About the paper of Touikan [14]: background

We fix a nonabelian free group $\mathbb{F}$ of finite rank. Please distinguish $\mathbb{F}$ from F; the letter F will also denote a free group.

1 General facts on equations

The following holds for any finite set of variables. For simplicity we use only x and y.

• An equation in x, y with coefficients in $\mathbb{F}$ can be considered as an element of

$$\mathbb{F}[x,y] := \mathbb{F} * F(x,y).$$

Let S be a (possibly infinite) system of equations in x, y. According to this point of view, S is a subset of $\mathbb{F}[x,y]$. The set of solutions of S in $\mathbb{F}$ is the set

$$V(S) := \{(g_1,g_2) \in \mathbb{F}\times\mathbb{F} \,|\, S(g_1,g_2) = 1\},$$

which is traditionally called the *algebraic variety* associated with S.

We assume that S has at least one solution in $\mathbb{F}$. Then $\langle\langle S\rangle\rangle \cap \mathbb{F} = 1$ and hence $\mathbb{F}$ is naturally embedded into $\mathbb{F}[x,y]/\langle\langle S\rangle\rangle$. Moreover, the images of x, y in $\mathbb{F}[x,y]/\langle\langle S\rangle\rangle$ satisfy S. But we are looking for solutions in $\mathbb{F}$ and not in larger groups.

(Think on polynomial equations over a field K. They also can be considered as elements of $K[x]$. Solutions of irreducible $f(x) \in K[x]$ live in the field $K[x]/\langle f(x)\rangle$ which contains K.)

• We will work in the category of $\mathbb{F}$-groups and $\mathbb{F}$-homomorphisms. Thus, for two $\mathbb{F}$-extensions $\mathbb{F} \leqslant G_1$ and $\mathbb{F} \leqslant G_2$, we will consider $\mathbb{F}$-homomorphisms $G_1 \to G_2$, i.e. homomorphisms which are identity on $\mathbb{F}$. The set of all $\mathbb{F}$-homomorphisms from G_1 to G_2 is denoted by $\mathrm{Hom}_{\mathbb{F}}(G_1, G_2)$.

• There is a one-to-one correspondence

$$\mathbf{Hom}_{\mathbb{F}}\big(\mathbb{F}[x,y]/\langle\langle S\rangle\rangle, \mathbb{F}\big) \;\longleftrightarrow\; V(S).$$

• Recall Hilbert's Nullstellensatz. Let $f \in K[x]$ and $\overline{K}$ be an algebraic closure of K. By definition, $\mathbf{Rad}(f)$ is the set of all polynomials $g \in K[X]$ which vanish on all solutions of $f(x)$ in $\overline{K}$. This set is an ideal in $K[x]$. Hilbert's Nullstellensatz says that

$$\mathbf{Rad}(f) = \langle g \in K[x] \,|\, \exists n \in \mathbb{N} : g^n = f\rangle.$$

• For a system of equations $S \subset \mathbb{F}[x,y]$, we define the radical $\mathbf{Rad}(S)$ in exactly the same way: Let $\mathbf{Rad}(S)$ be the set of all equations $g \in \mathbb{F}[x,y]$ which vanish on all solutions of S. Nobody knows how sounds Hilbert's Nulstellensatz in this situation.

• Since $V(S) = V(\mathbf{Rad}(S))$, we have one-to-one correspondences:

$$V(S) \;\longleftrightarrow\; \mathbf{Hom}_{\mathbb{F}}\big(\mathbb{F}[x,y]/\langle\langle S\rangle\rangle, \mathbb{F}\big) \;\longleftrightarrow\; \mathbf{Hom}_{\mathbb{F}}\big(\mathbb{F}[x,y]/\mathbf{Rad}(S), \mathbb{F}\big).$$

Clearly, if $V(S) \neq \emptyset$, then $\mathbb{F} \cap \mathbf{Rad}(S) = 1$ and hence $\mathbb{F}$ embeds into $\mathbb{F}[x,y]/\mathbf{Rad}(S)$. And if $V(S) = \emptyset$?

- We introduce the following important definitions:

(1) The group $\mathbb{F}_{R(S)} := \mathbb{F}[x,y]/\mathbf{Rad}(S)$ is called the *coordinate group* of S.

(2) The algebraic variety $V(S)$ is called *reducible* if it a union $V(S) = V(S_1) \cup V(S_2)$ of algebraic varieties with $V(S_1) \neq V(S)$ and $V(S_2) \neq V(S)$.

(3) An $\mathbb{F}$-group G is called *fully residually* $\mathbb{F}$ if for every finite subset $P \subset G$ there exists $f \in \mathbf{Hom}_{\mathbb{F}}(G, \mathbb{F})$ such that the restriction of f to P is injective.

(In particular, such groups are limit groups.)

The following theorems are of general character.

Theorem 1.4. [**1**] S is irreducible if and only if $\mathbb{F}_{R(S)}$ is fully residually $\mathbb{F}$.

Theorem 1.5. [**1**] Either $\mathbb{F}_{R(S)}$ is fully residually $\mathbb{F}$, or

$$V(S) = V(S_1) \cup \cdots \cup V(S_n),$$

where each $V(S_i)$ is irreducible and there are canonical epimorphisms

$$\pi_i : \mathbb{F}_{R(S)} \to \mathbb{F}_{R(S_i)}$$

such that each $f \in \mathbf{Hom}_{\mathbb{F}}\big(\mathbb{F}_{R(S)}, \mathbb{F}\big)$ factors through some π_i.

The following is a slight modification of Corollary 1.6 from the paper of Touikan. He gives it without proof. It seems that the condition "fully residually $\mathbb{F}$" can be replaced by "residually $\mathbb{F}$".

Proposition. If $\mathbb{F}[x,y]/\langle\langle S\rangle\rangle$ is fully residually $\mathbb{F}$, then $\langle\langle S\rangle\rangle = \mathbf{Rad}(S)$.

Proof. It is a general fact that each $\mathbb{F}$-homomorphism $\varphi : \mathbb{F}[x,y]/\langle\langle S\rangle\rangle \to \mathbb{F}$ factors through $\mathbb{F}[x,y]/\mathbf{Rad}(S)$. Suppose that there is $g \in \mathbf{Rad}(S) \setminus \langle\langle S\rangle\rangle$. By assumption, there exists an $\mathbb{F}$-homomorphism $\varphi : \mathbb{F}[x,y]/\langle\langle S\rangle\rangle \to \mathbb{F}$ such that $\varphi(g\langle\langle S\rangle\rangle) \neq 1$. This contradicts the fact that φ factors through $\mathbb{F}[x,y]/\mathbf{Rad}(S)$. □

2 Splittings of groups. Moves of splittings

2.1 Fundamental groups of graphs of groups

This subsection is written to make notation precise. All material can be found in the classical book of J.-P. Serre "Trees". Touikan is not accurate in defining the fundamental groups of graphs of groups that can lead to misunderstanding later. Therefore we want to recall precise definitions.

(A) (A *graph of groups* $\mathcal{G}(\Gamma)$)

Let Γ be a connected directed graph with the set of vertices $V\Gamma$ and the set of edges $E\Gamma$. The graph is directed in the sense that there are functions $i : E\Gamma \to V\Gamma$ and $t : E\Gamma \to V\Gamma$ corresponding to the initial and terminal vertices of edges. To Γ we associate the following:

- a vertex group G_v for each $v \in V\Gamma$ and an edge G_e for each $e \in E\Gamma$;
- monomorphisms $\sigma_e : G_e \to G_{i(e)}$ and $\tau_e : G_e \to G_{t(e)}$ for each edge $e \in E\Gamma$.

The maps σ_e and τ_e are called *boundary monomorphisms* and the images of these maps are called *boundary subgroups*. The set of these data (i.e the graph Γ, the vertex groups, the edge groups, and the boundary morphisms), denoted $\mathcal{G}(\Gamma)$, is called a *graph of groups*.

(B) Let T be a maximal tree in Γ. The *fundamental group* $\pi_1(\mathcal{G}(\Gamma), T)$ is constructed in two steps:

(1) We take the free products of all vertex groups G_v, $v \in V\Gamma(= VT)$, and put additional relations for each $e \in T$:
$$\sigma_e(g) = \tau_e(g), \quad g \in G_e.$$
In other words, the resulting group is the amalgamated product of vertex groups of Γ, where the amalgamation goes through the edge groups G_e, where $e \in T$.

(2) For each $e \in E\Gamma \setminus ET$, we add the stable letter t_e and the following relations:
$$t_e^{-1}(\sigma_e(g))t_e = \tau_e(g) \quad g \in G_e.$$
In other words, we take consecutive HNN extensions of the group obtained in (1).

We say that G *splits* as the fundamental group of a graph of groups if there is an isomorphism $\varphi : G \to \pi_1(\mathcal{G}(\Gamma), T)$. The data $D = (G, \mathcal{G}(\Gamma), T, \varphi)$ are called *splitting data* of G.

2.2 How changes the isomorphism φ if we choose another maximal tree T in Γ

Example. For brevity we write α instead of σ_e and β instead of τ_e. Other notations below are also unusual, but clear from the context. Maximal trees are colored in blue.

Figure 1.

Let T_1, T_2 be the distinguished maximal trees in Γ. Then
$$\pi_1(\mathcal{G}(\Gamma), T_1) = \langle G_1, G_2, G_3, G_4, t_1, t_2 \mid \ \alpha(G_{12}) = \beta(G_{12}),\ \alpha(G_{23}) = \beta(G_{23}),\ \alpha(G_{34}) = \beta(G_{34}),$$
$$t_1^{-1}\alpha(G_{13})t_1 = \beta(G_{13}),\ t_2^{-1}\alpha(G_{14})t_2 = \beta(G_{14})\rangle,$$

$$\pi_1(\mathcal{G}(\Gamma), T_2) = \langle G_1, G_2, G_3, G_4, s_1, s_2 \mid \ s_1^{-1}\alpha(G_{12})s_1 = \beta(G_{12}),\ \alpha(G_{23}) = \beta(G_{23}),$$
$$s_2^{-1}\alpha(G_{34})s_2 = \beta(G_{34}),\ \alpha(G_{13}) = \beta(G_{13}),\ \alpha(G_{14}) = \beta(G_{14})\rangle.$$

The following map is an isomorphism:

$$\psi : \pi_1(\mathcal{G}(\Gamma), T_1) \to \pi_1(\mathcal{G}(\Gamma), T_2), \quad G_1 \mapsto s_1^{-1}G_1s_1,\ G_2 \mapsto G_2,\ G_3 \mapsto G_3,\ G_4 \mapsto s_2G_4s_2^{-1},$$
$$t_1 \mapsto s_1^{-1},\ t_2 \mapsto s_1^{-1}s_2^{-1}.$$

In the general case an isomorphism $\psi : \pi_1(\mathcal{G}(\Gamma), T_1) \to \pi_1(\mathcal{G}(\Gamma), T_2)$ can be defined as follows:

- Choose an arbitrary $v \in V\Gamma$ (in the above example, I choose v_2). For two vertices $u, w \in V\Gamma$ let $[u, w]$ be the unique reduced path in T_1 from u to w.
- To every edge $e \in E\Gamma$, we associate the closed path $[v, i(e)] \cdot e \cdot [t(e), v]$. Let $e_1, e_2, \dots, e_k$ be the consecutive edges of this path which do not lie in T_2. Then we send t_e to $t_{e_1} t_{e_2} \dots t_{e_k}$.
- To every vertex $w \in V\Gamma$, we associate the path $[v, w]$. Let $e_1, e_2, \dots, e_k$ be the consecutive edges of this path which do not lie in T_2. Then we send G_w to $t_{e_1} t_{e_2} \dots t_{e_k} \cdot G_w \cdot (t_{e_1} t_{e_2} \dots t_{e_k})^{-1}$.

2.3 Elementary moves on graphs of groups

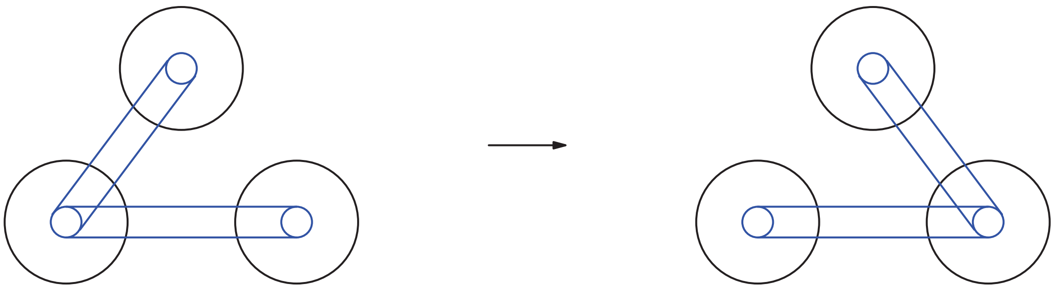
There are the following elementary moves:

(1) Conjugation of boundary monomorphisms (or local conjugation)

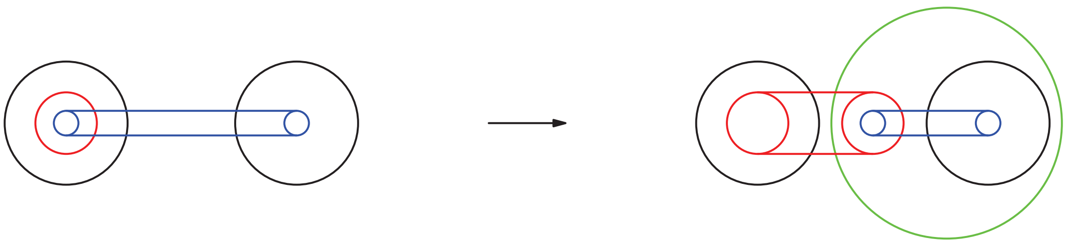
$A *_C B \rightsquigarrow A_{C^a} B^a$,

$\langle A, t \,|\, t^{-1} C_1 t = C_2 \rangle \rightsquigarrow \langle A, t_1 \,|\, t_1^{-1} C_1^a t_1 = C_2 \rangle$, where $t_1 = a^{-1} t$.

(2) Slide



(3) Folding



(4) Collapse an edge

These moves are well known. Formal definitions can be found in the paper of Touikan. Suppose that $\mathcal{G}(\Gamma_2)$ is obtained from $\mathcal{G}(\Gamma_1)$ by an elementary move. It is a useful exercise to establish an isomorphism between $\pi_1(\mathcal{G}(\Gamma_1), T_1)$ and $\pi_1(\mathcal{G}(\Gamma_2), T_2)$ for appropriately chosen maximal trees T_1 and T_2.

3 JSJ splittings

3.1 Theorem of Rips and Sela

Definitions 1.14 and 1.15.

(1) A splitting of G as the fundamental group of a graph of groups is called *cyclic* if all the edge groups are infinite cyclic groups.

(2) Let $G = \pi_1(\mathcal{G}(\Gamma), T)$. A subgroup $H \leqslant G$ is called *elliptic* if H is conjugate into a vertex subgroup of G; otherwise H is called *hyperbolic*.

(3) A splitting of G is called *elementary* if the underlying graph is either a segment or a loop with one edge. In this case G is an amalgamated product or an HNN-extension.

(4) Suppose that G has two elementary splittings D and D', say

$$G = A \underset{C_1=C_2}{*} B \qquad \text{and} \qquad G = \langle A', t \mid t^{-1}C_1't = C_2' \rangle.$$

We say that D *is elliptic in* D' if the edge group C_1 is elliptic in D', i.e. if C_1 is conjugate into A'.

(5) Let G be an $\mathbb{F}$-group. A splitting $\varphi : G \to \pi_1(\mathcal{G}(\Gamma), T)$ is said to be *modulo* $\mathbb{F}$ if $\varphi(\mathbb{F})$ is contained in a vertex group.

Theorem 1.16.

(1) Let G be freely indecomposible modulo $\mathbb{F}$ and let D and D' be two elementary cyclic splittings of G modulo $\mathbb{F}$. Then D is elliptic in D' if and ony if D' is elliptic in D.

(2) If D' is hyperbolic in D, then G admits a splitting D'' such that one of its vertex groups Q can be identified with the fundamental group $\pi_1(S)$ of a punctured surface S such that the boundary subgroups of Q correspond to the puncture subgroups. Moreover, if $\langle d \rangle$ and $\langle d' \rangle$ are the cyclic edge subgroups from D and D', then d and d' are conjugate to elements q and q' of $Q = \pi_1(S)$, which correspond to simple closed loops in S.

Definition 1.17. A subgroup $Q \leqslant G$ is a *quadratically hanging* (QH) subgroup if there are elementary cyclic splittings D and D' of G such that Q is a vertex group of a new cyclic splitting that arises as in Theorem 3.1.2 (2).

Remark. Not every surface with punctures can yield a QH subgroup. By [6, Theorem 3], the projective plane with puncture(s) and the Klein bottle with puncture(s) cannot give QH subgroups. Surfaces that can give QH subgroups must admit pseudo-Anosov homeomorphisms. (Why?)

Definitions 1.18 and 1.19.

(1) A QH subgroup Q of G is a *maximal* QH (MQH) subgroup if Q is not properly contained in another QH subgroup of G.

(2) Having a splitting of G with a QH vertex subgroup $Q = \pi_1(S)$, one can produce a *refinement* of this splitting by using simple loops in S.

(3) A splitting D is called *almost reduced* if vertices of valency 1 and 2 properly contain images of edge subgroups, except vertices between two MQH subgroups that may coincide with one of the edge groups.

(4) A splitting D is called unfolded if D cannot be obtained from another splitting D' via a folding move.

3.2 JSJ splitting of a fully residually $\mathbb{F}$ group

Theorem 1.20. ([7, Proposition 2.15]). Let G be a freely indecomposable modulo $\mathbb{F}$ finitely generated fully residually free $\mathbb{F}$ group. Then there exists an almost reduced unfolded cyclic splitting D, called the cyclic JSJ splitting of G modulo $\mathbb{F}$, with the following properties.

(1) Every MQH subgroup of G can be conjugated to a vertex group of D. Every QH subgroup of G lies in a MQH subgroup of G. Vertex subgroups in D, which are non-MQH, are either maximal abelian, or nonabelian (in the latter case they are called *rigid*). Every non-MQH vertex subgroup is elliptic in every cyclic splitting of G modulo F.

(2) If an elementary cyclic splitting $G = A *_C B$ or $G = A*_C$ is hyperbolic in another elementary cyclic splitting, then C can be conjugated into some MQH subgroup.

(3) Every elementary cyclic splitting $G = A*_C B$ or $G = A*_C$ modulo $\mathbb{F}$ which is elliptic with respect to any other elementary cyclic splitting modulo $\mathbb{F}$ of G can be obtained from D by a sequence of elementary moves given in Definition 1.12.

(4) If D_1 is another cyclic splitting of G modulo $\mathbb{F}$ that has properties (1)-(3), then D_1 can be obtained from D by a sequence of slidings, conjugations, and local conjugations.

Theorem 1.22. ([6, 10]) If $\mathbb{F}_{R(S)} \neq \mathbb{F}$ and is freely indecomposable modulo $\mathbb{F}$, then it admits a nontrivial cyclic JSJ decomposition modulo $\mathbb{F}$.

4 Example of a JSJ splitting

Let $w(x, y) = x^2y^2x^4y^4$ and $u \in \mathbb{F}$, $u \neq 1$. The following is a JSJ splitting of the group $\mathbb{F}[x, y]/\langle\langle S\rangle\rangle = \mathbb{F} \underset{u=w(x,y)}{*} F(x, y)$ modulo $\mathbb{F}$:

$$\mathbb{F} \underset{u=w(x,y)}{*} \left(F(x^2, y^2) \underset{x^2}{*} \langle x\rangle\right) \underset{y^2}{*} \langle y\rangle.$$

5 Canonical $\mathbb{F}$-automorphisms of the group $\mathbb{F}_{R(S)}$

Definition 1.13. *Dehn twist* along an edge of a cyclic splitting is defined as follows:

- Let $G = A *_{\langle\gamma\rangle} B$. We set

$$\delta(x) = \begin{cases} x & \text{if } x \in A, \\ x^{\gamma} & \text{if } x \in B. \end{cases}$$

- Let $G = \langle A, t \mid t^{-1}\gamma t = \beta\rangle$, $\gamma, \beta \in A$. We set

$$\delta(x) = \begin{cases} x & \text{if } x \in A, \\ t\beta & \text{if } x = t. \end{cases}$$

Definition 1.21. Suppose that $\mathbb{F}_{R(S)}$ is freely indecomposable modulo $\mathbb{F}$. Let D be a cyclic JSJ splitting of $\mathbb{F}_{R(S)}$ modulo $\mathbb{F}$. The group Δ of *canonical* $\mathbb{F}$*-automorphisms* of $\mathbb{F}_{R(S)}$ is generated by the following:

(1) Dehn twists along edges of D or along closed simple curves in MQH subgroups; these must fix $\mathbb{F} \leqslant \mathbb{F}_{R(S)}$.

(2) automorphisms of the abelian vertex groups that fix its peripheral subgroups.

6 The structure of $\mathrm{Hom}_{\mathbb{F}}\big(\mathbb{F}_{R(S)}, \mathbb{F}\big)$

The first theorem is general and the second is special.

Theorem. Let G be a finitely generated non-free group and $\mathbb{F}$ be a free group. There is a finite tree of epimorphisms $(\varphi_i)_{i\in I}$ with <u>nontrivial kernels</u> as on Figure 2 such that

- each group in this tree is a limit group with possible exception of G,
- all bottom groups are free groups,
- for any homomorphism $f : G \to \mathbb{F}$, there exists a branch of epimorphisms from the top vertex to some bottom vertex

$$G \overset{\phi_1}{\twoheadrightarrow} \Gamma_1 \overset{\phi_2}{\twoheadrightarrow} \Gamma_2 \overset{\phi_3}{\twoheadrightarrow} \dots \overset{\phi_k}{\twoheadrightarrow} \Gamma_k$$

and there exist automorphisms $\alpha_0 \in \mathrm{Aut}(G)$, $\alpha_i \in \mathrm{Aut}(\Gamma_i)$, $i = 1, \dots, k$, such that

$$f = \psi \circ \alpha_k \circ \phi_k \circ \dots \circ \alpha_2 \circ \phi_2 \circ \alpha_1 \circ \phi_1 \circ \alpha_0$$

for some homomorphism $\psi : \Gamma_k \to \mathbb{F}$.

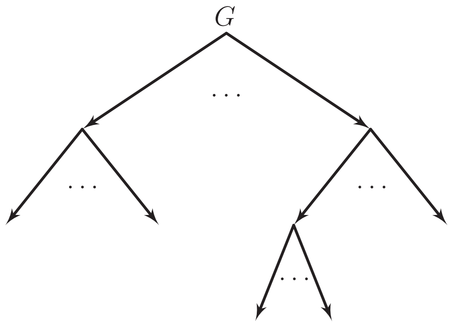


Figure 2.

Theorem. ([6, 10]) Let $G = \mathbb{F}_{R(S)}$. There is a finite tree of $\mathbb{F}$-epimorphisms $(\varphi_i)_{i\in I}$ with nontrivial kernels as on Figure 2 such that

- each group in this tree is of kind $\mathbb{F}_{R(S_i)}$ and is a limit group with possible exception of G,
- all bottom groups are free groups of kind $\mathbb{F} * F(Y_i)$ for some finite Y_i,
- for any $\mathbb{F}$-homomorphism $f : G \to \mathbb{F}$, there exists a branch of epimorphisms from the top vertex to some bottom vertex

$$G \overset{\phi_1}{\twoheadrightarrow} \Gamma_1 \overset{\phi_2}{\twoheadrightarrow} \Gamma_2 \overset{\phi_3}{\twoheadrightarrow} \dots \overset{\phi_k}{\twoheadrightarrow} \Gamma_k$$

and there exist $\alpha_0 \in \mathrm{Aut}_{\mathbb{F}}(G)$, $\alpha_i \in \mathrm{Aut}_{\mathbb{F}}(\Gamma_i)$, $i = 1, \dots, k$, such that

$$f = \psi \circ \alpha_k \circ \phi_k \circ \dots \circ \alpha_2 \circ \phi_2 \circ \alpha_1 \circ \phi_1 \circ \alpha_0$$

for some $\mathbb{F}$-homomorphism $\psi : \Gamma_k \to \mathbb{F}$. The $\mathbb{F}$-automorphisms $\alpha_1, \dots, \alpha_{k-1}$ are canonical, i.e. they appear from JSJ decompositions of $\Gamma_1, \dots, \Gamma_k$ as in Definition 1.21, and $\alpha_k = id$.

7 Existence of rank 2 solutions implies residual freeness of $\mathbb{F}_{R(S)}$

Consider one equation $S := w(x, y)u^{-1}$ with $u \in \mathbb{F}$. We have $S \in F[x, y] = \mathbb{F} * F(x, y)$. We say that S has a *rank n solution*, if there is a solution that generates F_n.

Theorem. ([14, Lemma 2.6]) Let $u \in \mathbb{F}$, $u \neq 1$. Suppose that $w(x, y) = u$ has a rank 2 solution in $\mathbb{F}$. We set $S := \{w(x, y)u^{-1}\}$. The group

$$\mathbb{F}[x, y]/\langle\langle S\rangle\rangle = \mathbb{F} \underset{u=w(x,y)}{*} F(x, y)$$

is fully residually free modulo $\mathbb{F}$.

In particular, this group coincides with the coordinate group $\mathbb{F}_{R(S)}$. If, additionally, w is not primitive and not a proper power, then $\mathbb{F}_{R(S)}$ is freely indecomposable modulo $\mathbb{F}$, and hence has a nontrivial cyclic JSJ splitting modulo $\mathbb{F}$.

Proof. Let X, Y be a rank 2 solution. Consider the $\mathbb{F}$-embedding

$$\mathbb{F} \underset{u=w(x,y)}{*} F(x, y) \underset{\substack{x \mapsto t^{-1}Xt \\ y \mapsto t^{-1}Yt}}{\longrightarrow} \mathbb{F} \underset{u=t^{-1}ut}{*} t^{-1}\mathbb{F}t \quad \leqslant \langle \mathbb{F}, t \,|\, t^{-1}ut = u\rangle$$

The latter HNN extension is fully residually free modulo $\mathbb{F}$, so the embedded group too. By the last proposition from Section 1, we have $\mathbb{F}[x, y]/\langle\langle S\rangle\rangle = \mathbb{F}_{R(S)}$. The free indecomposability of $\mathbb{F}_{R(S)}$ modulo $\mathbb{F}$ follows from Kurosh's theorem. □

8 Swarup's theorem on splittings of free groups

Swarup's theorem [13] (see also [2, 3, 5, 8, 12]) describes splittings of a free group F. They appear in a natural way. Figure 3 shows how amalgams appear from free splittings. Similarly HNN extensions appear from free splittings.

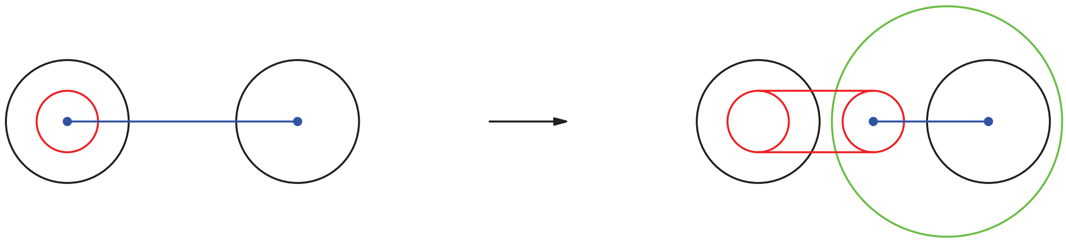


Figure 3.

Lemma. ([14, Lemma 2.10]) Let G be a free group of rank 2. Then the only possible almost reduced (see Def. 1.18 and 1.19) nontrivial splittings of G as the fundamental group of a graph of groups are as follows:

(1) G is a star of groups with the central group free of rank 2 and each edge group is nontrivial, cyclic and is a proper finite index subgroup of the associated peripheral group.

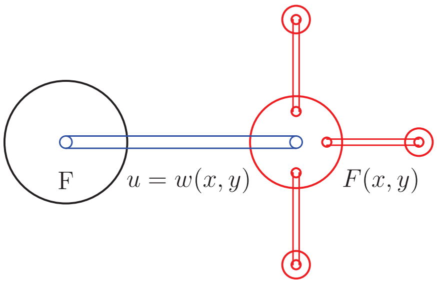
(2) G is an HNN extension $\langle H, t \,|\, t^{-1}pt = q^n \rangle$, where H is free of rank 2 with free generators p, q and $n \in \mathbb{N}$. In particular, $G = \langle q, t \rangle$.

Application. ([14, Corollary 2.12]) Suppose that $w \in F(x, y)$ is not primitive and not a proper power. Let $u \in \mathbb{F}$ and suppose that $w(x, y) = u$ has a rank 2 solution in $\mathbb{F}$. As we have seen in Section 7, the coordinate group

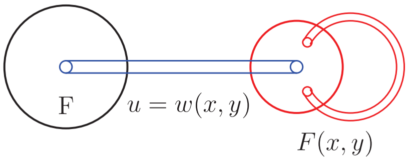
$$\mathbb{F}_{R(S)} = \mathbb{F} \underset{u=w(x,y)}{*} F(x, y)$$

has a nontrivial cyclic JSJ splitting modulo $\mathbb{F}$. There are only three possible classes of such splittings (see figures below). In each case the group of canonical $\mathbb{F}$-automorphisms can be computed easily.

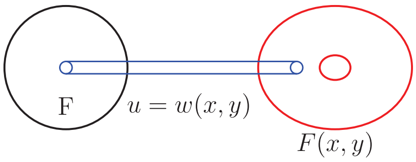
(1)



(2)



(3)



It seems, that in (3) the word $w(x, y)$ is conjugate to $[x, y]^{\pm 1}$ in $F(x, y)$.

9 Solutions of rank 1 in the situation, where there are also solutions of rank 2

Solutions of rank 1 for $w(x, y) = u$ can be described easily. However, it is instructive to describe (to parameterize) them by using Makanin-Razborov diagrams. We will do that under the following assumption.

Assumption. Suppose that $w(x, y) = u$ has solutions of rank 1 and of rank 2, and that w is neither primitive nor a proper power.

Then, by Baumslag (see [14, Theorem 2.4]), u is not a proper power. Thus, there are integers p, q such that

$$p\sigma_x(w) + q\sigma_y(w) = 1.$$

Let $S = \{w(x, y)u^{-1}\}$ and $S_1 = \{w(x, y)u^{-1}, [x, y]\}$. Clearly $V(S)$ is strictly larger than $V(S_1)$. Then the canonical epimorphism

$$\theta : \mathbb{F}_{R(S)} \to \mathbb{F}_{R(S_1)}$$

has a nontrivial kernel.

Lemma. ([14, Lemma 2.13]) Under above Assumption, $\mathbb{F}_{R(S_1)}$ is isomorphic to the fully residually free group

$$F_1 := \langle \mathbb{F}, s \mid [u, s] = 1 \rangle.$$

The $\mathbb{F}$-morphism $\pi : \mathbb{F}_{R(S_1)} \to F_1$ given by

$$\pi(x) = u^p s^{\sigma_y(w)} = \overline{x}; \quad \pi_1(y) = u^q s^{-\sigma_x(w)} = \overline{y},$$

where p, q are as above, is an isomorphism.

Proposition. ([14, Proposition 2.14]) Under above Assumption and using notation of above Lemma, we have the following.

Consider the $\mathbb{F}$-morphisms $\pi_1 = \pi \circ \theta : \mathbb{F}_{R(S)} \to F_1$ and $\pi_2 = F_1 \to \mathbb{F}$ given by $\pi_2(s) = u$.

(1) If $\mathbb{F}_{R(S)}$ is as in (1) in Application, then $V(S_1)$ is represented by the following branch in Diag$(\mathbb{F}_{R(S)}, \mathbb{F})$:

$$\mathbb{F}_{R(S)} \overset{\pi_1}{\to} \overset{\overset{\sigma}{\curvearrowright}}{F_1} \overset{\pi_2}{\to} \mathbb{F},$$

where $\sigma \in \Delta_1$ is a canonical $\mathbb{F}$-automorphism of $\mathbb{F}_1$.

(2) If $\mathbb{F}_{R(S)}$ is as in (2) in Application, then $V(S_1)$ is represented by the following branch in Diag$(\mathbb{F}_{R(S)}, \mathbb{F})$:

$$\overset{\overset{\sigma}{\curvearrowright}}{F_{R(S)}} \overset{\pi_3}{\to} \mathbb{F},$$

where $\sigma \in \Delta$ is a canonical $\mathbb{F}$-automorphism of $\mathbb{F}_{R(S)}$ and $\pi_3 = \pi_2 \circ \pi_1$.

Note that $\mathbb{F}_{R(S)}$ cannot be as in (3) in Application because of rank 1 solutions.

Proof. Observe that each rank 1 solution considered as an element of $\mathrm{Hom}_{\mathbb{F}}(\mathbb{F}_{R(S)}, \mathbb{F})$ factors through $\mathbb{F}_{R(S_1)} \cong F_1$. We use F_1 instead of $\mathbb{F}_{R(S_1)}$ since it has clear $\mathbb{F}$-morphisms into $\mathbb{F}$. □

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