INTRODUCTION TO TRAIN TRACKS

The train track technique invented by Bestvina and Handel in 1992, enables to solve difficult problems on $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$.

- Tits alternative for $\mathrm{Out}(F_n)$. (Bestvina, Feighn, Handel)
- Scott problem: $\mathrm{rk}(\mathrm{Fix}(\alpha)) \leqslant n$. (Bestvina, Handel)
- Computing a basis of $\mathrm{Fix}(\alpha)$. (Bogopolski, Maslakova)
- The conjugacy problem for $\mathrm{Aut}(F_n)$???

A difficulty in studying of automorphisms $\varphi : F_n \to F_n$

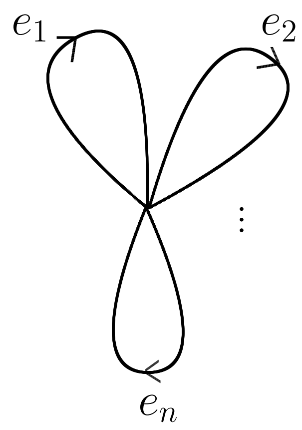
Unpredictable cancelations in computing of $x, \varphi(x), \varphi^2(x), \ldots$

$$\varphi : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_3 x_1^{-1} \end{cases}$$

$$\begin{aligned} x_3 &\mapsto x_3 x_1^{-1} \\ &\mapsto x_3 x_1^{-1} x_2^{-1} \\ &\mapsto x_3 x_1^{-1} x_2^{-1} x_3^{-1} \\ &\mapsto x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_1 x_3^{-1} \\ &\mapsto x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_1 x_3^{-1} x_2 x_1 x_3^{-1} \\ &\mapsto x_3 x_1^{-1} x_2^{-1} x_3^{-1} x_1 x_3^{-1} x_2 x_1 x_3^{-1} \cdot x_3 x_2 x_1 x_3^{-1}. \end{aligned}$$

A rose with n petals

$\mathcal{R}_n$:



$$\pi_1(\mathcal{R}_n) \cong F(x_1, \ldots, x_n)$$
$$[e_i] \mapsto x_i$$

The standard topological representative of $\alpha \in \mathrm{Aut}(F_n)$

$$F_n \xrightarrow{\alpha} F_n \qquad x_i \mapsto w_i(x_1, \ldots, x_n)$$

$$\mathcal{R}_n \xrightarrow{\alpha_{st}} \mathcal{R}_n \qquad e_i \mapsto w_i(e_1, \ldots, e_n)$$

A topological representative of $\alpha \in \mathrm{Aut}(F_n)$

Definition. Let Γ be a finite connected graph. A homotopy equivalence $f : \Gamma \to \Gamma$ is called a *topological representative* of α if

- $f(\Gamma^0) \subseteq \Gamma^0$,
- f is locally injective on the interior of each edge of Γ,
- there exist two "mutually inverse" homotopy equivalences $\tau : \mathcal{R}_n \to \Gamma$ and $\sigma : \Gamma \to \mathcal{R}_n$ such that the following diagram is commutative.

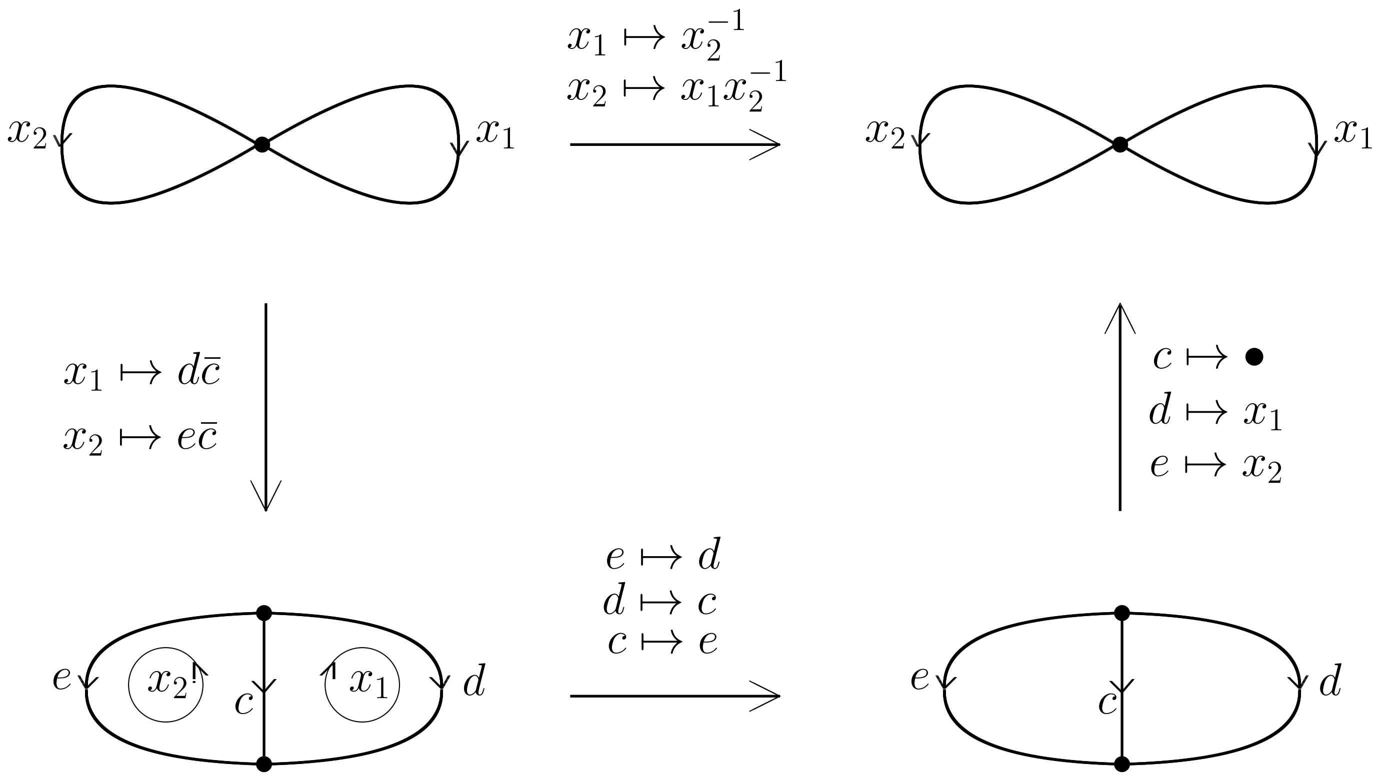
$$\begin{array}{ccc} \mathcal{R}_n & \xrightarrow{\alpha_{st}} & \mathcal{R}_n \\ \sigma \downarrow & & \uparrow \tau \\ \Gamma & \xrightarrow{f} & \Gamma \end{array}$$

We want to find a good topological representative $f : \Gamma \to \Gamma$ of α, i.e. such that, for each edge E of Γ, the paths $E, f(E), f^2(E), \ldots$ are reduced.

Example

$$\varphi : \begin{cases} x_1 \mapsto x_2^{-1} \\ x_2 \mapsto x_1 x_2^{-1} \end{cases}$$

$$x_2 \mapsto x_1 x_2^{-1} \mapsto x_2^{-1} \cdot x_2 x_1^{-1}.$$



Irreducible outer automorphisms

Definition. An automorphism $\mathcal{O} \in \mathrm{Out}(F_n)$ is *irreducible* if one of the following equivalent conditions is satisfied:

Geo.

Each topological representative $f : \Gamma \to \Gamma$ of $\mathcal{O}$
which has no hanging edges,
has no proper f-invariant subgraphs different from forests.

Alg.

There does not exist a decomposition of F_n of the form

$$F_n \neq H_1 * H_2 * \cdots * H_{\underset{\geqslant 1}{k}} * L$$

where $1 < H_i < F_n$, and $\mathcal{O}([H_i]) = [H_{i+1}] \bmod k$.

A sufficient condition for the irreducibility of $\mathcal{O}$

If
- the char. pol. of $\mathcal{O}_{ab} : \mathbb{Z}^n \to \mathbb{Z}^n$ is irreducible over $\mathbb{Z}$ and
- $\mathrm{Trace}(\mathcal{O}_{ab}) \neq 0$,

then $\mathcal{O}$ is irreducible.

Example. $[\phi], [\psi]$ are irreducible, $[\theta]$ is reducible.

$$\varphi : \begin{cases} x_1 \mapsto x_2^{-1} \\ x_2 \mapsto x_1 x_2^{-1} \end{cases} \qquad \psi : \begin{cases} x_1 \mapsto x_2 \\ x_2 \mapsto x_3 \\ x_3 \mapsto x_3 x_1^{-1} \end{cases} \qquad \theta : \begin{cases} x_1 \mapsto x_1 \\ x_2 \mapsto x_1 x_2 \end{cases}$$

Train tracks

Definition. A topological representative $f : \Gamma \to \Gamma$ of $\mathcal{O}$ is called *a train track* for $\mathcal{O}$ if, for each edge E of Γ, all the paths $E, f(E), f^2(E), \ldots$ are reduced.

Theorem (Bestvina, Handel; 92). Each irreducible automorphism $\mathcal{O} \in \mathrm{Out}(F_n)$ *has* a topological representative which is a train track map.

Remark. Given a topological representative $f : \Gamma \to \Gamma$, one can verify whether f is a train track or not.

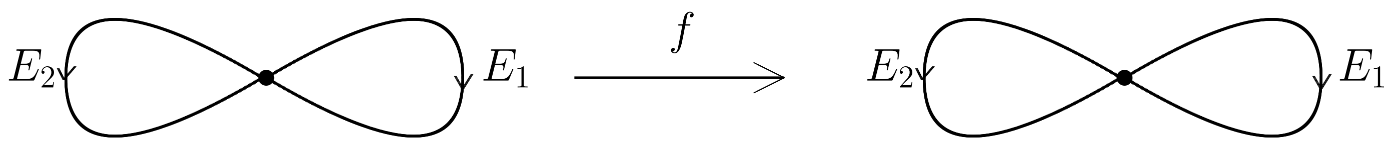
Transition matrix of the map $f : \Gamma \to \Gamma$

From each pair of mutually inverse edges of Γ we choose one edge. Let $\{E_1, \dots, E_k\}$ be the set of chosen edges.

The *transition matrix* of the map $f : \Gamma \to \Gamma$ is the matrix $M(f)$ of size $k \times k$ such that the ij^{th} entry of $M(f)$ is equal to the total number of occurrences of E_i and $\overline{E_i}$ in the path $f(E_j)$.

Ex.:

$$E_1 \mapsto \overline{E}_2$$
$$E_2 \mapsto E_1\overline{E}_2$$



$$M(f) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\mathrm{PF}(f) = \frac{1+\sqrt{5}}{2}$$

Theorem (Frobenius). If $M \geqslant 0$ is a nonzero irreducible integer matrix, then $\exists\, \vec{v} > 0$ and $\lambda \geqslant 1$ such that $M\vec{v} = \lambda\vec{v}$.

If $\lambda = 1$, then M is a permutation matrix.
v is unique up to a positive factor.
$\lambda =$ max of absolute values of eigenvalues of M

Remark. If $f : \Gamma \to \Gamma$ is an irreducible topological representative, Γ does not have hanging edges and nontrivial invariant forests, then $M(f)$ is irreducible.

The idea of the proof of Bestvina – Handel Theorem

A topological representative $f : \Gamma \to \Gamma$ is called *good* if $M(f)$ is irreducible and the $deg(v) \geqslant 3$ for each vertex of $v \in \Gamma^0$.

Note, if $rk(\Gamma) = n$, then the number of edges in Γ is at most $3n - 3$. In particular, $size(M(f)) \leqslant 3n - 3$.

$$\begin{array}{ccc}
\mathcal{R}_n & \xrightarrow{f_0 = \alpha_{st}} & \mathcal{R}_n \\
\sigma_1 \downarrow & & \uparrow \tau_1 \\
\Gamma_1 & \xrightarrow{f_1} & \Gamma_1 \\
\sigma_2 \downarrow & & \uparrow \tau_2 \\
\Gamma_2 & \xrightarrow{f_2} & \Gamma_2 \\
\vdots & & \vdots
\end{array}$$

Let $\alpha \in \mathrm{Aut}(F_n)$ be irreducible. In particular, f_0 is good.

The algorithm constructs a tower of topological representatives as above such that

- $\mathrm{PF}(f_0) \geqslant \mathrm{PF}(f_1) \geqslant \mathrm{PF}(f_2) \geqslant \ldots$,
- if f_i is not a train track, but f_i is good, then $f_{i+\ell}$ is also good and $\mathrm{PF}(f_i) > \mathrm{PF}(f_{i+\ell})$ for some universally bounded $\ell = \ell(i)$.

Clearly, if the algorithm stops on f_k, then f_k is a train track.

The algorithm stops, since, for every $\lambda > 0$, there is only finitely many nonnegative integer irreducible matrices M of bounded size ($\leqslant 3n - 3$) with $\mathrm{PF}(M) < \lambda$.

TOOLS

Transformations $f \to f_1$

<table>
<tr><td>1. tightening

2. collapsing f-invar. forest</td><td colspan="2">$M(f)$ and $M(f_1)$ are irr. $\Longrightarrow PF(f_1) < PF(f)$</td></tr>
<tr><td>3. subdivision

4. folding</td><td colspan="2">$M(P)$ irr. $\Longrightarrow M(f_1)$ irr. $\Longrightarrow PF(f_1) = PF(f)$</td></tr>
<tr><td>5. valency-one homotopy
+ cleaning</td><td rowspan="2">$M(f)$ and $M(f_1)$ are irr. $\Longrightarrow$</td><td>$PF(f_1) < PF(f)$</td></tr>
<tr><td>6. valency-two homotopy
+ cleaning</td><td>$PF(f_1) \leqslant PF(f)$</td></tr>
</table>

Cleaning means tightening and collapsing of the maximal f-invariant forest.

Perron-Frobenius metric on Γ

Let $f : \Gamma \to \Gamma$ be a topological representative of $\mathcal{O}$. We enumerate the edges $e_1, \dots, e_m$ of Γ and construct the transition matrix $M(f)$.

Suppose that $M(f)$ is irreducible.

Then $\vec{v}M(f) = \lambda\vec{v}$ for $\lambda = \mathrm{PF}(f)$ and some $\vec{v} = (v_1, \dots, v_m) > 0$. We define the PF-metric on Γ by $L(e_i) = v_i$.

Then $L(f(\rho)) = \lambda L(\rho)$ for any path ρ in Γ.