On the Congruence Subgroup Problem for Branch Groups

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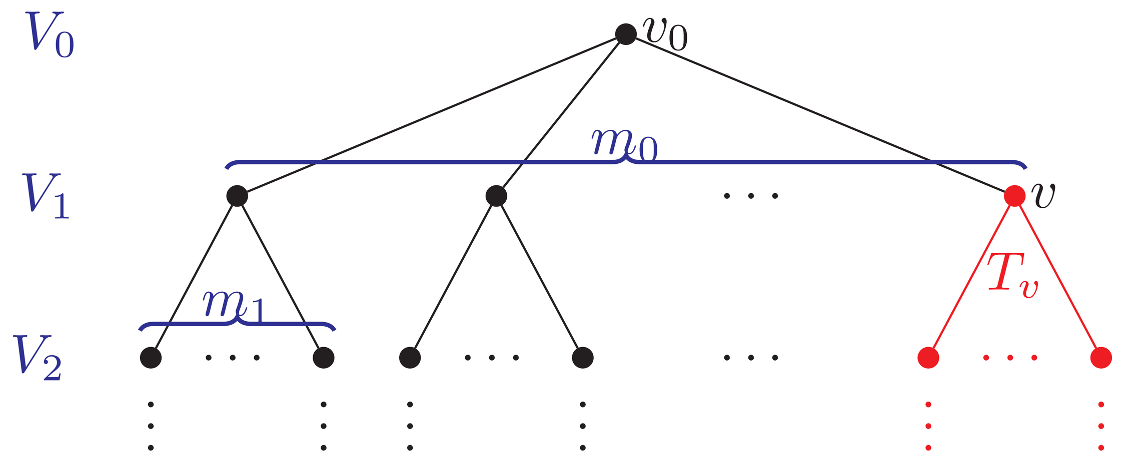
Branch groups

The class of branch groups consists of groups that act faithfully on rooted trees. It contains examples of groups with striking properties: finitely generated infinite torsion groups, groups of intermediate word growth, amenable but not elementary amenable groups, etc. (see [1]). Branch groups are just non-(virtually abelian) – they are not virtually abelian but all their proper quotients are – and have a nice subgroup structure ([6]). Many branch groups are just infinite (infinite but all proper quotients finite). The class of just infinite groups splits into three classes, just infinite branch groups being one of them.

Rooted trees

Let $(m_n)_{n\geq 0}$ be a sequence of integers with $m_n \geq 2$. A **spherically homogeneous rooted tree of type $(m_n)_n$** is a tree T with root v_0 of degree m_0 such that every vertex at distance $n \geq 1$ from v_0 has degree $m_n + 1$.



$V_n :=$ vertices at distance n from root;
$T_v :=$ subtree rooted at v

Branch actions

Let G act faithfully on T (in particular, G is residually finite). Define

- $\mathrm{St}_G(v) := \{g \in G : v^g = v\}$, the **stabilizer** of v
- $\mathrm{St}_G(n) := \bigcap_{v\in V_n} \mathrm{St}_G(v)$, the nth **level stabilizer**
- $\mathrm{rist}_G(v) := \{g \in G : g \text{ fixes } T \setminus T_v \text{ pointwise}\}$, the **rigid stabilizer** of v
- $\mathrm{rist}_G(n) := \prod_{v\in V_n} \mathrm{rist}_G(v)$, the nth **level rigid stabilizer**.

This faithful action is a **branch action** if for all n

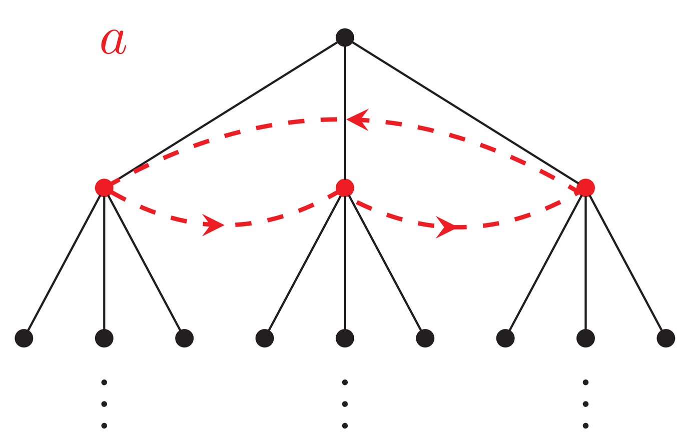
(i) the action is transitive on V_n

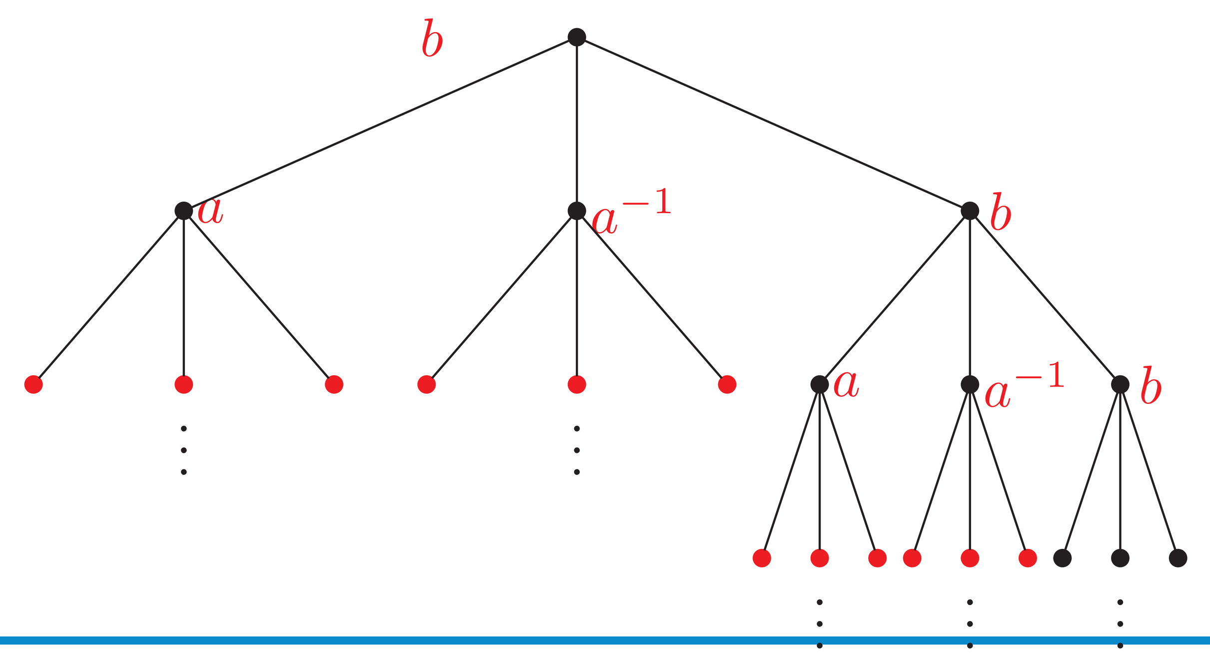
(ii) $|G : \mathrm{rist}_G(n)|$ is finite.

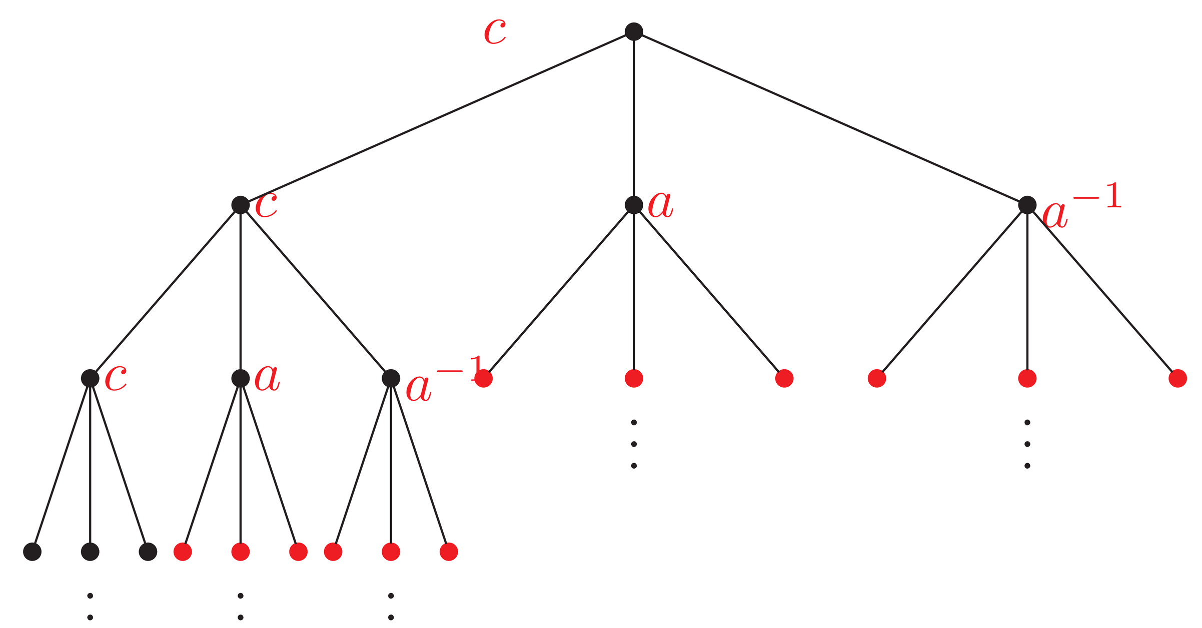
A group G is a **branch group** if it has a branch action on some T as above.

Examples

- $\mathrm{Aut}(T)$: acts transitively on each V_n with kernel $\mathrm{rist}_{\mathrm{Aut}(T)}(n) = \mathrm{St}_{\mathrm{Aut}(T)}(n)$.
- **Gupta–Sidki p-groups**, $GS(p)$ for each prime $p > 2$ ([4]).
$$GS(p) := \langle a, b\rangle \leq \mathrm{Aut}(T)$$
where T is the p-regular tree, a acts as a p-cycle on V_1 and $b := (a, a^{-1}, 1, \ldots, 1, b) \in \mathrm{St}_{GS(p)}(1)$. $GS(p)$ is a just infinite p-group.
- Variation: **Pervova groups**, $\Gamma(p)$ [5]
$$\Gamma(p) := \langle a, b, c\rangle \leq \mathrm{Aut}(T)$$
with T, a, b as above and $c = (c, a, a^{-1}, 1, \ldots, 1)$. $\Gamma(p)$ is also a just infinite p-group.







Congruence Subgroup Problem

Some motivation: subgroup growth

For a group G with finitely many subgroups of each finite index (e.g. G finitely generated), the **subgroup growth** function is given by
$$s_n(G) := |\{H \leq G : |G : H| \leq n\}|.$$
It is natural to ask about $s_n(G)$ when G is just infinite and, in particular, when G is branch.

Let G be a branch group. In general, it is difficult to calculate $s_n(G)$, so we focus on subgroups which are easier to control. We say that $H \leq G$ is a **congruence subgroup** if $\mathrm{St}(n) \leq H$ for some n. The **congruence subgroup growth** of G is
$$c_n(G) := |\{H \leq G : |G : H| \leq n, \; H \text{ is a congruence subgroup}\}|.$$
If all finite index subgroups are congruence subgroups ($c_n(G) = s_n(G)$) we say that G has the **congruence subgroup property (CSP)**.

How much can $s_n(G)$ and $c_n(G)$ differ?

Note: $G \hookrightarrow \widehat{G}$ (profinite completion of G), $G \hookrightarrow \overline{G}$ (completion with respect to $\{\mathrm{St}_G(n)\}$) and there is a homomorphism $\varphi\colon \widehat{G} \twoheadrightarrow \overline{G}$. We have $s_n(G) = s_n(\widehat{G})$ (*open* finite index subgroups of $\widehat{G}$) and $c_n(G) = s_n(\overline{G})$. Thus $s_n(G) = c_n(G)$ iff φ is injective.

Congruence subgroup problem: calculate the **congruence kernel** $\ker\varphi$

Examples:

- $GS(p)$ has CSP ([1]).
- $\Gamma(p)$ does *not* have CSP ($\ker\varphi = (\mathbb{Z}/p\mathbb{Z})[[\partial T]]$, [2, 5]).

Independence of branch action

Question ([2]): Does $\ker\varphi$ depend on the branch action $G \hookrightarrow \mathrm{Aut}(T)$?
NO

Theorem ([3]). *Let $\rho\colon G \hookrightarrow \mathrm{Aut}(T_\rho)$ and $\sigma\colon G \hookrightarrow \mathrm{Aut}(T_\sigma)$ be two branch actions of G. Let $\overline{G_\rho}$ (resp. $\overline{G_\sigma}$) denote the completions of G with respect to the topologies induced by taking $\{\mathrm{St}_G(n)\}$ with respect to ρ (resp. σ) as a neighbourhood basis of the identity. Then $\ker(\widehat{G} \to \overline{G_\rho}) = \ker(\widehat{G} \to \overline{G_\sigma})$.*

We can ask the same questions replacing $\{\mathrm{St}_G(n)\}$ by $\{\mathrm{rist}_G(n)\}$. The resulting kernel will also be independent of the branch action ([3]).

Proof sketch

Observation: For every branch action of G on T, and every $v \in T$, we have

- $\mathrm{rist}_G(v^g) = g^{-1}\mathrm{rist}_G(v)g$ for every $g \in G$,
- if $\mathrm{rist}_G(v^g) \cap \mathrm{rist}_G(v) \neq 1$ then $\mathrm{rist}_G(v^g) = \mathrm{rist}_G(v)$.

Lemma. *For every $u \in T_\rho$ there exists $v \in T_\sigma$ such that $\mathrm{rist}_\rho(u) \geq \mathrm{rist}_\sigma(v)$.*

We show that for every n there exists m such that $\mathrm{St}_\rho(n) \geq \mathrm{St}_\sigma(m)$ (and vice-versa by the same argument). Let $u \in V_n \subset T_\rho$. By the lemma, there exists $v \in V_m \subset T_\sigma$ such that $\mathrm{rist}_\rho(u) \geq \mathrm{rist}_\sigma(v)$. For any $g \in \mathrm{St}_\sigma(m)$, we have $1 \neq \mathrm{rist}_\sigma(v^g) = \mathrm{rist}_\sigma(v) \leq \mathrm{rist}_\rho(u^g) \cap \mathrm{rist}_\rho(u)$, so $\mathrm{rist}_\rho(u^g) = \mathrm{rist}_\rho(u)$ and $g \in \mathrm{St}_\rho(u)$. Claim follows by transitive action of G on V_n and V_m.

References

[1] L. Bartholdi, R. I. Grigorchuk and Z. Sunik. Branch groups. In *Handbook of Algebra*, vol. 3 (North-Holland, Amsterdam, 2003) pp. 989–1112.

[2] L. Bartholdi, O. Siegenthaler and P. Zalesskii. The congruence subgroup problem for branch groups, Isr. J. Math. **187** (2012), 419–450.

[3] A. Garrido. On the congruence subgroup problem for branch groups. Pre-print `arXiv:1405.3237`.

[4] N.D. Gupta and S. Sidki. On the Burnside problem for periodic groups, Math. Z. **182** (1983), 385–388.

[5] E. Pervova. Profinite completions of some groups acting on trees, J. Algebra **310** (2007), 858–879.

[6] J.S. Wilson. Structure theory for branch groups. In *Geometric and Homological Topics in Group Theory*, London Math. Soc. Lecture Note Ser. 358 (Cambridge University Press, 2009), 306–320.