Simple curves on surfaces and an analog of a theorem of Magnus for surface groups*

Oleg Bogopolski[1], **Elena Kudryavtseva**[2], **Heiner Zieschang**[3,4]

[1] Institute of Mathematics, 630090 Novosibirsk – Russia (e-mail: `groups@math.nsc.ru`)
[2] Department of Mathematics, Moscow State Lomonossov-University, 119992 Moscow, Russia (e-mail: `eakudr@mech.math.msu.su`)
[3] Fakultät für Mathematik, Ruhr-Universität Bochum, 44780 Bochum, Germany (e-mail: `heiner.zieschang@ruhr-uni-bochum.de`)
[4] Department of Mathematics, Moscow State Lomonossov-University, 119992 Moscow, Russia (e-mail: `zieschan@mech.math.msu.su`)

Abstract Magnus proved that, if $\mathcal{G}$ is a free group and u, v are elements of $\mathcal{G}$ with the same normal closure, u is a conjugate of v or v^{-1} [9]. We prove the analogous result in the case that $\mathcal{G}$ is the fundamental group of a closed surface S and u, v are elements of $\pi_1(S)$ containing simple closed two-sided curves on S. As a corollary we prove that, if S is not a torus and is not a Klein bottle, each automorphism of $\pi_1(S)$ which maps every normal subgroup of $\pi_1(S)$ into itself is an inner automorphism.

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1 Introduction

In [9, §6 No. 2] Magnus proved that, *if elements u, v of a free group $\mathcal{F}$ have the same normal closure, then u is conjugate to $v^{\pm 1}$*. We know the following two generalizations of this theorem.

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1) In [5] Greendlinger proved that, if two subsets $\mathcal{U}$ and $\mathcal{V}$ of a free group $\mathcal{F}$ satisfy some small cancellation conditions and have the same normal closure, then there is a bijection $\psi\colon\ \mathcal{U} \to \mathcal{V}$ such that u is conjugate to $\psi(u)^{\pm 1}$.
2) A group is said to be locally indicable if each of its non-trivial, finitely generated subgroups admits an epimorphism onto the infinite cyclic group. Let $\mathcal{A}$ and $\mathcal{B}$ be two non-trivial locally indicable groups. In [2] Edjvet proved that if $u, v \in \mathcal{A} * \mathcal{B}$ are cyclically reduced words each of length at least two, and if the normal closures of u and v coincide, then u is a conjugate of $v^{\pm 1}$.

In view of the Magnus' theorem the following problems seem to be of interest.

Problem 1. *Does a (one-relator) group $\mathcal{G}$ have the following property: if u, v are elements of $\mathcal{G}$ with the same normal closure, then u is a conjugate of $v^{\pm 1}$?*

Problem 2. *Does an element u of a (one-relator) group $\mathcal{G}$ have the following property: if φ is an automorphism of $\mathcal{G}$ sending the normal closure of u into itself then $\varphi(u)$ is a conjugate of $u^{\pm 1}$?*

It is clear that not each one-relator group has the property from Problem 1. For example, if $\mathcal{G} = \langle a, b \mid b^{-1}a^2b = a^3\rangle$ then the normal closures of a and a^2 coincide but a^2 is not conjugate to $a^{\pm 1}$. However, it seems to be very likely that the fundamental group of a closed surface has this property. Our main result is the following weaker assertion which implies that every element $g \in \pi_1(S)$ containing a simple closed two-sided curve is a solution of Problem 2.

Theorem 1.1. *Let S be a closed surface and $g,\ h$ non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves γ and χ, resp. If h belongs to the normal closure of g then h is conjugate to g^{ε} or to $(gug^{\eta}u^{-1})^{\varepsilon}$, $\varepsilon, \eta \in \{1, -1\}$; here u is a homotopy class containing a simple closed curve μ which properly intersects γ exactly once.*

Moreover, if h is not conjugate to g^{ε} then $\eta = 1$ if μ is one-sided and $\eta = -1$ elsewise, and χ is homotopic to the boundary of a regular neighbourhood of $\gamma \cup \mu$.

A direct consequence is the following analog of the Magnus' theorem [9].

Corollary 1.2. *Let S be a closed surface and $g,\ h$ be non-trivial elements of $\pi_1(S)$ both containing simple closed two-sided curves. If the normal closures of g and h coincide then h is conjugate to g or g^{-1}.*

The proof of Theorem 1.1 in section 2 is geometrical and uses coverings, intersection numbers of curves, and Brouwer's fixed-point theorem. As a corollary we obtain in Section 3 the following Theorem 1.3 concerning *normal* automorphisms (an automorphism of a group $\mathcal{G}$ is called normal if it maps each normal subgroup of $\mathcal{G}$ into itself).

Theorem 1.3. *If S is a closed surface different from the torus and the Klein bottle, then every normal automorphism of $\pi_1(S)$ is an inner automorphism.*

The proof of this theorem in section 3 uses Theorem 1.1 and purely algebraic techniques (amalgamated free products, HNN-extensions, Magnus' solution of the word problem for one-relator groups), as well as the Bass-Serre theory of groups acting on trees.

Earlier Lubotsky [6] and Lue [7] proved that every normal automorphism of a free group of rank at least 2 is an inner automorphism. In [13] Neshchadim proved that any normal automorphism of the free product of two non-trivial groups is an inner automorphism.

Remark that Theorem 1.1 and Corollary 1.2 admit some analogs for non-simple curves satisfying additional assumptions. By the *complexity* $c(x)$ of an element $x \in \pi_1(S)$ we understand the minimal self-intersection number of curves representing the free homotopy class of x. Under some additional conditions on g, it can be shown that, if h belongs to the normal closure of g, then $c(h) \geq c(g)$. If $c(h) = c(g)$ then h is conjugate to g or g^{-1}. A direct consequence is that the element g is a solution of Problem 2. These assertions can be proved, for example, under the following assumptions: $c(g) > 1$ and the curve γ representing the element g is in general position and is allowed to be non-simple or orientation-reversing, but each connected component of the complement of γ in S must be different from a disk.

It would be interesting to solve Problem 1 for the surface groups $\pi_1(S)$ and to find other solutions $g \in \pi_1(S)$ of Problem 2, as well as to obtain analogous results for other one-relator groups. Even for free groups there remain open questions, for instance, in [11] McCool wrote that "... no general result classifying elements of (relatively) small length in a normal closure of a single element of a free group is known, and such a result would be of great interest in the theory of one-relator groups". Here we obtain a direct approach to this problem for special one-relator groups (surface groups).

2 Proof of the main result

Let us first explain the notions used in Theorem 1.1 and Corollary 1.2. We consider a closed connected surface S, orientable or non-orientable. A closed curve γ on S either preserves the local orientation when moving along γ or reverses it; in the first case γ is called *orientation preserving*, otherwise *orientation reversing*. This is a property of the homotopy class of γ. If γ is simple then γ is *two-sided* when it preserves orientation, otherwise it is *one-sided*.

In the proof of Theorem 1.1 the following lemma will be used.

Lemma 2.1. *Let S be a compact surface with $k \geq 0$ boundary components. Let $\mathcal{N}$ be the smallest normal subgroup of $\pi_1(S)$ containing the homotopy classes of curves homotopic to a boundary component. Consider the covering $p\colon P \to S$ corresponding to the normal subgroup $\mathcal{N}$.*

(a) If $\chi(S) \leq -k$ then P is obtained from the plane by removing an infinite number of open disks which do not accumulate in the plane; if $\chi(S) = 1 - k$ (resp. $2-k$) then P is the sphere minus $2k$ (resp. k) disjoint open disks. The boundary of such a disk is homeomorphically mapped to a boundary curve of S.

(b) Every simple closed curve in int P *decomposes P into two components one of which is a disk with a finite number of removed open disks. (Both components*

are of this type iff $\chi(S) > -k$.*) The same is true for a simple path in* int P *with both endpoints on the same component of* ∂P.

Proof. From the restrictions on S and the properties of $\mathcal{N}$, in particular its normality, it follows that the group $\pi_1(S)/\mathcal{N}$ operates as the group $\mathcal{A}$ of covering transformations on P and that each lift of a boundary component of S is closed, that is, every boundary component of P is homeomorphically mapped to a boundary component of S.

Glue a disk into every boundary component of S and into every preimage of it. Then the covering can be extended to the new surfaces, and we obtain a covering $\hat{p}\colon \hat{P} \to \hat{S}$ where $\hat{S}$ is a closed surface with $\chi(\hat{S}) = \chi(S) + k \leq 2$. Since every closed path of P is mapped to a homotopy class which is a product of conjugates of the homotopy classes of curves homotopic to a boundary component, the fundamental group of $\hat{P}$ is trivial. Moreover we obtain the following commutative diagram of coverings and inclusions

$$\begin{array}{ccc} P & \xrightarrow{i_0} & \hat{P} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \hat{p}} \\ S & \xrightarrow[i]{} & \hat{S}. \end{array}$$

A consequence of the construction above is that $\hat{p}\colon \hat{P} \to \hat{S}$ is the universal cover of $\hat{S}$. By the restrictions on S, the surface $\hat{P}$ is either a sphere or a plane and, thus, P is obtained by removing discretely posed disks from the sphere or the plane. More precisely, if $\chi(S) = 2 - k$ then $\hat{S} = \hat{P}$ is the sphere, thus P is the sphere minus k disks; if $\chi(S) = 1 - k$ then $\hat{S}$ is the projective plane, thus P is the sphere minus $2k$ disks. Let $\chi(S) \leq -k$. Then $\chi(\hat{S}) \leq 0$, thus P is the plane minus infinitely many disks which are discretely distributed. By the Jordan curve theorem, every closed curve in $\hat{P}$ separates $\hat{P}$ such that one component is a disk D^2. Since the added disks do not accumulate in $\hat{P}$ it follows that D^2 contains only a finite number of added disks. This implies (b) in the case $\chi(S) \leq -k$. The similar assertion in the case $\chi(S) \geq 1 - k$ is obvious, since any simple curve in a sphere bounds two disks. □

Let us show that the first part of Theorem 1.1 implies the second part. More precisely, we will show that, under the hypothesis of Theorem 1.1, if h is conjugate to $(gug^{\eta}u^{-1})^{\varepsilon}$ then $\eta = 1$ if μ is one-sided and $\eta = -1$ elsewise, and χ is homotopic to the boundary of a regular neighbourhood of $\gamma \cup \mu$.

Let S be a closed surface, γ a simple closed curve on S, and $g \in \pi_1(S)$ its homotopy class. Assume that μ is another simple closed curve properly intersecting γ in exactly one point, in the basepoint. Let u be the homotopy class of μ. *If at least one of the curves* γ, μ *is two-sided then the boundary of a regular neighbourhood* U *of* $\gamma \cup \mu$ *is a simple closed curve from the homotopy class* $gug^{\eta}u^{\varepsilon}$ where $\varepsilon, \eta \in \{1, -1\}$. There are the following possibilities where the torus and the Klein bottle have a hole indicated by *:

γ	μ	ε	η	U	$gug^{\eta}u^{\varepsilon}$
two-sided	two-sided	-1	-1	torus*	$gug^{-1}u^{-1}$
two-sided	one-sided	-1	1	Klein bottle*	$gugu^{-1}$
one-sided	two-sided	1	-1	Klein bottle*	$gug^{-1}u$

If both curves are one-sided, the regular neighbourhood has two boundary curves and is a projective plane minus two disks. Let us remark that $\varepsilon = -1$, η satisfies the last assertion of Theorem 1.1, and $gug^{\eta}u^{\varepsilon}$ lies in the normal closure of g in both cases when γ is two-sided.

Suppose that, under the hypothesis of Theorem 1.1, h is conjugate to $(gug^{\eta}u^{-1})^{\varepsilon}$, where $\eta = 1$ if μ is one-sided and $\eta = -1$ elsewise. We may assume that $h = gug^{\eta}u^{-1}$. For each $x \in \pi_1(S)$ which vanishes in the group $H = H_1(S, \mathbb{Z}_2)$, we define the self-intersection index $\mu(x) \in \mathbb{Z}_2[H]/\mathbb{Z}_2[\{e\}]$ similarly to [16] or [4] where the ring of coefficients is $\mathbb{Z}_2 = \{0, 1\}$, the operation in H is multiplicatively written, and e is the neutral element of H. Namely, $\mu(x)$ is the formal sum of homology classes of loops corresponding to (transversal) self-intersection points of a closed curve ξ representing x; this group ring element does not depend on the choice of the curve ξ. Observe that h contains a closed curve $\chi'\colon [0, 1] \to S$ with a unique proper self-intersection point $\chi'(t_1) = \chi'(t_2)$, $0 \le t_1 < t_2 \le 1$ such that the loop $\chi'|_{[0,t_2]}\chi'|^{-1}_{[0,t_1]}$ is from the homotopy class g; this can be seen either geometrically or using the algebraic calculation from [4, Propositions 4.21, 4.24, 4.25]. Since the projection of h to H is trivial, but the projection $[g]$ of g to H is non-trivial, it follows that $\mu(h) = [g] + \mathbb{Z}_2[\{e\}] \neq 0$ in $\mathbb{Z}_2[H]/\mathbb{Z}_2[\{e\}]$; hence, the self-intersection point of χ' is "essential" and h does not contain a simple closed curve. The obtained contradiction shows that $\eta = 1$ if μ is one-sided and $\eta = -1$ elsewise. Hence, by the table above, the conjugacy class of h contains the boundary of a regular neighbourhood of $\gamma \cup \mu$. This proves that the first part implies the second one.

Proof of Theorem 1.1. By assumption, S is neither the sphere nor the projective plane. Denote by $\mathcal{N}$ the normal closure of g in $\pi_1(S)$. Let γ be a simple closed curve in S representing the conjugacy class of g.

Let U be the interior of a regular neighbourhood of γ and $\bar{U}$ be its closure. We assume that the basepoint lies on the boundary of U. Let $\tilde{S}$ be the covering of S corresponding to the subgroup $\mathcal{N}$. Consider two cases:

Case A: γ does not split S.

Step 1. Let us describe the structure of $\tilde{S}$. The complement of U in S is a connected compact surface with two boundary components, say α, β, where α contains the basepoint. Let $p\colon P \to S \setminus U$ be the covering corresponding to the subgroup of $\pi_1(S \setminus U)$ which is generated by the union of the conjugacy classes in $\pi_1(S \setminus U)$ corresponding to the free homotopy classes of α and β in $S \setminus U$. A boundary component of P and $\bar{U}$ will be colored by white if it is projected onto α and by black if it is projected onto β. By Lemma 2.1 (a) for $k = 2$, the covering space P can be considered as either a sphere minus 2 disjoint open disks (white and black), or a sphere minus 4 disjoint open disks (2 white and 2 black), or a plane minus infinitely many disjoint open disks which do nowhere accumulate, thus, it is a two-dimensional manifold with boundary. Let Γ be the universal covering of the wedge of one or two or infinitely many circles, respectively, in dependence on the type of P.

Now, take countably many copies of the covering P and countably many copies of the "tube" $\bar{U}$ each with a "projection" to P or $\bar{U}$, respectively. Glue them together along their boundary components into a connected surface R without boundary in such a way that the following conditions are satisfied (see Figure 2.1).

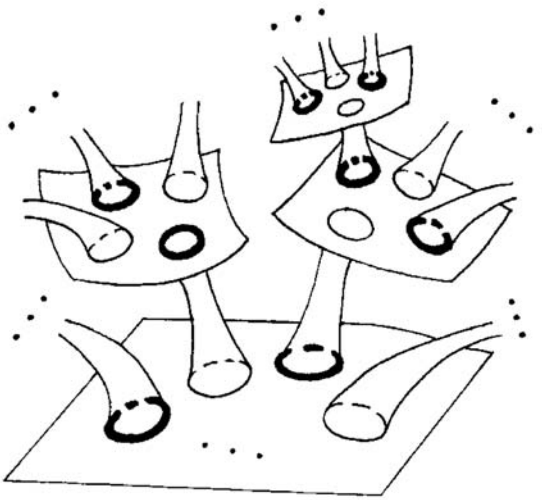

Fig. 2.1. The surface R for $\chi(S) \leq -2$

1) Glued boundary components have the same color.
2) Different points of each copy of P (respectively, $\bar{U}$) are not identified. Different copies of P (respectively, $\bar{U}$) in R are disjoint. The gluing of boundary components α_i, β_i of copies of $\bar{U}$ and P respects the projections to S.
3) After collapsing in R each copy of P into a point and each copy of $\bar{U} = S^1 \times [0, 1]$ into a segment $[0, 1]$ we obtain a tree which is isomorphic to Γ.

A consequence is that any copy of $\bar{U}$ connects exactly two copies of P.

The obtained space R is a connected surface which obviously covers S. We choose a basepoint in R over the basepoint of S, and denote by $\mathcal{H}$ the subgroup of $\pi_1(S)$ corresponding to this covering. We claim that $\mathcal{H} = \mathcal{N}$.

$\mathcal{H} \supset \mathcal{N}$: Generators of $\mathcal{N}$ have the form $[\sigma\alpha\sigma^{-1}]$ where σ is a closed curve in S starting at the basepoint of S. Any lift of the curve $\sigma\alpha\sigma^{-1}$ in R is closed, since any lift of the curve α in R is closed. Hence, $[\sigma\alpha\sigma^{-1}] \in \mathcal{H}$.

$\mathcal{H} \subset \mathcal{N}$: Consider a closed path σ in R starting at the basepoint. We may assume that σ has the following property: if σ enters a tube then σ leaves it at the other boundary component. The projection of σ into Γ is a closed path in the tree

Γ consisting of "full" edges. Therefore, it is either a point or admits a "peak", that is, it passes an edge and returns directly through the same edge. In the first case, σ lies in a copy of P; hence, by the construction of P, $[\sigma] \in \mathcal{N}$. In the second case, divide σ into three parts $\sigma_1, \sigma_2, \sigma_3$ such that the projection of σ_2 to Γ is the peak, that is, a vertex. So, σ_2 lies in one copy of P and its endpoints are on the same boundary component; thus we may deform σ such that the endpoints of σ_2 coincide. Now

$$[\sigma] = [\sigma_1\sigma_2\sigma_3] = [\sigma_1\sigma_2\sigma_1^{-1}][\sigma_1\sigma_3].$$

The first factor lies in $\mathcal{N}$, and the second is represented by a curve σ' whose projection into Γ is shorter. By induction, the claim follows.

Hence, we can identify R and $\tilde{S}$. In the following we will use the decomposition of $\tilde{S}$ into tubes and pieces of type P which are glued together respecting the structure of the tree Γ. In Step 3, Γ will be used to apply *peak-reduction arguments*.

Step 2. Consider a simple closed curve χ representing the conjugacy class of h. Without loss of generality, we may (and do) assume that χ transversally intersects the colored curves in a minimal number of points among all simple curves homotopic to χ. Actually we shall only use that there is no disk in S whose boundary can be divided into two parts one of them lies in χ and the other one in a colored curve.

The factor-group $\mathcal{A} = \pi_1(S)/\mathcal{N}$ acts freely on $\tilde{S}$ by homeomorphisms which project to the identity of S and transitively permute the preimages of the white curves. In other words, $\mathcal{A}$ is the group of covering transformations of the covering $\tilde{S} \to S$. In particular, $\mathcal{A}$ acts freely and transitively on the set of tubes and transitively on the set of copies of P, and preserves the coloring of the curves.

Let $\tilde{\chi}$ be a lifting of χ to $\tilde{S}$. It is a simple closed curve in $\tilde{S}$, since the conjugacy class of h is contained in $\mathcal{N}$. All liftings of χ to $\tilde{S}$ can be obtained by applying the covering transformations to $\tilde{\chi}$. They are pairwise disjoint simple closed curves.

Step 3. Let us show that $\tilde{\chi}$ has no intersections with the colored curves, i.e. with the liftings of α, β. Assume the contrary. Take the projection of $\tilde{\chi}$ into the graph Γ. If it is a vertex we go to Step 4. Suppose it is not a vertex. Since Γ is a tree there is a peak in this projection. The intersection of $\tilde{\chi}$ with the copy of P, say P_1, corresponding to the vertex of this peak contains a simple path $\tilde{\chi}_1 \subset \tilde{\chi}$ on P_1 with endpoints on the same boundary component which is, say, white; let us denote it by α_1. By Lemma 2.1 (b), $\tilde{\chi}_1$ divides P_1 into two pieces such that the closure D_1 of one piece is compact and therefore contains finitely many boundary components. We claim that they all are black. Otherwise we apply the covering transformation moving α_1 to a white boundary component in the interior of D_1. The image of D_1 under this transformation lies in P_1 and, thus, in the interior of D_1 (see Figure 2.2); hence, it contains less boundary components than D_1, a contradiction.

If P is compact then D_1 can be chosen by two ways, since we may replace D_1 by $\overline{P \setminus D_1}$. If P is a sphere minus 2 disjoint open disks (white and black), we can choose D_1 so that it does not contain colored curves, thus the number of intersection points of χ with colored curves is not minimal, a contradiction. If P is a sphere minus 4 disjoint open disks (2 white and 2 black), we can choose D_1 so that it contains a white curve, a contradiction with arguments from above.

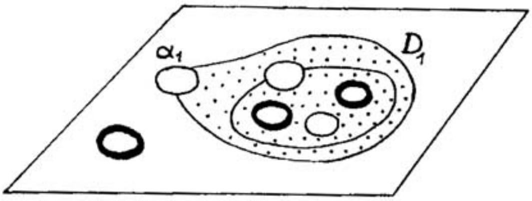

Fig. 2.2. The domain D_1 and its image D_2 under the covering transformation

So, we may assume that P is not compact and that D_1 contains only black curves. If there are several black curves in D_1, apply a covering transformation moving one of them to another one. The image of D_1 contains or is contained in D_1. Hence α_1 is mapped to itself. This is a contradiction, since the transformation is not the identity.

It remains to consider the situation where D_1 contains only one black curve β_1. Let U_1 be the tube with α_1 as a boundary component. Let β_2 be the other boundary component of U_1. Denote by P_2 the copy of P which contains β_2. Since β_2 is the black curve, there is a covering transformation which moves β_1 to β_2. Let D_2, α_2 and $\tilde{\chi}_2$ be the images of D_1, α_1 and $\tilde{\chi}_1$ under this transformation, respectively (see Figure 2.3). Since $\tilde{\chi}$ intersects α_1 at least twice, it passes U_1 and intersects β_2 at least twice. Since $\tilde{\chi}$ does not intersect $\tilde{\chi}_2$, at least two arcs of $\tilde{\chi} \cap P_2$ lie in D_2 and connect β_2 and α_2. Using induction we can prove that there exists an infinite sequence of pieces $P_1, P_2, \ldots$ (copies of P in $\tilde{S}$) and white and black curves α_i and β_i in P_i such that all P_i are different (since Γ is a tree), and $\tilde{\chi}$ intersects α_i and β_i in at least two points. This is a contradiction to the compactness of $\tilde{\chi}$.

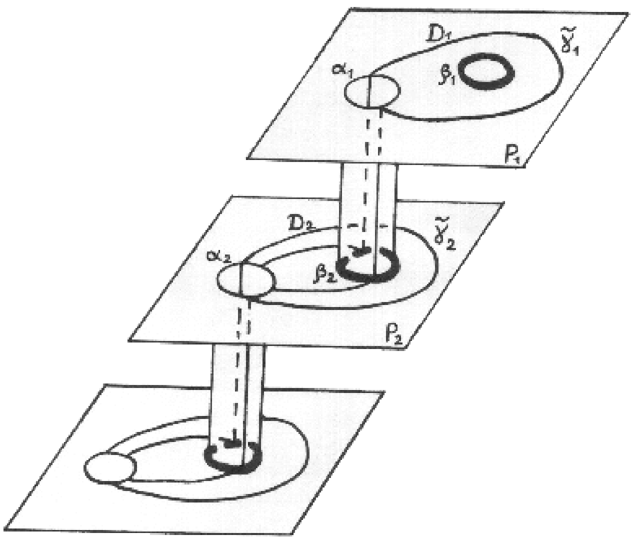

Fig. 2.3. The domain D_1 contains only one black curve β_1

Step 4. By Step 3, $\tilde{\chi}$ lies either in a copy of U or in a copy of P, say in P'. In the first case $\chi \subset U$ and, thus, χ is isotopic to γ or its inverse. This means that h is conjugate to g or g^{-1}.

Consider the second case $\tilde{\chi} \subset P'$. Let D be the closure of a component of $P' \setminus \tilde{\chi}$ containing a finite number of colored curves, see Lemma 2.1 (b). If D contains at least two curves of the same color, then there is a covering transformation moving one of them to the other. The image of D contains or is contained in D. It follows from Brouwer's fixed-point theorem [15, 11.1.2] that either there is a fixed point in the interior of D or some colored curve is invariant. This is a contradiction. If D contains exactly one white and one black curve then the projection of D to S is a regular neighbourhood of $\gamma \cup \mu$ where μ is a simple closed curve properly intersecting γ exactly ones. Therefore h is conjugate to $(gug^{\eta}u^{-1})^{\varepsilon}$ for some $\varepsilon, \eta \in \{1, -1\}$, where u is the homotopy class of μ. From the table above we get that $\eta = 1$ if and only if μ is one-sided.

If D contains exactly one closed colored curve then χ is isotopic to a colored curve and, thus, h is conjugate to g or g^{-1}, similarly to the case $\chi \subset U$.

Case B: γ splits S.

Step 1. Let us describe the covering surface $\tilde{S}$. Now $S \setminus U = A^{\alpha} \cup A^{\beta}$, the disjoint union of two compact surfaces A^{α} and A^{β}, each has one boundary component, say α and β, which are again colored by white and black, respectively, and $\chi(A^{\iota}) \leq 0$, $\iota \in \{\alpha, \beta\}$. Take P^{ι} as the covering of the surface A^{ι} corresponding to the normal subgroup of $\pi_1(A^{\iota})$ generated by the conjugacy class of $[\iota]$, $\iota \in \{\alpha, \beta\}$. By Lemma 2.1 (a) with $k = 1$, P^{ι} is either a cylinder or is obtained from a plane by removing an infinite number of open disks which do not accumulate in the plane (depending on whether A^{ι} is the Möbius band or not); all boundary components of P^{ι} have the same color, $\iota \in \{\alpha, \beta\}$. Similarly to Case A, the covering surface $\tilde{S}$ is a connected surface without boundary which is glued from infinitely many copies of P^{α}, P^{β} and the "tube" $\bar{U}$ in such a way that each tube connects a (white) boundary component of a copy of P^{α} and a (black) boundary component of some copy of P^{β}, and the graph Γ of the gluing is a tree.

Step 2 is the same as in Case A.

Step 3 is also as in Case A and even simpler. Namely, since the boundary components of a piece P_1 corresponding to a peak have the same color, say white, there is no need to consider the cases where D_1 contains black curves. Thus $\tilde{\chi}$ does not intersect colored curves and, hence, lies in a copy of P^{α}, P^{β} or of the tube $\bar{U}$.

Step 4. If $\chi \subset U$, we are done. If $\chi \subset S \setminus U$, define D as in Case A. By the above arguments, D contains at most one colored curve, and the result follows. □

Proof of Corollary 1.2. By Theorem 1.1, h is conjugate either to g^{ε} or to $(gug^{\eta}u^{-1})^{\varepsilon}$ where $u \in \pi_1(S)$ contains a simple closed curve μ which properly intersects γ exactly once, $\varepsilon, \eta \in \{1, -1\}$. If $h \sim g^{\varepsilon}$ we are done. Suppose that $h \sim (gug^{\eta}u^{-1})^{\varepsilon}$. Then h represents the trivial element of the group $H_1(S, \mathbb{Z}_2)$, thus each element of the normal closure of h also represents the trivial element of $H_1(S, \mathbb{Z}_2)$. On the other hand, it follows from the properties of μ that γ does not split S, thus g represents a non-trivial element of the group $H_1(S, \mathbb{Z}_2)$, a contradiction. □

Remark 2.2. It is possible to apply the above arguments to the case where γ is one-sided. However these arguments lead to a final answer only if S is the projective plane or the Klein bottle. For these surfaces, one obtains the following result: *If, under hypothesis of Theorem* 1.1*, γ is one-sided and h belongs to the normal closure of g, then h is conjugate to g^{ε} for some* $\varepsilon \in \{1, -1, 2, -2\}$. For arbitrary non-orientable surfaces, the steps 1, 2, and 4 can be done by arguments as in the proof of Theorem 1.1; however, in general they fail in Step 3. So, we are not yet able to prove an analog of Theorem 1.1 for a one-sided γ in the general case.

3 Applications to normal automorphisms

In this section we prove Theorem 1.3. For this we prove some lemmas. In the proof of Lemmas 3.3 and 3.6 we use that the fundamental group of a closed surface can be represented as an amalgamated free product of its free subgroups and that each subgroup of infinite index is free (these well known facts are obtained by topological arguments).

An automorphism of a group $\mathcal{G}$ is called *normal* (resp. *i-normal*) if it maps each normal subgroup of $\mathcal{G}$ onto itself (resp. into itself). Actually, we will prove the generalisation of Theorem 1.3 for i-normal automorphisms. We use the following notation for commutator: $[a, b] = aba^{-1}b^{-1}$. A closed surface of genus n will be denoted by T_n if it is orientable and by S_n if not. Furthermore we fix canonical presentations of their fundamental groups:

$$\pi_1(T_n) = \langle a_1, b_1, \dots, a_n, b_n \mid [a_1, b_1] \cdot \dots \cdot [a_n, b_n] \rangle,$$

$$\pi_1(S_n) = \langle c_1, \dots, c_n \mid c_1^2 \cdot \dots \cdot c_n^2 \rangle.$$

Note that the elements $a_i, b_i, a_ib_i, a_ia_{i+1}, b_ib_{i+1}$ are automorphic images of a_1 since their conjugacy classes can be represented by nonseparating simple closed curves on T_n. Analogously the elements c_i, $c_i^2c_{i+1}$, and $c_i^2c_{i+2}$ are automorphic images of c_1.

Let $\mathcal{G}$ be a group simplicically acting on a simplicial tree Γ. An element of $\mathcal{G}$ is called *elliptic* (with respect to this action) if it has a fixed vertex in Γ. The following lemma is proved in [14, Section I.6.5, Corollary 1].

Lemma 3.1. *Let $\mathcal{G}$ be a group simplicially acting on a simplicial tree Γ. If g_1 and g_2 are two elliptic elements in $\mathcal{G}$ then the element g_1g_2 is elliptic if and only if g_1 and g_2 have a common fixed vertex in Γ.* □

Lemma 3.2. *Let $\mathcal{G} = \mathcal{G}_1 *_{\mathcal{G}_3} \mathcal{G}_2$ where $\mathcal{G}_1$ is a free group. Let $a, b, ab \in \mathcal{G}_1 \setminus \bigcup_{g\in\mathcal{G}} g^{-1}\mathcal{G}_3 g$ and let a, b be powers of elements $a_1, b_1 \in \mathcal{G}_1$, respectively. Suppose that $\{a_1, b_1\}$ is a part of some free basis of $\mathcal{G}_1$ and*

$$x^{-1}ax \cdot y^{-1}by = z^{-1}abz \tag{3.1}$$

for some $x, y, z \in \mathcal{G}$. Then $xy^{-1} = a_1^t b_1^s$ for some $t, s \in \mathbb{Z}$. In particular,

$$x^{-1}ax = c^{-1}ac \text{ and } y^{-1}by = c^{-1}bc \text{ for } c = a_1^{-t}x.$$

Proof. Let Γ be the Bass–Serre tree associated to the decomposition $\mathcal{G} = \mathcal{G}_1 *_{\mathcal{G}_3} \mathcal{G}_2$ [14]. Remind that the vertices of Γ are in a one-to-one correspondence to the left cosets $g\mathcal{G}_i$ of $\mathcal{G}$ modulo $\mathcal{G}_i$, $i \in \{1, 2\}$, the edges of Γ are in a one-to-one correspondence to the left cosets $g\mathcal{G}_3$ of $\mathcal{G}$ modulo $\mathcal{G}_3$ and the incidences of edges and vertices respect inclusions of the corresponding left cosets. The group $\mathcal{G}$ acts on Γ by multiplication from the left.

Let v be the vertex of Γ corresponding to the coset $\mathcal{G}_1$. Then the elements $x^{-1}ax$ and $y^{-1}by$ stabilize the vertices $x^{-1}v$ and $y^{-1}v$, respectively and do not stabilize any other vertex (otherwise a or b stabilizes an edge and, hence, is conjugate to an element of $\mathcal{G}_3$). By Lemma 3.1, $x^{-1}v = y^{-1}v$; hence xy^{-1} stabilizes the vertex v and, thus, $xy^{-1} \in \mathcal{G}_1$. By analogy we obtain $xz^{-1} \in \mathcal{G}_1$. Now the lemma follows from an analysis in the free group $\mathcal{G}_1$ of the equality

$$a \cdot (xy^{-1})b(yx^{-1}) = (xz^{-1})ab(zx^{-1})$$

which is equivalent to (3.1). In fact, if xy^{-1} does not have the required form then, with respect to a free basis of $\mathcal{G}_1$ containing $\{a_1, b_1\}$, the cyclic syllable length of the left side of this equality is at least 4, whereas the right side has the cyclic syllable length 2. □

Lemma 3.3. *Let φ be an automorphism of $\pi_1(T_n)$ which sends each element a_i, b_i, a_ib_i, a_ia_{i+1}, b_ib_{i+1} to a conjugate of itself or of its inverse. If $\varphi(a_1)$ is conjugate to a_1 then φ is an inner automorphism. If $\varphi(a_1)$ is conjugate to a_1^{-1} then either $n = 1$ and φ sends each element of $\pi_1(T_1)$ to its inverse, or $n = 2$ and φ is the composition of an inner automorphism and φ_0 sending*

$$a_1 \mapsto b_1\, a_1^{-1}\, b_1^{-1}, \quad b_1 \mapsto b_1a_1\, b_1^{-1}\, a_1^{-1}b_1^{-1},$$

$$a_2 \mapsto a_2b_2\, a_2^{-1}\, b_2^{-1}a_2^{-1}, \quad b_2 \mapsto a_2\, b_2^{-1}\, a_2^{-1}.$$

Proof. The lemma is clear for $n \leqslant 1$. So suppose that $n \geqslant 2$. By assumption, φ sends each $x \in \{a_i, b_i, a_ib_i, a_ia_{i+1}, b_ib_{i+1}\}$ to a conjugate of x^{ε} where $\varepsilon = 1$ or -1. By abelianizing $\pi_1(T_n)$, it follows that ε is independent of x.

Consider the case $\varepsilon = 1$. Let $n = 2$. In this subcase

$$\pi_1(T_2) = \langle a_1, b_1 \mid\rangle \underset{[a_1,b_1]=[b_2,a_2]}{*} \langle a_2, b_2 \mid\rangle.$$

By Lemma 3.2, there are elements $v, u \in \pi_1(T_2)$ such that $\varphi(x) = v^{-1}xv$ if $x \in \langle a_1, b_1\rangle$ and $\varphi(x) = u^{-1}xu$ if $x \in \langle a_2, b_2\rangle$. Multiplying φ by an inner automorphism we may assume that $\varphi(a_1) = a_1, \varphi(b_1) = b_1$ and $\varphi(a_2) = u^{-1}a_2u$, $\varphi(b_2) = u^{-1}b_2u$ for some $u \in \pi_1(T_2)$. Since $[b_1, a_1] = [a_2, b_2]$, the element $[a_2, b_2]$ is fixed by φ. On the other hand, $\varphi([a_2, b_2]) = u^{-1}[a_2, b_2]u$. Hence $u = [a_2, b_2]^k$ for some $k \in \mathbb{Z}$ (this follows, for example, from the normal form theorem for amalgamated free products, see [8, Chapter IV, Theorem 2.6]). By assumption $a_1 \cdot [a_2, b_2]^{-k}a_2[a_2, b_2]^k$ is conjugate to a_1a_2. By the conjugacy theorem for amalgamated free products (see [8, Chapter IV, Theorem 2.8]), this is possible only if $k = 0$ and, thus, $\varphi = \mathrm{id}$.

Let $n \geqslant 3$. Again by Lemma 3.2, we may assume that $\varphi(a_1) = a_1$, $\varphi(b_1) = b_1$ and $\varphi(a_i) = u_i^{-1} a_i u_i$, $\varphi(b_i) = u_i^{-1} b_i u_i$ for some u_i, $i = 2, \dots, n$. Consider the following decomposition

$$\pi_1(T_n) = \langle a_1, b_1, a_2, b_2 \mid \rangle \underset{([a_1,b_1][a_2,b_2])^{-1}=\prod_{i=3}^{n}[a_i,b_i]}{*} \langle a_3, b_3, \dots, a_n, b_n \mid \rangle.$$

By the conditions of Lemma 3.3, we have the equality $a_1 \cdot u_2^{-1} a_2 u_2 = v^{-1} a_1 a_2 v$ for some v. By Lemma 3.2, $u_2 \in \langle a_1, a_2 \rangle$. If we consider the pair (b_1, b_2) instead of the pair (a_1, a_2), we obtain $u_2 \in \langle b_1, b_2 \rangle$. Hence $u_2 = 1$. By analogy, $u_i = 1$ for each $i = 2, \dots, n$.

Consider the case $\varepsilon = -1$. The subcase $n \geqslant 3$ can be considered as above and we get that $\varphi(a_i) = a_i^{-1}$ and $\varphi(b_i) = b_i^{-1}$ for every i. But

$$\prod_{i=1}^{n} [a_i^{-1}, b_i^{-1}] \neq 1 \quad \text{in } T_n \text{ for } n \geqslant 3,$$

as follows, for instance, from Dehn's solution of the word problem or by a computation in the amalgamated free product.

Now consider the subcase $n = 2$. As above we may assume that

$$\varphi(a_1) = a_1^{-1},\ \varphi(b_1) = b_1^{-1},\ \varphi(a_2) = u^{-1} a_2^{-1} u,\ \varphi(b_2) = u^{-1} b_2^{-1} u. \tag{3.2}$$

Applying φ to the equality $[b_1, a_1] = [a_2, b_2]$ we obtain

$$[b_1^{-1}, a_1^{-1}] = u^{-1} [a_2^{-1}, b_2^{-1}] u$$

which implies that $[a_2, b_2] = w^{-1} [a_2, b_2] w$ for $w = a_2 b_2 u b_1^{-1} a_1^{-1}$. As above $w = [a_2, b_2]^k$ for some $k \in \mathbb{Z}$; hence

$$u = b_2^{-1} a_2^{-1} [b_1, a_1]^k a_1 b_1. \tag{3.3}$$

Next we use the reduction to the case $\varepsilon = 1$. Since φ^2 sends each element $a_i, b_i, a_i b_i$ to a conjugate of itself we get that φ^2 is an inner automorphism. It follows from (3.2) that φ^2 is the identity and $\varphi(u) = u^{-1}$. From this, (3.2), and (3.3) it directly follows that $[b_1, a_1]^{2k-2} = 1$ and, thus, $k = 1$. Therefore φ is the composition of an inner automorphism and φ_0. □

Lemma 3.4. *The automorphism φ_0 of $\pi_1(T_2)$ from Lemma* 3.3 *is not normal.*

Proof. We will show that the normal closure $\mathcal{N}$ of $a_1 a_2^{-1}$ is not invariant under φ_0. The factor group $\pi_1(T_2)/\mathcal{N}$ is an amalgamated free product:

$$\begin{aligned}\pi_1(T_2)/\mathcal{N} &= \langle \tilde{a}_1, \tilde{b}_1, \tilde{a}_2, \tilde{b}_2 \mid [\tilde{a}_1, \tilde{b}_1][\tilde{a}_2, \tilde{b}_2], \tilde{a}_1 \tilde{a}_2^{-1} \rangle \\ &= \langle \tilde{a}_1, \tilde{b}_1 \mid \rangle \underset{\substack{[\tilde{a}_1, \tilde{b}_1] = [\tilde{b}_2, \tilde{a}_2] \\ \tilde{a}_1 = \tilde{a}_2}}{*} \langle \tilde{a}_2, \tilde{b}_2 \mid \rangle\end{aligned}$$

since $[\tilde{a}_1, \tilde{b}_1]$ and $\tilde{a}_1$ generate a free subgroup of rank 2 in the left factor, and $[\tilde{b}_2, \tilde{a}_2]$ and $\tilde{a}_2$ generate a free subgroup of rank 2 in the right factor. Let $p\colon \pi_1(T_2) \to$

$\pi_1(T_2)/\mathcal{N}$ be the natural projection $a_i \mapsto \tilde{a}_i$, $b_i \mapsto \tilde{b}_i$. Then the element $p \circ \varphi_0(a_1 a_2^{-1})$ of $\pi_1(T_2)/\mathcal{N}$ is not trivial:

$$p \circ \varphi_0(a_1 a_2^{-1}) = p(b_1 a_1^{-1} b_1^{-1}\, a_2 b_2\, a_2\, b_2^{-1} a_2^{-1}) = p(b_1 a_1^{-1} b_1^{-1} a_1 b_2\, a_2\, b_2^{-1} a_2^{-1})$$
$$= p(b_1 a_1^{-1} b_1^{-1}\, a_1 \cdot a_1\, b_1\, a_1^{-1} b_1^{-1}) \neq 1.$$

Therefore $\varphi_0(a_1 a_2^{-1}) \notin \mathcal{N}$; hence, φ_0 is not normal. □

Remark 3.5. Consider the surface S_n, $n \geqslant 2$. Let $x \in \{c_i, c_i^2 c_{i+1}, c_i^2 c_{i+2}\}$ and let an automorphism φ of $\pi_1(S_n)$ map the normal closure of x^2 into itself. The conjugacy class of x^2 (but not x) can be represented by a two-sided simple closed curve. Hence, by Theorem 1.1, $\varphi(x^2)$ is conjugate to x^2 or to x^{-2}. An element r of a group $\mathcal{G}$ is called a *primitive root* of $x \in \mathcal{G}$ if $x = r^k$ for some $k \geq 1$, and if an equality $x = y^l$, $y \in \mathcal{G}$, $l > 0$ implies $l \leq k$. For free groups and surface groups – except the fundamental group of the Klein bottle – the centralizer of any non-trivial element is cyclic: this follows for free groups from the solution of the word problem and for surface groups from the presentation as a group of linear fractional transformations of the upper half plane $\{x + iy \mid y > 0\}$ where each non-trivial transformation is hyperbolic and, thus, admits a unique invariant axis. Actually, this fact is even valid for torsion-free hyperbolic groups (see [1, Ch. III Γ, Cor. 3.10] and [17, Lemma 3.2]). Hence, the primitive root of any element is uniquely determined. From this it follows that $\varphi(x)$ is conjugate to x or to x^{-1} if $n \geq 3$. For $n = 2$, suppose that φ maps the normal closure $\mathcal{N}$ of x into itself. By simple arguments it can be shown that each automorphism of $\pi_1(S_2)$ sends x to a conjugate of $x^{\pm 1}$ or $(c_1 c_2 x)^{\pm 1}$. Since $c_1 c_2 x \notin \mathcal{N}$, it follows that $\varphi(x)$ is conjugate to x or x^{-1}.

Lemma 3.6. *Let φ be an automorphism of $\pi_1(S_n)$, $n \geq 2$ which sends each element c_i, $c_i^2 c_{i+1}$, and $c_i^2 c_{i+2}$ to a conjugate of itself or of its inverse. If $\varphi(c_1)$ is conjugate to c_1 then φ is an inner automorphism. If $\varphi(c_1)$ is conjugate to c_1^{-1} then $n \in \{2, 3\}$ and φ is the composition of an inner automorphism and φ_n with $\varphi_n(c_1) = c_1^{-1}$, $\varphi_n(c_2) = c_2^{-1}$, $\varphi_3(c_3) = c_1^{-2} c_3^{-1} c_1^2$.*

Proof. The proof of this lemma is similar to the proof of Lemma 3.3. Again, by assumption, φ sends each $x \in \{c_i, c_i^2 c_{i+1}\}$ to a conjugate of x^{ε} where $\varepsilon = 1$ or -1 and, by abelianization, it follows that ε is independent of x. Consider the case $\varepsilon = 1$. Let $n = 2$. We may assume that $\varphi(c_1) = c_1$. It can be proved by a direct calculation that $c_1 c_2$ is the unique primitive root of $(c_1 c_2)^2$. Since $(c_1 c_2)^2$ generates the commutator subgroup of $\pi_1(S)$, the cyclic subgroup generated by $c_1 c_2$ is characteristic. It follows that $\varphi(c_1 c_2) = c_1 c_2$ or $\varphi(c_1 c_2) = c_2 c_1 = c_1^{-1}(c_1 c_2) c_1$. In both cases φ is an inner automorphism. Let $n = 3$. In this subcase

$$\pi_1(S_3) = \langle c_1, c_2 \mid \rangle \underset{c_1^2 c_2^2 = c_3^{-2}}{*} \langle c_3 \mid \rangle.$$

By Lemma 3.2, it follows that, multiplying φ by an inner automorphism, we may assume that $\varphi(c_1) = c_1$, $\varphi(c_2) = c_2$ and $\varphi(c_3) = u^{-1} c_3 u$ for some $u \in \pi_1(S_3)$. Since $c_1^2 c_2^2 = c_3^{-2}$, the element c_3^2 is fixed by φ. Hence the element c_3 is fixed by φ. Therefore $\varphi = \mathrm{id}$.

Let $n \geqslant 4$. Again by Lemma 3.2 we may assume that $\varphi(c_1) = c_1$, $\varphi(c_2) = c_2$ and $\varphi(c_i) = u_i^{-1} c_i u_i$ for $i = 3, \dots, n$. Consider the following decomposition:

$$\pi_1(S_n) = \langle c_1, c_2, c_3 \mid \rangle \underset{c_1^2 c_2^2 c_3^2 = c_n^{-2} \dots c_4^{-2}}{*} \langle c_4, \dots, c_n \mid \rangle.$$

By hypothesis, $c_1^2 \cdot u_3^{-1} c_3 u_3 = v^{-1} c_1^2 c_3 v$ for some v. By Lemma 3.2, $u_3 \in \langle c_1, c_3 \rangle$. Analogously, $c_2^2 \cdot u_3^{-1} c_3 u_3 = w^{-1} c_2^2 c_3 w$ for some w, and we get that $u_3 \in \langle c_2, c_3 \rangle$. Hence $u_3 \in \langle c_3 \rangle$ and we may assume that $u_3 = 1$. In a similar way it follows that $u_i = 1$ for $i = 3, \dots, n$.

Consider the case $\varepsilon = -1$. The subcase $n \geqslant 4$ can be considered as above and we get that $\varphi(c_i) = c_i^{-1}$ for each i. This is impossible. For $n = 2, 3$, $\varphi\varphi_n^{-1}$ is an inner automorphism because of the case $\varepsilon = 1$. □

Lemma 3.7. *Let φ_2, φ_3 be the automorphisms from Lemma* 3.6. *Then φ_2 is normal, and φ_3 is not.*

Proof. To show that φ_2 is i-normal, it is enough to prove that any element $g \in \pi_1(S_2)$ is sent to a conjugate of its inverse. Since g can be written in the form $g = c_1^k (c_1 c_2)^\ell$, we have

$$\begin{aligned} \varphi_2(g) &= c_1^{-k} (c_1^{-1} c_2^{-1})^\ell = c_1^{-k-1} (c_1 c_2)^{-\ell} c_1 \\ &= c_1^{-k-1} \cdot (c_1 c_2)^{-\ell} c_1^{-k} \cdot c_1^{k+1} = c_1^{-k-1} g^{-1} c_1^{k+1}. \end{aligned}$$

Hence $\varphi_2(g)$ is conjugate to g^{-1}; thus φ_2 is i-normal. Actually φ_2 is normal.

Now we will show that the i-normal closure of $c_1 c_3 c_2$ is not invariant under φ_3. What we need is to show that the element $\varphi_3(c_1 c_3 c_2) = c_1^{-3} c_3^{-1} c_1^2 c_2^{-1}$ is nontrivial in the group $\mathcal{G} = \langle\, c_1, c_2, c_3 \mid c_1^2 c_2^2 c_3^2,\ c_1 c_3 c_2 \,\rangle$. Substituting $c_3 = c_1^{-1} c_2^{-1}$ and applying Tietze transformation we have to show that $w = c_1^{-3} c_2 c_1^3 c_2^{-1}$ is nontrivial in $\mathcal{G} = \langle\, c_1, c_2 \mid c_1^2 c_2^2 c_1^{-1} c_2^{-1} c_1^{-1} c_2^{-1} \,\rangle$. Next we will follow the approach of Magnus to the solution of the word problem for one-relator groups [10]. Note that $c_1^2 c_2^2 c_1^{-1} c_2^{-1} c_1^{-1} c_2^{-1} = c_1^2 \cdot c_2^2 c_1^{-1} c_2^{-2} \cdot c_2 c_1^{-1} c_2^{-1}$. Using the notations $t = c_2$, $b_i = c_2^i c_1 c_2^{-i}$ it is easy to see that $\mathcal{G} = \langle\, t, b_0, b_1, b_2 \mid b_0^2 b_2^{-1} b_1^{-1},\ t b_0 t^{-1} = b_1,\ t b_1 t^{-1} = b_2 \,\rangle$ is an HNN-extension with the base $\mathcal{H} = \langle\, b_0, b_1, b_2 \mid b_0^2 b_2^{-1} b_1^{-1} \,\rangle = \langle\, b_0, b_1 \mid \,\rangle$ and associated subgroups $\langle\, b_0, b_1 \rangle$ and $\langle\, b_1, b_2 \rangle = \langle\, b_0^2, b_1 \rangle$. It remains to note that $w = b_0^{-3} b_1^3 \in \mathcal{H}$ is nontrivial in $\mathcal{H}$ and, hence, in $\mathcal{G}$. □

Proof of Theorem 1.3. Let g be a non-trivial element of $\pi_1(S)$ containing a simple closed two-sided curve in S. Let φ be a normal automorphism of $\pi_1(S)$. Then $\varphi(g)$ lies in the normal closure of g. It follows from Theorem 1.1 and abelianization that $\varphi(g)$ is conjugate to g or to g^{-1}. Now Theorem 1.3 follows from Lemmas 3.3, 3.4 in the orientable case, and from Remark 3.5 and Lemmas 3.6, 3.7 in the non-orientable case. □

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