Subgroup structure of branch groups

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Outline

1. What is a branch group?
2. Why do we study them?
3. Subgroup structure

What is a branch group?

2 definitions:

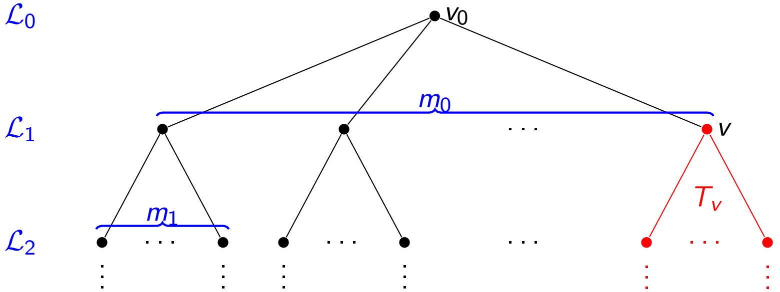
Algebraic Groups whose lattice of subnormal subgroups is similar to the structure of a regular rooted tree.

Geometric Groups acting level-transitively on a spherically homogeneous rooted tree T and having a subnormal subgroup structure similar to that of $Aut(T)$.

Definition

$(m_n)_{n\geq 0}$ sequence of integers ≥ 2.
T is a **spherically homogeneous rooted tree of type** $(m_n)_n$ if T is a tree with root v_0 of degree m_0 s.t. every vertex at distance $n \geq 1$ from v_0 has degree $m_n + 1$.

$\mathcal{L}_n$ = vertices at distance n from root



T_v is subtree rooted at v

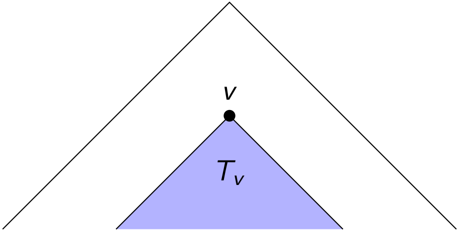
Rigid Stabilisers

Definition

T - spherically homogeneous tree of type $(m_n)_n$. G acts faithfully on T.

$rist_G(v) := \{g \in G : g$ fixes all vertices outside $T_v\}$ is the **rigid stabiliser** of $v \in T$.

$rist_G(n) := \prod_{v \in \mathcal{L}_n} rist_G(v)$ is the rigid stabiliser of level n.



Branch Group
Definition and Example

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G **acts as a branch group** on T iff for every n:

1. G acts transitively on $\mathcal{L}_n$ ('acts level-transitively on T')
2. $|G : rist_G(n)| < \infty$

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Example

For all n, $A = Aut(T)$ acts transitively on $\mathcal{L}_n$ with kernel $rist_A(n)$.

Example: Gupta–Sidki p-groups

$T = T(p)$, p - odd prime

$a := (1\,2\,\ldots\,p)$ on $\mathcal{L}_1$
$b := (a, a^{-1}, 1, \ldots, 1, b)$.

$G := \langle a, b \rangle$

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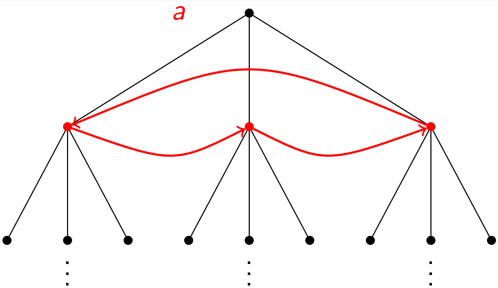

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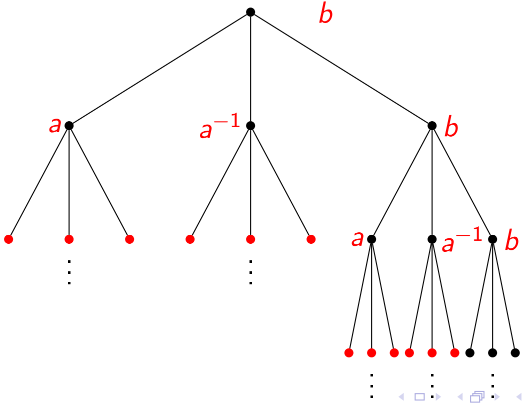

Example: (First) Grigorchuk Group

$T = T(2)$ - binary tree ($m_n = 2$)

$a := (12)$ on $\mathcal{L}_1$
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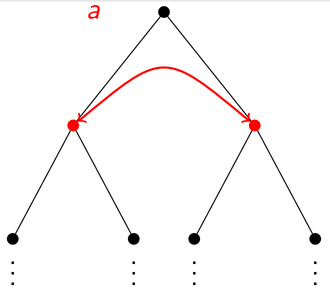

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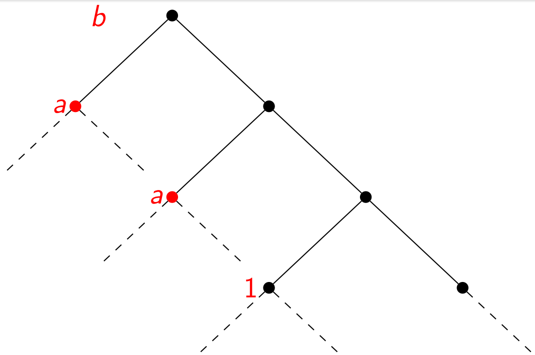

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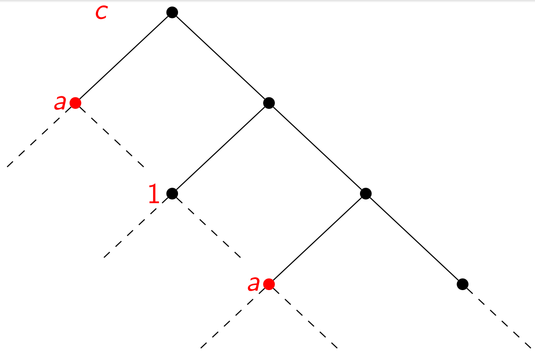

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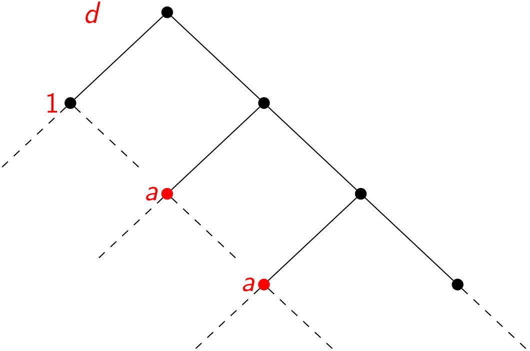

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Solve Open Problems

General Burnside Problem

Is every finitely generated torsion group finite?

Gupta–Sidki, '83 Gupta–Sidki p-groups are just infinite p-groups.
Every finite p-group is contained in the Gupta–Sidki p-group.

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Other problems

Grigorchuk group is first group shown to have intermediate word growth.

Also first group shown to be amenable but not elementary amenable.

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Definition

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Lemma

Every infinite finitely generated (f.g.) group has a just infinite quotient.

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Theorem (Wilson, '70)

The class of f.g. just infinite groups splits into 3 classes:

- *(just infinite) branch groups*
- *groups with a finite index subgroup H s.t. $H = \prod_k L$ for some k and L is*
  - *hereditarily just infinite (all subgroups of finite index are just infinite)*
  - *simple*

Proved by looking at lattice of subnormal subgroups.

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Are these classes different?

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If G is branch and $K \trianglelefteq G$ then there exists n such that $rist_G(n)' \leq K$.

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For $n \gg 1$ there is some H-orbit X on $\mathcal{L}_n$ such that

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We can use this to give an isomorphism invariant for H:

Definition

$b(H) :=$ maximum number of infinite normal subgroups of H that generate their direct product.

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Remark

The number of H-orbits on any layer is bounded (by $|G : H|$).
Say $\mathcal{L}_n = X_1 \sqcup \ldots \sqcup X_r$, each X_i an H-orbit.
Then $rist_G(X_i)' \triangleleft H$ and $rist_G(n)' = \prod rist_G(X_i)' \triangleleft H$.

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Corollary

$b(H) =$ maximum number of orbits of H on any layer of T.

How it all fits together

$b(H)$ behaves well under direct products

Let $H \leq_f H_1 \times \ldots \times H_r$ be subdirect; $b(H_i)$ finite.
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