Model theory of pseudo-finite fields – SoSe 2021

We will mostly follow the 2005-notes by Chatzidakis [1]; all Sections, Theorem numbers, etc. refer to that.

The dates below might still be shifted a bit.

1 The Theory PSF

20.4. – Pablo

- Introduce the notion of pseudo-finite field (Defn. (6.1)), using Theorem (6.18) to motivate it.

  In particular, introduce the notions appearing in Defn. (6.1). Be careful that what's called "variety" there is maybe not exactly what one would expect (often, this is rather called "absolutely irreducible variety"): State the definition from the beginning of Sect. 5.

- Prove that pseudo-finite fields are first-order axiomatizable (Lemma (6.2)). Choose how much details to give about P2 and P3; details for P3 are in Section 5.

- Prove one direction of Theorem (6.18), namely the one given in Theorem (6.4). Use the Lang-Weil estimates (6.5) as a black box.

2 The embedding Lemma: statement

27.4. – Nadja

- State the embedding Lemma (6.8).

- Introuce the notion of regular field extensions and explain some things around it (4.5)–(4.7). (The part in (4.6) about the algebraic set V might need also speaking about $(4.n)$ for some $n \leq 4$)

- Explain the meaning and the role of the continuity of the isomorphism Φ in the embedding lemma. (See Section 3 for an intro to infinte Galois theory; but don't do all of Section 3.)

- Show how the embedding Lemma implies Corollary (6.10).

- If possible, give a very sketchy/vague motivation for the embedding Lemma, by stating how (6.10) is used in the proof of (6.11).

3 The embedding Lemma: proof

4.5. – Tristan

- Prove the embedding Lemma (6.8)

4 Completions of PSF

11.5. – David

- The central (motivating) goal for this talk is Theorem (6.13), which can be considered as characterizing all completions of the theory PSF. Also state the version given in Theorem (6.14).
- On the way to that theorem, prove Proposition (6.11) and also Corollary (6.12) (which characterizes elementary extensions).

5 Quantifier elimination

18.5. – Florian S.

- Prove Theorem (6.15), which states that every formula is (modulo PSF), equivalent to a boolean combintation of certain simple existential formulas.
- Also deduce model completeness (6.16) of Psf_c (and recall what model completeness means)
- Prove Theorem (6.17) – a refined version of (6.15)

6 PSF vs. finite fields

1.6. – Leon

- Prove the missing direction of Theorem (6.18), i.e., that every pseudo-finite field is a model of the theory of finite fields.

7 Dimension and measure: statement and applications

8.6. – Hamed

- State Theorem (7.1)
- State (some of the) Remarks (7.2)
- Present (some of the) Applications (7.3)
- Introduce the notions of dimension and measure in a pseudo-finite field (7.5) and prove the proposition in (7.5).

8 Dimension and measure: proof

15.6. – Immi

- Sketch the proof of Theorem (7.1), either following (7.4), or also by looking into the original article [2].

9 Galois covers

22.6. – Florian F.

- Explain how definable sets in pseudo-finite fields can be defined in terms of Galois covers, e.g. following the book Field Arithmetic by Fried and Jarden. (Introduce all notions necessary for that.)
- Give examples.
- Relate this to the dimension and measure. (How does one obtain the dimension and measure from a Galois cover?)

10 Bonus talk: tba

20.7. – Simone Ramello

References

[1] Z. Chatzidakis, *Notes on the model theory of finite and pseudo-finite fields v2005*. `http://www.logique.jussieu.fr/~zoe/papiers/Madrid05.ps`.

[2] Z. Chatzidakis, L. van den Dries, and A. Macintyre, *Definable sets over finite fields*, J. Reine Angew. Math., 427 (1992), pp. 107–135.