentioned earlier this is in fact why it was first invented.
It might be expected that the Reidemeister torsion counts something geometric (like the Euler characteristic). D. Fried showed that it counts the periodic orbits of a flow and the periodic points of a map. We will show that the Reidemeister torsion counts the periodic point classes of a map (fixed point classes of the iterations of the map).

Some further properties of τ describe its behavior under bundles. Let $p : X \to B$ be a simplicial bundle with fiber F where F, B, X are finite complexes and p^{-1} sends subcomplexes of B to subcomplexes of X over the circle S^1. We assume here that E is a flat, complex vector bundle over B . We form its pullback p^*E over X. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in B$ form a flat vector bundle over B, which we denote H^iF. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iF$ is acyclic for all i. Under these conditions D. Fried has shown that the bundle p^*E is acyclic, and

$$\tau(X; p^*E) = \prod_i \tau(B; E \otimes H^iF)^{(-1)^i}.$$

Mapping torus of map f. Let $f : X \to X$ be a homeomorphism of a compact polyhedron X. Let $T_f := (X \times I)/(x, 0) \sim (f(x), 1)$ be the mapping torus of f.
We shall consider the bundle $p : T_f \to S^1$ over the circle S^1. We assume here that E is a flat, complex vector bundle with finite dimensional fibre and base S^1. We form its pullback p^*E over T_f. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in S^1$ form a flat vector bundle over S^1, which we denote H^iF. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle $E \otimes H^iF$ is acyclic for all i. Under these conditions the bundle p^*E is acyclic,

and we have

$$\tau(T_f; p^*E) = \prod_i \tau(S^1; E \otimes H^i F)^{(-1)^i}. \qquad (28)$$

Let g be the preferred generator of the group $\pi_1(S^1)$ and let $A = \rho(g)$ where $\rho : \pi_1(S^1) \to GL(V)$. Then the holonomy around g of the bundle $E \otimes H^i F$ is $A \otimes (f^*)^i$. Since $\tau(S^1; E) = |\det(I - A)|$ it follows from (28) that

$$\tau(T_f; p^*E) = \prod_i |\det(I - A \otimes (f^*)^i)|^{(-1)^i}.$$

We now consider the special case in which E is one-dimensional, so A is just a complex scalar λ of modulus one. Then in terms of the rational function $L_f(z)$ we have :

$$\tau(T_f;p^*E)=\prod_i\mid\det(I-\lambda(f^*)^i)\mid^{(-1)^i}=\mid L_f(\lambda)\mid^{-1} \qquad (29)$$

This means that the special value of the Lefschetz zeta function is given by the Reidemeister torsion of the corresponding mapping torus.

Theorem (F. - Jong Bum Lee, 2013)

Let $f : M \to M$ be a homeomorphism of an infra-nilmanifold M. Assume that $N(f) = |L(f)|$. Then

$$\tau(T_f; p^*E) = \mid L_f(\lambda) \mid^{-1} = \mid N_f(\sigma\lambda) \mid^{(-1)^{r+1}} = \mid R_f(\sigma\lambda) \mid^{(-1)^{r+1}}$$

where $\sigma = (-1)^p$, p is the number of real eigenvalues of F^* in the region $(-\infty, -1)$ and r is the number of real eigenvalues of F^* whose absolute value is greater that 1.

Theorem (F. - Jong Bum Lee, 2013)

Let f be a homeomorphism on an infra-nilmanifold $\Pi\backslash G$ with an affine homotopy lift (d, D). Then

$$|R_f((-1)^n\lambda)^{(-1)^{p+n}}| = |R_\phi((-1)^n\lambda)^{(-1)^{p+n}}| = |N_f((-1)^n\lambda)^{(-1)^{p+n}}|$$
$$= \begin{cases} |L_f(\lambda)| = \tau(T_f; p^*E)^{-1} & \textit{when } \Pi = \Pi_+; \\ |L_{f_+}(\lambda)L_f(\lambda)^{-1}| = \tau(T_f; p^*E)\tau(T_{f_+}; p_+^*E)^{-1} & \textit{when } \Pi \neq \Pi_+, \end{cases}$$

where p is the number of real eigenvalues of D_* which are >1 and n is the number of real eigenvalues of D_* which are <-1.

Theorem (F.- Jong Bum Lee, 2013)

Let f be a homeomorphism on an infra-solvmanifold $\Pi\backslash S$ of type $\mathbb{R}$ with an affine homotopy lift (d, D). Then

$$|N_f((-1)^n\lambda)^{(-1)^{p+n}}| = \begin{cases} |L_f(\lambda)| = \tau(T_f; p^*E)^{-1} & \text{when } \Pi = \Pi_+; \\ |L_{f_+}(\lambda)L_f(\lambda)^{-1}| = \tau(T_f; p^*E)\tau(T_{f_+}; p_+^*E)^{-1} & \text{when } \Pi \neq \Pi_+, \end{cases}$$

where p is the number of real eigenvalues of D_* which are > 1 and n is the number of real eigenvalues of D_* which are < -1.

Gauss congruences

Theorem (F. - Jong Bum Lee, 2013)

Let f be any continuous map on an infra-solvmanifold of type $\mathbb{R}$ such that all $R(f^n)$ are finite. Then we have

$$\sum_{d|n} \mu(d)\ R(f^{n/d}) = \sum_{d|n} \mu(d)\ N(f^{n/d}) \equiv 0 \mod n$$

for all $n > 0$.

Asymptotic behavior of the sequence $\{N(f^k)\}$

The growth rate of a sequence a_n of complex numbers is defined by

$$\mathrm{Growth}(a_n) := \max\left\{1, \limsup_{n\to\infty} |a_n|^{1/n}\right\}.$$

We define the asymptotic Nielsen number and the asymptotic Reidemeister number to be the growth rate $N^\infty(f) := \mathrm{Growth}(N(f^n))$ and $R^\infty(f) := \mathrm{Growth}(R(f^n))$ correspondingly. These asymptotic numbers are homotopy type invariants.

We denote by $sp(A)$ the spectral radius of the matrix or the operator A, $sp(A) = \lim_n \sqrt[n]{\|A^n\|}$ which coincide with the largest modulus of an eigenvalue of A. We denote by $\bigwedge F_* := \bigoplus_{\ell=0}^{m} \bigwedge^{\ell} F_*$ a linear operator induced in the exterior algebra $\bigwedge^* \mathbb{R}^m := \bigoplus_{\ell=0}^{m} \bigwedge^{\ell} \mathbb{R}^m$ of $\mathfrak{G}$ considered as the linear space $\mathbb{R}^m$.

Theorem

Let f be a continuous map on an infra-solvmanifold of type $\mathbb{R}$ with an affine homotopy lift (d, D). Then we have

$$N^{\infty}(f) = sp(\bigwedge D_*)$$

provided that 1 is not in the spectrum of D_*.

Theorem (F.- J.B. Lee, 2014)

Let f be a continuous map on an infra-solvmanifold M of type $\mathbb{R}$ with an affine homotopy lift (d, D). If 1 is not in the spectrum of D_*, then

$$h(f) \geq \log(sp(f)).$$

If $\bar{f}$ is the map on M induced by the affine map (d, D), then

$$h(f) \geq h(\bar{f}) \geq \log sp(f),$$
$$h(\bar{f}) = \log sp(\bigwedge D_*) = \log N^\infty(\bar{f}) = \log N^\infty(f).$$

Hence $\bar{f}$ minimizes the entropy in the homotopy class of f.

We denote by R the radius of convergence of the zeta functions $N_f(z)$ or $R_f(z)$.

Theorem (F.- J.B. Lee, 2014)

Let f be a continuous map on an infra-nilmanifold with an affine homotopy lift (d, D). Then the Nielsen zeta function $N_f(z)$ and the Reidemeister zeta function $R_f(z)$, whenever it is defined, have the same positive radius of convergence R which admits following estimation

$$R \geq \exp(-h) > 0,$$

where $h = \inf\{h(g) \mid g \simeq f\}$.
If 1 is not in the spectrum of D_*, the radius R of convergence of $R_f(z)$ is

$$R = \frac{1}{N^\infty(f)} = \frac{1}{\exp h(\bar{f})} = \frac{1}{sp(\bigwedge D_*)}.$$

Essential periodic orbits

$N_f(z)$ is a rational function with coefficients in $\mathbb{Q}$.
On the other hand, since $N_f(0) = 1$ by definition, $z = 0$ is not a zero nor a pole of the rational function $N_f(z)$. Thus we can write

$$N_f(z) = \frac{u(z)}{v(z)} = \frac{\prod_i (1-\beta_i z)}{\prod_j (1-\gamma_j z)} = \prod_{i=1}^{r} (1-\lambda_i z)^{-\rho_i}$$

with all λ_i distinct nonzero algebraic integers and ρ_i nonzero integers.This induces

$$N(f^k) = \sum_{i=1}^{r(f)} \rho_i \lambda_i^k. \qquad (N1)$$

We define $\lambda(f) := \max\{|\lambda_i| \mid i = 1, \cdots, r(f)\}$. If $r(f) = 0$, i.e., if $N(f^k) = 0$ for all $k > 0$, then $N_f(z) \equiv 1$ and $1/R = 0$. In this case, we define customarily $\lambda(f) = 0$.

In this section, we study the asymptotic behavior of the Nielsen numbers of iterates of maps on infra-solvmanifolds of type $\mathbb{R}$.

Theorem (F.- J.B. Lee, 2014)

For a map f of an infra-solvmanifold of type $\mathbb{R}$, one of the following three possibilities holds:

(1) $\lambda(f) = 0$, *which occurs if and only if* $N_f(z) \equiv 1$.

(2) *The sequence* $\{N(f^k)/\lambda(f)^k\}$ *has the same limit points as a periodic sequence* $\{\sum_j \alpha_j \epsilon_j^k\}$ *where* $\alpha_j \in \mathbb{Z}, \epsilon_j \in \mathbb{C}$ *and* $\epsilon_j^q = 1$ *for some* $q > 0$.

(3) *The set of limit points of the sequence* $\{N(f^k)/\lambda(f)^k\}$ *contains an interval.*

In this section, we shall give a linear lower bound for the number of *essential periodic orbits* of maps on infra-solvmanifolds of type $\mathbb{R}$, which sharpens well-known results of Shub and Sullivan for periodic points and of Babenko and Bogatyĭ for periodic orbits.

We denote by $\mathcal{O}(f,k)$ the set of all essential periodic orbits of f with length $\leq k$. Thus $\mathcal{O}(f,k)=\{\langle\mathbb{F}\rangle\mid$ $\mathbb{F}$ is a essential fixed point class of f^m with $m\leq k\}$.

Theorem (F. - Jong Bum Lee, 2014)

Let f be a map on an infra-solvmanifold of type $\mathbb{R}$. Suppose that the sequence $N(f^k)$ is unbounded. Then there exists a natural number N_0 such that

$$k \geq N_0 \Longrightarrow \#\mathcal{O}(f,k) \geq \frac{k-N_0}{r(f)}.$$

Using Reidemeister/Nielsen zeta function we can also study the set of (homotopy) minimal periods of maps f on infra-nilmanifolds and on infra-solvmanifolds of type $\mathbb{R}$.

R_∞ groups: examples

Definition

A group G is an R_∞-*group* if for any automorphism ϕ the number $R(\phi)$ is infinite.

In contrast with the case of automorphisms, we have a plenty of classes of groups and endomorphisms for which the Reidemeister zeta function is well defined, also among groups with R_∞ property. The problem of determining which classes of discrete infinite groups have the R_∞ property is an area of active research initiated in 1994.

Later, it was shown by various authors that the following groups have the R_∞-property:

- non-elementary Gromov hyperbolic groups (F., Levitt-Lustig); relatively hyperbolic groups (F.);
- Baumslag-Solitar groups $BS(m, n)$ except for $BS(1, 1)$ (F.–D.Gonçalves), generalized Baumslag-Solitar groups, that is, finitely generated groups which act on a tree with all edge and vertex stabilizers infinite cyclic (Levitt); the solvable generalization Γ of $BS(1, n)$ given by the short exact sequence $1 \to \mathbb{Z}[\frac{1}{n}] \to \Gamma \to \mathbb{Z}^k \to 1$, as well as any group quasi-isometric to Γ (Taback–Wong);

R_∞ groups: examples

- a wide class of saturated weakly branch groups (including the Grigorchuk group and the Gupta-Sidki-Sushchanskyy groups) (F. - Yu.Leonov - E.T.), Thompson's group F (Bleak – F. – Gonçalves); generalized Thompson's groups $F_{n,\,0}$ and their finite direct products (Gonçalves – Kochloukova);

R_∞ groups: examples

- symplectic groups $Sp(2n,\mathbb{Z})$, the mapping class groups Mod_S of a compact oriented surface S with genus g and p boundary components, $3g+p-4>0$, and the full braid groups $B_n(S)$ on $n>3$ strings of a compact surface S in the cases where S is either the compact disk D, or the sphere S^2 (Damani – F. – Gonçalves);

R_∞ groups

- extensions of $SL(n,\mathbb{Z})$, $PSL(n,\mathbb{Z})$, $GL(n,\mathbb{Z})$, $PGL(n,\mathbb{Z})$, $Sp(2n,\mathbb{Z})$, $PSp(2n,\mathbb{Z})$, $n>1$, by a countable abelian group, and normal subgroup of $SL(n,\mathbb{Z})$, $n>2$, not contained in the centre (Mubeena – Sankaran);
- $GL(n,K)$ and $SL(n,K)$ if $n>2$ and K is an infinite integral domain with trivial group of automorphisms, or K is an integral domain, which has a zero characteristic and for which $\mathrm{Aut}(K)$ is torsion (Nasybullov);
- irreducible lattice in a connected semi simple Lie group G with finite center and real rank at least 2 (Mubeena-Sankaran);

Dynamic representation theory zeta functions

Suppose, ϕ is an endomorphism of a discrete group Γ. Generally the correspondence $\widehat{\phi} : \rho \mapsto \rho \circ \phi$ does not define a dynamical system (an action of the semigroup of positive integers) on the unitary dual $\widehat{\Gamma}$ or its finite-dimensional $\widehat{\Gamma}_f$ part, or finite $\widehat{\Gamma}_{ff}$ part, because in contrast with the authomorphism case, the representation $\rho \circ \phi$ may be reducible. Here the *unitary dual* is the space of equivalence classes of unitary irreducible representations of Γ, equipped with the *hull-kernel* topology, $\widehat{\Gamma}_f$ is its subspase formed by finite-dimensional representations, and $\widehat{\Gamma}_{ff}$ is formed by *finite* representations. Nevertheless we can consider representations ρ such that $\rho \sim \rho \circ \phi$.

Definition

A *representation theory Reidemeister number* $RT(\phi)$ is the number of all $[\rho] \in \widehat{\Gamma}$ such that $\rho \sim \rho \circ \phi$. Taking $[\rho] \in \widehat{\Gamma}_f$ (respectively $[\rho] \in \widehat{\Gamma}_{ff}$) we obtain $RT^f(\phi)$ (respectively $RT^{ff}(\phi)$). Evidently $RT(\phi) \geq RT^f(\phi) \geq RT^{ff}(\phi)$.

Definition

If these numbers are finite for all powers of ϕ, we define the corresponding *dynamic representation theory zeta functions*

$$RT_\phi(z) := \exp\left(\sum_{n=1}^\infty \frac{RT(\phi^n)}{n} z^n\right), \quad RT_\phi^f(z) := \exp\left(\sum_{n=1}^\infty \frac{RT^f(\phi^n)}{n} z^n\right),$$

$$RT_\phi^{ff}(z) := \exp\left(\sum_{n=1}^\infty \frac{RT^{ff}(\phi^n)}{n} z^n\right).$$

The importance of these numbers is justified by the following dynamical interpretation. The following "dynamical part" of the dual space, where $\widehat{\phi}$ and all its iterations $\widehat{\phi}^n$ define a dynamical system, was defined [F.-Troitsky].

Definition

A class $[\rho]$ is called a $\widehat{\phi}$-**f**-point, if $\rho \sim \rho \circ \phi$ (so, these are the points under consideration in above definitions).

Definition

An element $[\rho] \in \widehat{\Gamma}$ (respectively, in $\widehat{\Gamma}_f$ or $\widehat{\Gamma}_{ff}$) is called ϕ-*irreducible* if $\rho \circ \phi^n$ is irreducible for any $n = 0, 1, 2, \ldots$. Denote the corresponding subspaces of $\widehat{\Gamma}$ (resp., $\widehat{\Gamma}_f$ or $\widehat{\Gamma}_{ff}$) by $\widehat{\Gamma}^\phi$ (resp., $\widehat{\Gamma}^\phi_f$ or $\widehat{\Gamma}^\phi_{ff}$).

Lemma

Suppose, the representations ρ and $\rho \circ \phi^n$ are equivalent for some $n \geq 1$. Then $[\rho] \in \widehat{\Gamma}^{\phi}$.

Corollary

Generally, there is no dynamical system defined by $\widehat{\phi}$ on $\widehat{\Gamma}$ (resp., $\widehat{\Gamma}_f$, or $\widehat{\Gamma}_{ff}$). We have only the well-defined notion of a $\widehat{\phi}^n$-**f**-point. A well-defined dynamical system exists on $\widehat{\Gamma}^{\phi}$ (resp, $\widehat{\Gamma}^{\phi}_f$, or $\widehat{\Gamma}^{\phi}_{ff}$). Its n-periodic points are exactly $\widehat{\phi}^n$-**f**-points.

The following statement evidently follows from the definitions.

Proposition

Suppose, $\phi : \Gamma \to \Gamma$ is an endomorphism and $R(\phi) < \infty$. If TBFT (resp., TBFT$_f$) is true for Γ and ϕ, then $R(\phi) = RT(\phi)$ (resp, $R(\phi) = RT^f(\phi) = RT^{ff}(\phi)$).
If the suppositions keep for ϕ^n, for any n, then $R_\phi(z) = RT_\phi(z)$ (resp., $R_\phi(z) = RT^f_\phi(z) = RT^{ff}_\phi(z)$).

Theorem (F. - E.Troitsky - M. Zietek, 2018)

Suppose, TBFT (resp., $TBFT_f$) is true for Γ and ϕ^n; and $R(\phi^n) < \infty$ for any n. If $R_\phi(z)$ is rational, then $RT_\phi(z)$ (resp., $RT^f_\phi(z) = RT^{ff}_\phi(z)$) is rational. In particular, $RT^f_\phi(z) = RT^{ff}_\phi(z)$ is rational in the following cases:

1. *Γ is a finitely generated abelian group;*
2. *Γ is a finitely generated torsion free nilpotent group;*
3. *Γ is a crystallographic group with diagonal holonomy $\mathbb{Z}_2$ and ϕ is an automorphism.*