A surface groups analogue of a theorem of Magnus

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Abstract. We prove that the fundamental group G of a closed orientable surface possesses the Magnus property. This means that if u, v are two elements in G with the same normal closure, then u is conjugate to v or v^{-1}. We show that the Magnus property does not hold for generalized Baumslag-Solitar groups, for noncyclic one-relator groups with torsion, and for infinitely many one-relator torsion-free hyperbolic groups.

Introduction

In 1930 W. Magnus published a very important (for combinatorial group theory and logic) article where he proved the so-called *Freiheitssatz* and the following

Theorem 0.1 ([**M**]). *Let F be a free group and $u, v \in F$. If the normal closures of u and v coincide, then u is conjugate to v or v^{-1}.*

The main result of the present article is the following

Theorem 1.2. *Let G be the fundamental group of a closed orientable surface and $r, s \in G$. If the normal closures of r and s coincide, then r is conjugate to s or s^{-1}.*

We will say that a group G possesses *the Magnus property*, if for any two elements u, v of G with the same normal closures we have that u is conjugate to v or v^{-1}. So, the fundamental group of a compact orientable surface possesses the Magnus property. The following theorem shows that the Magnus property does not hold for many one-relator groups, including generalized Baumslag-Solitar groups, noncyclic one-relator groups with torsion, and infinitely many one-relator torsion-free hyperbolic groups.

Theorem 2.4. *The Magnus property does not hold for the following one-relator groups:*

(1) $G = \langle a, b \,|\, a^k b^n \rangle$, *where* $|k| \geqslant 2, |n| \geqslant 2$ *and* $(|k|, |n|) \neq (2, 2)$,
(2) $G = \langle a, b \,|\, a^{-k} b^n a^k b^{-m} \rangle$ *where* k, n, m *are nonzero and not all equal to* ± 1,
(3) $G = \langle X \,|\, R^n \rangle$, *where* $|X| > 1$, $n \geqslant 2$,

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(4) $G_n = \langle a, b \,|\, b^{-1}ab^2a^n\rangle$, *where* $n \geqslant 3$. *The groups* G_n $(n \geqslant 3)$ *are pairwise non-isomorphic torsion-free hyperbolic.*

Remark 0.2. The groups $\langle a, b \,|\, a^{-1}ba = b^n\rangle$, where $n \in \{-1, 0, 1\}$, possess the Magnus property. The groups from (1) and (2) are not hyperbolic. The groups from (3) have torsion, and they are hyperbolic since they satisfy the Dehn property by Newman's Spelling Theorem [**N**]. For $n \geqslant 3$ they satisfy the small cancellation condition $C(2n)$ by [**P**]. The groups $G_n = \langle a, b \,|\, b^{-1}ab^2a^n\rangle$ are torsion-free and hyperbolic for $n \notin \{-2, -1, 0, 1\}$ by [**IS**, Theorem 3].

The normal closure of a subset X of a group G will be denoted by $\langle\langle X\rangle\rangle$.

Problems. 1) There are three traditional algorithmic problems in group theory: the (generalized) word problem, the conjugacy problem, and the isomorphism problem. We suggest the following *normal closure problem.* Given a group G and two elements $g, h \in G$, decide, whether the normal closures of g and h coincide. Theorem 0.1 decides this problem for free groups, Theorem 1.2 for fundamental groups of closed orientable surfaces. Is this problem algorithmically decidable

a) for (indecomposable into free constructions) torsion-free hyperbolic groups,

b) for one-relator groups?

Note, that if the word problem for two-relator groups is decidable, then the normal closure problem for one-relator groups is decidable also. We do not know even, whether the word problem for surface groups factorized by one relation is decidable. The normal closure problem is undecidable in the class of all finitely presented groups. Indeed, if we take a finitely presented group G with an unsolvable word problem then in G it is undecidable if the normal closure of a word u is equal to the normal closure of 1.

2) Let U, V, R be three words in a free group F. Suppose that $\langle\langle R, U\rangle\rangle = \langle\langle R, V\rangle\rangle$. What can be said on these words? Theorem 2.4 shows, that U is not necessarily conjugate to $V^{\pm 1}$ modulo R (that is in $F/\langle\langle R\rangle\rangle$).

In the case $\langle\langle R, U\rangle\rangle = \langle\langle R, V\rangle\rangle = F$ the Andrews-Curtis conjecture states that $\{R, U\}$ is AC-equivalent to $\{R, V\}$. The following example of this type is considered in [**MMS**, Proposition 1.1]: the presentations $\langle x, y \,|\, x^{-1}y^2x = y^3, x^2 = yxy^{-1}\rangle$ and $\langle x, y \,|\, x^{-1}y^2x = y^3, x^2 = yxy\rangle$ define the trivial group and they are AC-equivalent.

3) Which one-relator groups satisfy the Magnus property?

Note some earlier related results.

1) In [**G**] M. Greendlinger proved that if two subsets U and V of a free group satisfy some metric small cancellation conditions and have the same normal closure, then there is a bijection $\varphi : U \to V$ such that u is conjugate to $\varphi(u)$ or $\varphi(u)^{-1}$ for each $u \in U$. This result was generalized by E. V. Kashintsev in [**K1, K2**] and by M. Palasinski in [**Pal**]. An interesting and apparently nontrivial open problem is: if two symmetrized, say, $C(8)$-sets in a free group with the same normal closure must be equal?

2) In [**E**] M. Edjvet proved the following result. Let $G = A * B$, where A and B are non-trivial locally indicable groups. If $u, v \in G$ are cyclically reduced words of length at least 2, and if the normal closures of u and v coincide, then u is conjugate to v or v^{-1}.

3) In [**BKZ**] O. Bogopolski, E. Kudrjavtseva and H. Zieschang proved a partial result, related to Theorem 1.2. Let T be a closed surface and g, h be non-trivial elements of $\pi_1(T)$ both containing simple closed two-sided curves. If the normal closures of g and h coincide, then h is conjugate to g or g^{-1}.

As a corollary the following result on normal automorphisms was proved in [**BKZ**]. (Remind, that an automorphism of a group G is called *normal* if it maps each normal subgroup of G into itself.) If T is a closed surface different from the torus and the Klein bottle, then every normal automorphism of $\pi_1(T)$ is an inner automorphism.

Notations. For $g, h \in G$ we will write $g \sim_G h$ (or simply $g \sim h$) if g is conjugate to h. Denote $g^h = h^{-1}gh$, $[g,h] = g^{-1}h^{-1}gh$. For a group word R over the alphabet X and a letter $x \in X$ the exponent sum of x in R is denoted by R_x. Let $\langle X \,|\, P\rangle$ be a presentation with $p_x = 0$ for every $p \in P$, $x \in X$, and let $G = F(X)/\langle\langle P\rangle\rangle$, where $F(X)$ is the free group with the basis X. For any element $g \in G$ denote by g_x the exponent sum of x in any word over X representing g.

Remark 0.3. As noted V. N. Gerasimov, Theorem 1.2 follows from the following result of Z. Sela: A finitely generated group is elementary equivalent to a non-abelian free group if and only if it is a non-elementary hyperbolic w-residually free tower [**S**, Theorem 6].

In particular, the fundamental group G of a hyperbolic surface different from the non-orientable surface of genus 3 is elementary equivalent to a non-abelian free group F. Let u and v be two elements of G with the same normal closure. Then there are numbers $\varepsilon_1, \dots, \varepsilon_n, \mu_1, \dots, \mu_m \in \{-1, 1\}$, and elements $g_1, \dots, g_n, h_1, \dots, h_m \in G$ such that

$$u^{\varepsilon_1 g_1} u^{\varepsilon_2 g_2} \dots u^{\varepsilon_n g_n} = v, \quad v^{\mu_1 h_1} v^{\mu_2 h_2} \dots v^{\mu_m h_m} = u.$$

For these $\varepsilon_1, \dots, \varepsilon_n, \mu_1, \dots, \mu_m$ write the formula

$$\forall w_1 \forall w_2 \forall x_1 \dots \forall x_n \forall y_1 \dots \forall y_m \exists g$$

$$(w_1^{\varepsilon_1 x_1} \dots w_1^{\varepsilon_n x_n} = w_2 \wedge w_2^{\mu_1 y_1} \dots w_2^{\mu_m y_m} = w_1 \Longrightarrow g^{-1} w_1 g = w_2 \vee g^{-1} w_1 g = w_2^{-1}).$$

This formula is valid in F by Theorem 0.1. Therefore it is valid in G by Sela's result, hence there is $f \in G$ such that $f^{-1}uf = v$ or $f^{-1}uf = v^{-1}$. So, G possesses the Magnus property. In Section 1 we give a proof (in the case of orientable surfaces), which is not based on the result of Sela. It is close to the original Magnus' proof for free groups.

Question. Is there a non-elementary hyperbolic group with the Magnus property which is not elementary equivalent to a non-abelian free group?

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1. The main theorem

In the proof of Theorem 1.2 we will use Theorem 0.1 and the following Theorem 1.1 of Magnus, which is called *Freiheitssatz*.

Theorem 1.1 ([**M**]). *Let F be a free group with a basis X, and r be a cyclically reduced word in F with respect to X. If s is a nontrivial word, which lies in the normal closure of r, then s contains all letters of X that occur in r. In particular, if x is a letter of r, then the subgroup generated by $X \setminus \{x\}$ in the one-relator group $\langle X \,|\, r\rangle$ is free.*

Theorem 1.2. *Let G be the fundamental group of a closed orientable surface and $r, s \in G$. If the normal closures of r and s coincide, then r is conjugate to s or s^{-1}.*

Proof. If the surface has genus 1, the theorem is clear. So, suppose that the surface has genus at least 2. Let $G = \langle a, b, c, d, \dots, e, f \,|\, [a,b][c,d]\dots[e,f] = 1\rangle$, where e, f are absent for genus 2, and let the normal closures of r and s in G coincide. We assume that r and s are nontrivial. Consider two cases.

Case 1. Suppose that $r_a = 0$. Then $s_a = 0$ also. Let $\varphi : G \to \mathbb{Z}$ be an epimorphism sending a to 1 and all other generators $b, c, \dots, f$ to 0. For any $g \in G$ denote $g_i = a^{-i}ga^i$. Then $\operatorname{Ker}\varphi$ is generated by the set $Y = \cup_{i\in\mathbb{Z}} Y_i$ where $Y_i = \{b_i, c_i, \dots, f_i\}$. It is known that any subgroup of infinite index in G is a free group. So, $N = \operatorname{Ker}\varphi$ is a free group, and it can be expressed as infinite amalgamated product:

$$N = \dots * \langle Y_{-1}\rangle *_{Z_0} \langle Y_0\rangle *_{Z_1} \langle Y_1\rangle * \dots, \tag{1}$$

where $\langle Y_i\rangle$ is the free group with the basis Y_i, and $\langle Y_i\rangle, \langle Y_{i+1}\rangle$ are amalgamated over the cyclic subgroup generated by $b_i[c_i, d_i]\dots[e_i, f_i]$ in $\langle Y_i\rangle$ and by b_{i+1} in $\langle Y_{i+1}\rangle$. We call these cyclic subgroups as *Z-subgroups*.

For each $i \leqslant j$ denote $N_{i,j} = \langle Y_i, Y_{i+1}, \dots, Y_j\rangle$. The group $N_{i,j}$ has two special free bases

$$\{b_i\} \cup \bigcup_{i\leqslant k\leqslant j} \{c_k, \dots, f_k\}$$

and

$$\bigcup_{i\leqslant k\leqslant j} \{c_k, \dots, f_k\} \cup \{b_j\}$$

which will be called *the left* and *the right basis* of $N_{i,j}$ respectively.

Let g be a nontrivial element of N. If g does not lie in the union of Z-subgroups, then there is a unique $N_{i,j}$ such that $g \in N_{i,j}$ and $j - i$ is minimal. Put $\alpha(g) = i$, $\omega(g) = j$. If g lies in some Z-subgroup, then there is a unique $N_{i,i}$ such that $g \in N_{i,i}$, and i is minimal. In this case put $\alpha(g) = \omega(g) = i$. In both cases the number $||g|| = \omega(g) - \alpha(g) + 1$ will be called *the width* of the element g. Note, that if $g \in N_{k,l}$ and g does not lie in the union of Z-subgroups, then $k \leqslant \alpha(g) \leqslant \omega(g) \leqslant l$. A nontrivial element g of N will be called *width-minimal* if it has minimal width among all its conjugates. We will use the following

Remark 1.3. Let g be a width-minimal element of N with $||g|| \geqslant 2$. Write g as a reduced word in the free group $N_{\alpha(g),\omega(g)}$ with respect to its left basis. Make in this word all cyclic reductions and denote the resulting word by g_L. Then g_L contains a letter from $\{c_{\omega(g)}, \dots, f_{\omega(g)}\}$. Analogously we can consider the right basis of $N_{\alpha(g),\omega(g)}$ and define the word g_R. Then g_R contains a letter from $\{c_{\alpha(g)}, \dots, f_{\alpha(g)}\}$.

Conjugating, we may assume that r_0, s_0 are width-minimal. Then r_i, s_i are width-minimal for each i. Note that $\alpha(r_{i+1}) = \alpha(r_i) + 1$, $\omega(r_{i+1}) = \omega(r_i) + 1$. In particular, all r_i have the same width. The same is valid for s_i.

It is clear that the sets $\mathcal{R} = \{\dots, r_{-1}, r_0, r_1, \dots\}$ and $\mathcal{S} = \{\dots, s_{-1}, s_0, s_1, \dots\}$ have the same normal closure in N. Unfortunately Y is not a free basis of N and we can not apply the theorem of Magnus about staggered presentations [**M, LS**].

Write s_0 as a product of some conjugates to elements of $\mathcal{R}$ and their inverses:

$$s_0 = \prod_{j=1}^{n} f_{i_j}^{-1} r_{i_j}^{\pm 1} f_{i_j},$$

where $f_{i_j} \in N$, $r_{i_j} \in \mathcal{R}$. Denote $\alpha = \min\{\alpha(f_{i_1}), \dots, \alpha(f_{i_n}), \alpha(r_{i_1}), \dots, \alpha(r_{i_n})\}$, $\omega = \max\{\omega(f_{i_1}), \dots, \omega(f_{i_n}), \omega(r_{i_1}), \dots, \omega(r_{i_n})\}$. We may assume that $\omega - \alpha$ is minimal over all such products. It is clear that $\omega - \alpha + 1 \geqslant ||r_0||$.

Subcase 1. Suppose that $||r_0|| \geqslant 2$.

First we will prove that $\alpha(s_0) = \alpha$, $\omega(s_0) = \omega$. Consider two subcases.

Subcase 1.1. Suppose that $\omega - \alpha + 1 = ||r_0||$.

Then $\alpha = \alpha(r_i)$, $\omega = \omega(r_i)$ for some $r_i \in \mathcal{R}$ and s_0 can be deduced in $N_{\alpha,\omega}$ from r_i. Write s_0 as the reduced word s_0' with respect to the left basis of $N_{\alpha,\omega}$. The word s_0' can be deduced in $N_{\alpha,\omega}$ from $(r_i)_L$. By Remark 1.3, $(r_i)_L$ contains a letter from $\{c_\omega, \dots, f_\omega\}$. By Magnus' Freiheitssatz it follows that s_0' contains this letter too. Hence $\omega(s_0) \geqslant \omega$. Considering the right basis of $N_{\alpha,\omega}$, we get $\alpha(s_0) \leqslant \alpha$. Since $s_0 \in N_{\alpha,\omega}$, these two inequalities are actually equalities.

Subcase 1.2. Suppose that $\omega - \alpha + 1 > ||r_0||$.

Let G_1 be the group, generated by $Y_\alpha \cup \dots \cup Y_{\omega-1}$, let G_2 be the group, generated by $Y_{\omega-||r_0||+1} \cup \dots \cup Y_\omega$, and let G_3 be the group, generated by $Y_{\omega-||r_0||+1} \cup \dots \cup Y_{\omega-1}$. It is clear that $\langle G_1, G_2 \rangle = G_1 *_{G_3} G_2$ is a nontrivial amalgamated product.

Let r_i, r_j be elements of $\mathcal{R}$ such that $\alpha(r_i) = \alpha$, $\omega(r_j) = \omega$. Denote by H_1 the quotient of G_1 by the normal closure of $\{r_i, \dots, r_{j-1}\}$ in G_1, and by H_2 the quotient of G_2 by the normal closure of r_j in G_2. Let $\varphi_1 : G_1 \to H_1$ and $\varphi_2 : G_2 \to H_2$ be the canonical epimorphisms.

Claim. The restrictions $\varphi_1|_{G_3}$ and $\varphi_2|_{G_3}$ are injective. In particular, the quotient of $G_1 *_{G_3} G_2$ by the normal closure of $\{r_i, \dots, r_j\}$ can be identified with $H_1 *_{G_3} H_2$.

Proof. The group H_2 is the quotient of the free group G_2 by the normal closure of the word $(r_j)_L$. This word is cyclically reduced with respect to the left basis of G_2 and contains a letter from $\{c_\omega, \dots, f_\omega\}$. The group G_3 is generated by a part of this basis, that does not contain these letters. Therefore $\varphi_2|_{G_3}$ is injective by the Magnus Freiheitssatz.

Let us prove that $\varphi_1|_{G_3}$ is injective. If $\omega - \alpha = ||r_0||$, then H_1 is a quotient of G_1 by the normal closure of $(r_i)_R$ and $\varphi_1|_{G_3}$ is injective by the same reason. Let $(\omega - 1) - \alpha + 1 > ||r_0||$. By induction H_1 can be identified with $A *_C B$, where A is the quotient of the group, generated by $Y_\alpha \cup \dots \cup Y_{\omega-2}$ by the normal closure of $\{r_i, \dots, r_{j-2}\}$, B is the quotient of the group, generated by $Y_{\omega-||r_0||} \cup \dots \cup Y_{\omega-1}$ by the normal closure of r_{j-1}, and C is the group, generated by $Y_{\omega-||r_0||} \cup \dots \cup Y_{\omega-2}$. Again by the Magnus Freiheitssatz G_3 is mapped injectively into B, and hence into H_1. The claim is proved.

Now we will finish the proof that $\alpha(s_0) = \alpha$, $\omega(s_0) = \omega$. Let

$$\varphi : G_1 *_{G_3} G_2 \to H_1 *_{G_3} H_3$$

be the factorization by the normal closure of $\{r_i, \dots, r_j\}$. It is clear that $\varphi|_{G_1} = \varphi_1$ and $\varphi|_{G_2} = \varphi_2$. Suppose that $\omega(s_0) < \omega$. Then $s_0 \in G_1$. Since $\varphi(s_0) = 1$, then $\varphi_1(s_0) = 1$. Hence s_0 lies in the normal closure of $\{r_i, \dots, r_{j-1}\}$ in G_1, a contradiction with the minimality of $\omega-\alpha$. So, $\omega(s_0)\geqslant\omega$. Analogously $\alpha(s_0)\leqslant\alpha$. Since $s_0\in N_{\alpha,\omega}$, the last two inequalities are actually equalities.

So, we have proved that if $||r_0||\geqslant 2$, then $||s_0||=\omega-\alpha+1\geqslant||r_0||$. By symmetry we get $||r_0||\geqslant||s_0||$, hence $||s_0||=\omega-\alpha+1=||r_0||$. As in Subcase 1.1 we get that s_0 can be deduced from some r_i in N with $\alpha(r_i)=\alpha(s_0)$, $\omega(r_i)=\omega(s_0)$, and analogously r_i can be deduced from some s_j in N, actually from s_0. Theorem 0.1 implies that s_0 is conjugate to r_i or r_i^{-1} in N, hence s is conjugate to r or r^{-1} in G.

Subcase 2. Suppose that $||r_0||=1$.

We may assume also that $||s_0||=1$, otherwise we interchange r and s and came to Subcase 1. We may assume also that neither r nor s is conjugate to a power of b. Otherwise $r_c=s_c=0$ and neither r nor s is conjugate to a power of d (we use here that the normal closures of r and s coincide), and we can consider c instead of a and d instead of b. This gives us that each r_i and each s_j belongs to only one subgroup of type $\langle Y_k\rangle$.

Suppose that $\omega-\alpha\geqslant 1$. We have that s_0 can be deduced from some $r_i,r_{i+1},\dots,r_{i+\omega-\alpha}$ in the subgroup $\langle G_1,G_2\rangle=G_1*_Z G_2$ where G_1 is generated by $Y_\alpha\cup\dots\cup Y_{\omega-1}$, G_2 is generated by Y_ω, and Z is the cyclic subgroup generated by b_ω. Since r is not conjugate to a power of b, we can proceed as in Subcase 1.2 and get $||s_0||=\omega-\alpha+1$, that contradicts to $||s_0||=1$.

So, $\alpha=\omega$. Hence, s_0 can be deduced from r_i in the free group $\langle Y_\alpha\rangle$. In particular s_0 and r_i lie in the same subgroup of type $\langle Y_k\rangle$. We can interchange r and s and define the numbers α',ω' analogous to α,ω. As above, $\alpha'=\omega'$ and r_0 can be deduced from some s_j in the group $\langle Y_{\alpha'}\rangle$. Therefore r_i can be deduced from s_{i+j} in $\langle Y_{i+\alpha'}\rangle$. In particular r_i and s_{i+j} lie in the same subgroup of type $\langle Y_k\rangle$. Since r_i,s_0,s_{i+j} each belongs to only one subgroup of type $\langle Y_k\rangle$, we get that $i+j=0$. Hence, r_i can be deduced from s_0 (and conversely) in N. By Magnus' Freiheitssatz r_i is conjugate to s_0 or s_0^{-1} in N. Hence, r is conjugate to s or s^{-1} in G.

Case 2. Now suppose that $r_u\neq 0$ for every letter $u\in\{a,b,\dots,f\}$. Let x be a new letter. Define a new group

$$\bar{G}=G\underset{a=x^{r_b}}{*}\langle x\,|\,\rangle.$$

It is easy to see that the normal closures of r and s in $\bar{G}$ coincide and that $r\underset{\bar{G}}{\sim}s\Leftrightarrow r\underset{G}{\sim}s$. So, we will now work with the group $\bar{G}$. This group has the following presentation:

$$\langle x,b,c,d,\dots,e,f\,|\,[x^{r_b},b][c,d]\dots[e,f]=1\rangle.$$

Let $\bar{b}=x^{r_a}b$. Using Tietze transformation we can rewrite this presentation as

$$\langle x,\bar{b},c,d,\dots,e,f\,|\,[x^{r_b},\bar{b}][c,d]\dots[e,f]=1\rangle.$$

We have $r_x=r_ar_b-r_br_a=0$. As above we can consider the epimorphism $\varphi:\bar{G}\to\mathbb{Z}$, sending x to 1 and all other generators $\bar{b},c,\dots,f$ to 0. Then $\operatorname{Ker}\varphi$ is generated by the set

$$Y=\bigcup_{j=0}^{r_b-1}\bigcup_{i\in\mathbb{Z}}Y_{j,i},$$

where $Y_{j,i} = \{\bar{b}_{j,i}, c_{j,i}, \dots, f_{j,i}\}$ and $g_{j,i}$ denotes $x^{-(j+ir_b)} g x^{j+ir_b}$. Again $N = \operatorname{Ker}\varphi$ is a free group, and it can be expressed in the form $N = N_0 * \dots * N_{r_b - 1}$ with

$$N_j = \dots * \langle Y_{j,-1}\rangle *_{Z_{j,0}} \langle Y_{j,0}\rangle *_{Z_{j,1}} \langle Y_{j,1}\rangle * \dots,$$

where the free groups $\langle Y_{j,i}\rangle$, $\langle Y_{j,i+1}\rangle$ are amalgamated over the cyclic subgroups generated by $\bar{b}_{j,i}[c_{j,i}, d_{j,i}]\dots[e_{j,i}, f_{j,i}]$ in $\langle Y_{j,i}\rangle$ and by $b_{j,i+1}$ in $\langle Y_{j,i+1}\rangle$. Next one can argue as in Case 1. □

2. Some one-relator groups without the Magnus property

To prove Theorem 2.4 we need a lemma of Collins about the conjugacy of elements in HNN-extensions [**C**], see also [**LS**]. Let $G = \langle H, t \,|\, t^{-1}At = B\rangle$ be an HNN-extension. Recall that a word $w = h_1 t^{\varepsilon_1} h_2 t^{\varepsilon_2} \dots t^{\varepsilon_n}$ with $h_i \in H$, $\varepsilon_i \in \{-1, 1\}$ for $i = 1, \dots, n$, is *cyclically reduced* if every its cyclic permutation does not contain a subword of type $t^{-1}at$, where $a \in A$ and of type tbt^{-1}, where $b \in B$. The *length* of w, written $|w|$ is the number of occurrences of $t^{\pm 1}$ in w.

LEMMA 2.1 ([**C**]). *Let $G = \langle H, t \,|\, t^{-1}At = B\rangle$ be an HNN-extension. If w and v are two conjugate cyclically reduced elements of G and w ends in t^{ε_n}, then $|w| = |v|$ and w can be obtained from v by taking a suitable cyclic permutation of v, which ends in t^{ε_n}, and then conjugating by an element z, where $z \in A$ if $\varepsilon_n = -1$, and $z \in B$ if $\varepsilon_n = 1$.*

We need also the following two lemmas. The proof of the first lemma relies on an algebraic trick, the second lemma is obvious.

LEMMA 2.2. *In the group $BS(n, sn) = \langle x, y \,|\, y^{-1}x^n y = x^{sn}\rangle$, where $n, s \in \mathbb{Z}$, $n > 0$, the elements $(yx)^n x^{-1} y x^2 (yx)^{-n} x^{-1}$ and y have the same normal closure.*

PROOF. Denote $r = (yx)^n x^{-1} y x^2 (yx)^{-n} x^{-1}$. From r one can deduce $(rx)^n x^{-n}$ that is equal to $(yx)^n x^{-1} (yx)^n x (yx)^{-n} \cdot x^{-n}$. In the given group the last element is equal to $(yx)^n x^{-1} (yx)^n x x^{-ns^n} (yx)^{-n}$. Hence it is conjugate to $(yx)^n x^{-ns^n}$. From r and $(yx)^n x^{-ns^n}$ one can deduce y. □

LEMMA 2.3. *Let $\varphi : G \to H$ be a homomorphism from a group G to a group H, and let u, v be two elements of G with the same normal closure. Then the elements $\varphi(u), \varphi(v)$ have the same normal closure in H.*

THEOREM 2.4. *The Magnus property does not hold for the following one-relator groups:*

(1) *$G = \langle a, b \,|\, a^k b^n\rangle$, where $|k| \geqslant 2, |n| \geqslant 2$ and $(|k|, |n|) \neq (2, 2)$,*
(2) *$G = \langle a, b \,|\, a^{-k} b^n a^k b^{-m}\rangle$ where k, n, m are nonzero and not all equal to ± 1,*
(3) *$G = \langle X \,|\, R^n\rangle$, where $|X| > 1$, $n \geqslant 2$,*
(4) *$G_n = \langle a, b \,|\, b^{-1} a b^2 a^n\rangle$, where $n \geqslant 3$. The groups G_n $(n \geqslant 3)$ are pairwise non-isomorphic torsion-free hyperbolic.*

PROOF. (1) Replacing b by b^{-1} if needed, we may assume that $n > 0$. Denote $r = (ab)^n b^{-1} a b^2 (ab)^{-n} b^{-1}$. By Lemma 2.2 the elements r and a have the same normal closure in the group $\langle a, b \,|\, a^{-1} b^n a = b^n\rangle$ and hence in its homomorphic image G (the homomorphism is induced by the identity map on a, b). Using normal forms in the amalgamated product $G = \langle a \,|\, \rangle \underset{a^k = b^{-n}}{*} \langle b \,|\, \rangle$, one can verify that r is conjugate to neither a nor a^{-1} in G.

(2) *Case 1.* Let $G = \langle a, b \,|\, a^{-1}ba = b^n\rangle$. Assume that $n > 1$ (the proof in the case $n < -1$ is analogous). In this group any two elements of the form $a^{-i}ba^i$ commute. Moreover, G is solvable and any element of G can be uniquely written in the form $a^k \cdot a^{-l}b^s a^l$. So, any element of G is conjugate to the element of the form $a^k b^s$. Let $g = a^k b^s$ and $h = a^k b^t$ be two elements of G. First we need to find a condition which is equivalent to g and h being conjugate. Suppose that $h = w^{-1}gw$ where $w = a^m \cdot a^{-l}b^r a^l$. Then

$$h = a^{-l}b^{-r}a^l a^{-m} \cdot a^k b^s \cdot a^m a^{-l} b^r a^l$$
$$= a^k \cdot (a^{-(l+k)}b^{-r}a^{(l+k)}) \cdot (a^{-m}b^s a^m) \cdot (a^{-l}b^r a^l) = a^k b^{-rn^{l+k}+sn^m+rn^l}.$$

Therefore the condition is that there exist $r, l, m \in \mathbb{Z}$ such that

$$t = -rn^l(n^k - 1) + sn^m. \tag{2}$$

Second, we find a condition which ensures that $\langle\langle a^k b^t\rangle\rangle = \langle\langle a^k, b^{n-1}\rangle\rangle$. One can compute that $[a^k b^t, a] = b^{t(n-1)}$, $[a^k b^t, b] = b^{-(n^k-1)}$. Hence, if

$$\Big(t, \frac{n^k - 1}{n - 1}\Big) = 1, \tag{3}$$

then $b^{n-1} \in \langle\langle a^k b^t\rangle\rangle$. If additionally

$$t = q(n-1) \ \text{ for some } \ q \in \mathbb{Z}, \tag{4}$$

then $\langle\langle a^k b^t\rangle\rangle = \langle\langle a^k, b^{n-1}\rangle\rangle$.

Now we will find k and q such that the elements $g = a^k b^{n-1}$ and $h = a^k b^{q(n-1)}$ have the same normal closure but not conjugate. Let k be a prime number larger than $\max\{n-1, 7\}$, let q be a prime number larger than n^k-1, and let $t = q(n-1)$. Then the condition (3) will be satisfied (the numbers $n-1$ and $\frac{n^k-1}{n-1}$ are coprime because $\frac{n^k-1}{n-1} = k \pmod{n-1}$) and we have

$$\langle\langle a^k b^{q(n-1)}\rangle\rangle = \langle\langle a^k, b^{n-1}\rangle\rangle.$$

The same arguments show that $\langle\langle a^k b^{n-1}\rangle\rangle = \langle\langle a^k, b^{n-1}\rangle\rangle$. Hence, for these k and q we have

$$\langle\langle a^k b^{q(n-1)}\rangle\rangle = \langle\langle a^k b^{n-1}\rangle\rangle.$$

It remains to find a prime number $q > n^k - 1$ such that the elements $a^k b^{n-1}$ and $a^k b^{q(n-1)}$ are not conjugate. According to (2) the following condition must be satisfied: there is no $r, l, m \in \mathbb{Z}$ such that

$$q(n-1) = -rn^l(n^k - 1) + (n-1)n^m.$$

Since n is invertible modulo $\frac{n^k-1}{n-1}$, this condition is equivalent to the the following: there is no $m \in \mathbb{Z}$ such that

$$q \equiv n^m \Big(\bmod\ \frac{n^k - 1}{n - 1}\Big). \tag{5}$$

In the following claim $\varphi(x)$ denotes the Euler function of x.

Claim. If $k \geqslant 7$, $n \geqslant 2$, then $\varphi\big(\frac{n^k-1}{n-1}\big) > k$.

Proof. It is easy to see that $\varphi(x) \geqslant \sqrt{x}$ for any natural $x > 6$. Hence $\varphi\big(\frac{n^k-1}{n-1}\big) > n^{\frac{k-1}{2}} \geqslant 2^{\frac{k-1}{2}} > k$.

We need to find a prime number $q > n^k - 1$ such that the congruence (5) is not valid for any $m \in \mathbb{Z}$. The order of n in the group of units of the ring $Z_{\frac{n^k-1}{n-1}}$ is a divisor of k. Hence, the number of units of the form n^m, where $m \in \mathbb{Z}$, is at most k. Since $k < \varphi\big(\frac{n^k-1}{n-1}\big)$, there is a unit u different from the units n^m, $m \in \mathbb{Z}$. By Dirichlet's theorem we can choose a prime number $q > n^k - 1$ in the arithmetic progression with the first term u and the difference $\frac{n^k-1}{n-1}$.

Case 2. Let $G = \langle a, b \,|\, a^{-k}b^n a^k = b^n\rangle$. Using replacements a by b and b by a, or b by b^{-1} if needed, we may assume that $n > 1$. By Lemma 2.2 the elements $r = (a^k b)^n b^{-1} a^k b^2 (a^k b)^{-n} b^{-1}$ and $w = a^k$ have the same normal closure in G. Using the normal form in the amalgamated product

$$G = \langle a, b_1 \mid a^{-k} b_1 a^k = b_1\rangle \underset{b_1=b^n}{*} \langle b \mid \rangle,$$

one can verify that r is conjugate to neither w nor w^{-1}.

Case 3. Let $G = \langle a, b \,|\, a^{-k}b^n a^k = b^{-n}\rangle$. Replacing b by b^{-1} if needed, we may assume that $n > 0$. If $n > 1$, we can choose r and w as in Case 2 and complete the proof analogously. If $n = 1$, then these elements are conjugate. In this subcase we have $G = \langle a, b \,|\, a^{2k} = (ba^k)^2\rangle$. Using Tietze transformations, we get $G = \langle a, b_1 \,|\, a^{2k} = b_1^2\rangle$. Now we can apply the assertion (1) of Theorem 2.4.

Case 4. Let $G = \langle a, b \,|\, a^{-k}b^n a^k = b^m\rangle$, where $m \neq \pm n$. Let d be the greatest common divisor of n and m. If $d < |n|$ and $d < |m|$, then the elements b^d and b^n have the same normal closure, but not conjugate up to inversion. Let for example $d = |n|$. Denote $b_1 = b^n$, $a_1 = a^k$, $n_1 = m/|d|$. Then $n_1 \notin \{-1, 0, 1\}$ and

$$G = (\langle a_1, b_1 \mid a_1^{-1} b_1 a_1 = b_1^{n_1}\rangle \underset{b_1=b^d}{*} \langle b \mid \rangle) \underset{a_1=a^k}{*} \langle a \mid \rangle.$$

Now, one can use Case 1 to complete the proof.

(3) Let $X = \{t, a, \dots, c\}$. Let w be an element of G and

$$r = (wR)^n R^{-1} w R^2 (wR)^{-n} R^{-1}.$$

There is a homomorphism from the group $\langle x, y \,|\, y^{-1}x^n y = x^n\rangle$ into $G = \langle X \mid R^n\rangle$ sending y to w and x to R. By Lemmas 2.2 and 2.3 the elements r and w have the same normal closure in G. We will choose w so that r will be not conjugate to $w^{\pm 1}$.

Case 1. Suppose that $R_z = 0$ for some $z \in X$, say $R_t = 0$. Put $w = t$. Rewrite R in terms x_i, where $x_i = t^{-i}xt^i$ ($x \in X \setminus \{t\}, i \in \mathbb{Z}$), and let $\alpha(x)$ (resp. $\omega(x)$) be the smallest (resp. the largest) subscript i such that x_i occurs in the expression for R. In this case G is the HNN-extension $\langle H, t \,|\, t^{-1}At = B\rangle$, where

$$H = \langle a_{\alpha(a)}, \dots, a_{\omega(a)}, \dots, c_{\alpha(c)}, \dots, c_{\omega(c)}\rangle,$$

$$A = \langle a_{\alpha(a)}, \dots, a_{\omega(a)-1}, \dots, c_{\alpha(c)}, \dots, c_{\omega(c)-1}\rangle,$$

$$B = \langle a_{\alpha(a)+1}, \dots, a_{\omega(a)}, \dots, c_{\alpha(c)+1}, \dots, c_{\omega(c)}\rangle.$$

By Lemma 2.1, r is not conjugate to $w^{\pm 1}$. Note that the element $w = t^k$, where $k \in \mathbb{Z} \setminus \{0\}$, is suitable also.

Case 2. Suppose that $R_x \neq 0$ for every $x \in X$. Let y be a new letter, $y \notin X$, and

$$G' = G \underset{t=y^{R_a}}{*} \langle y \mid \rangle.$$

Then G' can be generated by $X' = X \setminus \{t, a\} \cup \{y, z\}$, where $z = ay^{R_t}$. Moreover, $G' = \langle X' \,|\, (R')^n \rangle$, where R' is obtained from R by substituting y^{R_a} instead of t, and zy^{-R_t} instead of a. It is clear that $R'_y = 0$. Put $w = y^{R_a}$. As in Case 1, r and $w^{\pm 1}$ are not conjugate in G' and hence in G.

(4) Let $x = a^{-1}, y = ba^{n+1}$. Rewriting in terms x, y, we obtain the following presentation

$$G_n = \langle x, y \,|\, x^{-n}y^{-1}x^{-1}yx^{n+1}y \rangle.$$

In terms $y_i = x^{-i}yx^i$ we have $x^{-n}y^{-1}x^{-1}yx^{n+1}y = y_n^{-1}y_{n+1}y_0$. So, G_n is the HNN-extension with the base group $H = \langle y_0, y_1, \dots, y_{n+1} \,|\, y_n^{-1}y_{n+1}y_0 \rangle$, associated subgroups $C_1 = \langle y_0, y_1, \dots, y_n \,|\, \rangle$ and $C_2 = \langle y_1, y_2, \dots, y_{n+1} \,|\, \rangle$, and the stable letter x which acts as $x^{-1}y_ix = y_{i+1}$ for $0 \leqslant i \leqslant n-1$, and $x^{-1}y_nx = y_{n+1} = y_ny_0^{-1}$. It is clear that $H = C_1 = C_2$. So, G_n is the extension of the free group $H = \langle y_0, y_1, \dots, y_n \rangle$ of rank $n+1$ by the automorphism φ such that $\varphi(y_i) = y_{i+1}$ for $0 \leqslant i \leqslant n-1$, and $\varphi(y_n) = y_ny_0^{-1}$.

Take $r = y_0$, $w = y_0y_1 \dots y_{n-1}$. It is clear that the normal closure of r in G_n coincides with H. We prove that the normal closure of w in G_n coincides with H also. We have $w^\varphi = y_1y_2 \dots y_n$, hence $y_0 = y_n$ modulo w. Since $y_{n+1} = y_ny_0^{-1}$, we get $y_{n+1} = 1$ modulo w. This implies, that $y_0 = 1$ modulo w and we are done. Now we will prove that r is not conjugate to $w^{\pm 1}$ in G_n. We will use the induced action of φ on H/H'.

Let $\theta : H \to H/H'$ be the canonical homomorphism. The image of an element $h \in H$ under the action of θ will be denoted by $\bar{h}$. The free abelian group H/H' has the basis $\{\bar{y}_0, \dots, \bar{y}_n\}$. Let us identify $\bar{y}_i$ with the row of length $n+1$ whose $(i+1)$-th entry is equal to 1 and all other entries are equal to 0. The automorphism φ induces the automorphism of the group H/H', acting by the rule $\bar{h} \mapsto \bar{h}A$, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \dots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

The matrix A has the characteristic polynomial $p(\lambda) = \lambda^{n+1} - \lambda^n + 1$. This polynomial has $n+1$ different roots $\lambda_0, \dots, \lambda_n$. Not all λ_i are roots of 1. Indeed, suppose that $|\lambda_i| = 1$ for every i. Then $|\lambda_i - 1| = 1$, so $\lambda_i = e^{\pm \frac{\pi i}{3}}$. But all λ_i are different and $n \geqslant 3$, a contradiction.

Let $u_0, \dots, u_n$ be some eigenvectors corresponding to $\lambda_0, \dots, \lambda_n$. We can write $\bar{y}_0 = \alpha_0u_0 + \dots + \alpha_nu_n$ for some complex numbers $\alpha_0, \dots, \alpha_n$. Since $y_0, y_0A, \dots, y_0A^n$ are linearly independent, all α_i are nonzero.

Suppose that r is conjugate to w^ε in G for some $\varepsilon \in \{-1, 1\}$. Then $h^{-1}r^{\varphi^k}h = w^\varepsilon$ for some $h \in H$, $k \in \mathbb{Z}$. Since $\bar{w} = \bar{y}_0(E + A + \dots + A^{n-1}) = \bar{y}_0(A^n - E)(A - E)^{-1}$, then $\bar{y}_0A^k = \pm\bar{y}_0(A^n - E)(A - E)^{-1}$. Since all α_i are nonzero, then $\lambda_i^k = \pm\frac{\lambda_i^n - 1}{\lambda_i - 1}$ for each eigenvalue λ_i. First consider the case where $k \geqslant 0$. Then $p(\lambda)$ is a divisor of the polynomial $\frac{\lambda^n - 1}{\lambda - 1} \pm \lambda^k$. But $p(\lambda)\frac{\lambda^{n+1} - 1}{\lambda - 1} = \frac{\lambda^n - 1}{\lambda - 1} + \lambda^{2n+1}$. Hence, $p(\lambda)$ is a divisor of $\lambda^{2n+1} \pm \lambda^k$. Since the roots of $p(\lambda)$ are nonzero and not all of them are roots of 1, this is possible only if $k = 2n+1$. Now consider the case where $k < 0$. In this case $p(\lambda)$ is a divisor of the polynomial $\frac{\lambda^n - 1}{\lambda - 1}\lambda^{-k} \pm 1$. But $p(\lambda)\frac{\lambda^{n+1} - 1}{\lambda - 1}\lambda^{-k} = \frac{\lambda^n - 1}{\lambda - 1}\lambda^{-k} + \lambda^{2n+1-k}$. Hence $p(\lambda)$ is a divisor of $\lambda^{2n+1-k} \pm 1$, that is impossible. So, $k = 2n+1$. We have

$$r^{\varphi^{2n+1}} = y_0^{\varphi^{2n+1}} = y_n(y_{n-1}y_{n-2}\dots y_0)^{-1}y_n^{-1}.$$

For $n \geqslant 3$ this element is conjugate to neither w nor w^{-1} in H. Hence r is not conjugate to w^{ε} in G.

Now, we prove that the groups G_n are pairwise non-isomorphic. We have $G_n = H \underset{\varphi}{\rightthreetimes} \langle x\rangle$, where H is a free group of rank $n+1$. Thus, it is enough to prove that the commutator subgroup G_n' coincides with H. Recall that H has the basis $\{y_0,\dots,y_n\}$, and $\varphi(y_i) = y_{i+1}$ for $0 \leqslant i \leqslant n-1$, $\varphi(y_n) = y_n y_0^{-1}$. For $v = x^{-1}yxy^{-1}$ we have $v = y_1y_0^{-1}$, $\varphi(v) = y_2y_1^{-1},\dots,\varphi^{n-1}(v) = y_ny_{n-1}^{-1}$, $\varphi^n(v) = y_ny_0^{-1}y_n^{-1}$. Hence $y_0 \in G_n'$. It follows that $H \leqslant G_n'$. The converse $G_n' \leqslant H$ is clear. □

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