Bsple für K-Algebren:

- $K \subset L$

  $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q} \subset \mathbb{C}$ $\qquad$ $\mathbb{Q}$-Alg

  $\mathbb{R} \subset \mathbb{C} \subset \mathbb{C}(x) \subset \mathbb{C}(x,y)$ $\qquad$ $\mathbb{R}$-Alg

  $K[x] \subset K(x)$ = Körper der rationalen Fkt über K

  $$K(x) = \left\{ \frac{f}{g} \;\middle|\; f,g \in K[x],\ g \neq 0 \right\}$$

  analog zu $\mathbb{Z} \subset \mathbb{Q}$

  $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} \mid a,b \in \mathbb{Q} \}$

- $K[x_1, \dots, x_n]/\sim$
- K^n
- $K^{n \times m}$
- $T(V)/\sim$

für geeignetes $\sim$: erhalte VR/Alg. wo Differentialformen leben "dx"

$\rightsquigarrow$ Äußere Produkte

- Produkte von Algebren

$$T(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} = K \oplus \underset{\mathbb{R}^2}{V} \oplus \underset{\mathbb{R}^2\otimes\mathbb{R}^2}{(V\otimes V)} \oplus (V\otimes V\otimes V) \oplus \ldots$$

$$(\underset{1}{-}, \underset{v_i}{-}, \underset{v_i\otimes v_j}{-}, \underset{\ldots}{-}, -)$$

$$T(\mathbb{R}^2) \ni \left(\pi, \begin{pmatrix}1\\0\end{pmatrix}, 2\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} - 1\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}, 0, 0, \ldots\right)$$

$$\mathbb{R}^2 = \mathbb{R}\oplus\mathbb{R}$$
$$(-,-)$$

$$\lambda\otimes v = \lambda\cdot v$$

$$K\otimes V \cong V$$

$$V \subseteq T(V)$$
$$v \mapsto (0, v, 0, 0, \ldots)$$

$$f_A\otimes f_B(e_i\otimes e_j) = f_A(e_i)\otimes f_B(e_j)$$
$$\shortparallel$$
$$\ldots$$

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triv. Bsp.

$K \overset{i}{\subset} K[x]$ ist Homo. von K-Algebren

Wissen: i ist linear

Zu überprüfen:

$i(\lambda \cdot \mu) = i(\lambda) \cdot i(\mu)$

$i(1) = 1$

Nutze Bem 8.6.5:

Nehme Basis von K: 1

$$\begin{array}{ccc} i(1 \cdot 1) & \overset{?}{=} & \underbrace{\underbrace{i(1)}_{=1} \underbrace{i(1)}_{=1}}_{=1} \\ \shortparallel & & \\ i(1) & & \\ \shortparallel & & \\ 1 & = & \end{array}$$

Wirkliches Bsp.:

$$\mathbb{R}[x,y] \xrightarrow{\varphi} \mathbb{R}^2$$
$$x \mapsto (2,-1)$$
$$y \mapsto (1,0)$$

univ. Eig.

$\varphi(\sum a_{ij} x^i y^j) \mapsto \sum a_{ij} (2,-1)^i (1,0)^j$

Basis von $\mathbb{R}[x,y]$: $x^i y^j$, $i,j \in \mathbb{N}$

$$\varphi(x^i y^j \cdot x^m y^n) = \varphi(x^{i+m} y^{j+n})$$
$$= (2,-1)^{i+m} (1,0)^{j+n}$$

$$\varphi(x^i y^j) \cdot \varphi(x^m y^n) = (2,-1)^i (1,0)^j \cdot (2,-1)^m (1,0)^n$$

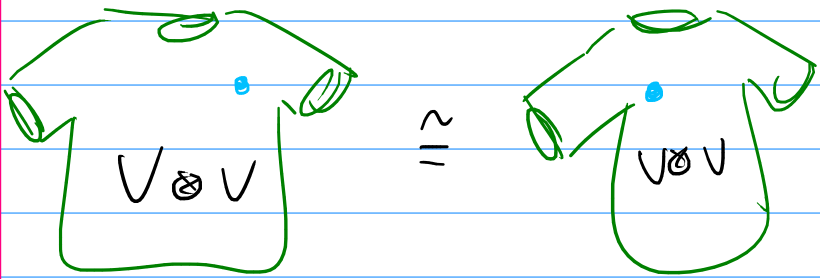
$$V \otimes W \cong W \otimes V$$

$$v \otimes w \underbrace{\mapsto}_{V} w \otimes v$$

$$V = W$$

$$V \otimes V \cong V \otimes V$$

$$v_1 \otimes v_2 \mathrel{\hat{=}} v_2 \otimes v_1$$



$$v \otimes 1 \mathrel{\hat{=}} v \qquad V \otimes \mathbb{K} \cong V$$

$T(\mathbb{R}^2)$

$$(v_n)_n = \Big(\underset{0}{-1}, \underset{1}{\binom{1}{0}}, \underset{2}{2\binom{2}{1} \otimes \binom{0}{1} + 3\binom{0}{1} \otimes \binom{1}{0}}, \underset{3}{0}, \underset{4}{0}, \_\_\Big)$$

$$(v'_n)_n = \Big(\underset{0}{2}, \underset{1}{0}, \underset{2}{0}, \underset{3}{\binom{1}{0} \otimes \binom{0}{1} \otimes \binom{1}{1}}, \underset{4}{0}, \underset{5}{0}, \_\_\Big) \quad \ldots$$

$$(w_n)_n = (v_n)_n \cdot (v_n')_n$$

$$w_0 = v_0 \otimes v_0' = -1 \otimes 2 = -2$$

$$0 = 0+0$$

$$w_1 = v_0 \otimes v_1' + v_1 \otimes v_0' = \underbrace{-1 \otimes 0}_{=0} + \underbrace{\begin{pmatrix}1\\0\end{pmatrix} \otimes 2}_{=\begin{pmatrix}2\\0\end{pmatrix}} = \begin{pmatrix}2\\0\end{pmatrix} \in \mathbb{R}^2$$

$$1 = 0+1 = 1+0$$

$$w_2 = v_0 \otimes \underbrace{v_2'}_{=0} + v_1 \otimes \underbrace{v_1'}_{=0} + v_2 \otimes v_0'$$

$$2 = 0+2 = 1+1 = 2+0$$

$$\| \quad \left(2\begin{pmatrix}2\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \otimes 2$$

$$= 4\begin{pmatrix}2\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + 6\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}$$

$\vdots$

$$w_5 = v_0 \otimes \underbrace{v_5'}_{=0} + v_1 \otimes \underbrace{v_4'}_{=0} + v_2 \otimes v_3' + \underbrace{v_3}_{=0} \otimes v_2' + \underbrace{v_4}_{=0} \otimes v_1' + \underbrace{v_5}_{=0} \otimes v_0'$$

$$5 = 0+5 = 1+4 = 2+3 = 3+2 = 4+1 = 5+0$$

$$= v_2 \otimes v_3' = \left(2\begin{pmatrix}2\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} + 3\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}\right) \otimes \left(\begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}\right)$$

$$= 2\begin{pmatrix}2\\1\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$

$$+ 3\begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}0\\1\end{pmatrix} \otimes \begin{pmatrix}1\\1\end{pmatrix}$$

$$T(V) = \boxed{\bigoplus_{n \geq 0} V^{\otimes n}}$$

$$\begin{array}{ccccc} V^{\otimes 2} & & V^{\otimes 3} & & \\ \cup & & \cup & & \\ v & \otimes & w & \in & V^{\otimes 5} \end{array}$$

Will Hom $\mathbb{Q}[x,y,z] \xrightarrow{\varphi} \mathbb{Q}(\sqrt{2})[\omega]$ von $\mathbb{Q}$-Alg.

8.6.10: $\varphi \mathrel{\hat{=}} \{x,y,z\} \longrightarrow \mathbb{Q}(\sqrt{2})[\omega]$

$$\left\{ \begin{array}{lll} x & \mapsto & 1+\sqrt{2} \\ y & \mapsto & \omega \\ z & \mapsto & \sqrt{2}\omega \end{array} \right.$$

$$\sum a_{i,j,k} x^i y^j z^k \overset{\varphi}{\mapsto} \sum a_{i,j,k} (1+\sqrt{2})^i \omega^j (\sqrt{2}\omega)^z$$

$$\begin{array}{rcl} \mathbb{Q}[x,y,z] & \longrightarrow & \mathbb{Q}(\sqrt{2})[\omega] \\ x & \mapsto & 1+\sqrt{2} \\ y & \mapsto & \omega \\ z & \mapsto & \sqrt{2}\omega \end{array}$$

$$\overset{\text{UVR}}{U} \subseteq \overset{\text{Alg.}}{A}$$

Will A/U

$\rightsquigarrow$ ist nicht wohldef. (als Algebra (i.A))

U muss erfüllen:

$$\underset{\underset{u \cdot a}{\rotatebox{90}{$\in$}}}{a \cdot u} \in U, \quad a \in A, \quad u \in U$$

$\rightsquigarrow$ A/U wohldef.

$$A \supset (f) = \{af \mid a \in A\}$$

$$K[x]/\underbrace{(x)}_{\text{von } x \text{ erzeugtes Ideal}} \cong K$$

$$(x) = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; a_0 = 0 \right\}$$

$$\mathbb{Q} \subset \underbrace{\mathbb{Q}[x]/(x^2-2)}_{= A \,(\cong \mathbb{Q}(\sqrt{2}))}$$

$x^2 - 2 = 0$

$\Leftrightarrow x^2 = 2$

$\pm \bar{x} \in A$ sind Wurzeln von 2

$$f \mapsto f(0)$$

$$K[x] \xrightarrow{\varphi} K$$

$$\ker(\varphi) = (x)$$

Isom. satz $\quad K[x]/(x) \cong \operatorname{im}(\varphi) = K$

$$\begin{pmatrix}1\\1\\1\end{pmatrix}^2 = \begin{pmatrix}1\\1\\1\end{pmatrix} \cdot \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \qquad \rightsquigarrow \qquad K^n$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \cdot \begin{pmatrix}x'\\y'\\z'\end{pmatrix} \mapsto \begin{pmatrix}xx'\\yy'\\zz'\end{pmatrix}$$

$$(x,y,z) \cdot (x',y',z') = (xx', yy', zz')$$

$$A \times S$$

$$(a,s) \cdot (a',s') = (\underbrace{a \cdot a'}_{\text{in } A}, \underbrace{s \cdot s'}_{\text{in } S})$$