Decompositions of fundamental groups of closed surfaces into free constructions

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§ 0. Introduction

A. Let T be a surface with a basepoint x. A compact subsurface S of T is called *incompressible* if no component of the closure of $T \setminus S$ is a 2-disk whose boundary is contained in ∂S (see [Sc]). If S is incompressible and $x \in S$, then the canonical map $\pi_1(S, x) \to \pi_1(T, x)$ is injective. So we will identify $\pi_1(S, x)$ with its image in $\pi_1(T, x)$. Let H be a subgroup of $\pi_1(T, x)$. We say that H is *realized* by an incompressible subsurface S in T if $x \in S$ and $H = \pi_1(S, x)$.

Definition 0.1. Let T be a surface with a basepoint x and let $\pi_1(T, x) = G_1 *_{G_3} G_2$ be a decomposition of its fundamental group into a free product with amalgamation. We say that this decomposition is *geometric* if there are incompressible subsurfaces S_1, S_2, S_3 in T such that $T = S_1 \cup S_2$, $S_1 \cap S_2 = S_3$, $x \in S_3$, and $G_i = \pi_1(S_i, x)$ for $i = 1, 2, 3$.

In [K] H. Zieschang formulated the following Problem 10.69.

Problem. *Let T_g be a closed orientable surface of genus $g \geqslant 2$ with a basepoint x. Is any decomposition $\pi_1(T_g, x) = G_1 *_{G_3} G_2$ geometric, provided $G_1 \neq G_3 \neq G_2$ and the subgroup G_3 is finitely generated?*

It is known that any such decomposition is geometric if G_3 is a cyclic group (see [HS], [Z] and [L]). In this case the decomposition is defined by a simple closed curve on T_g which separates T_g. There is only a finite number of such curves up to homeomorphisms of T_g. Therefore there is only a finite number of decompositions $\pi_1(T_g, x) = G_1 *_{G_3} G_2$ with $G_3 \cong Z$, up to automorphisms of $\pi_1(T_g, x)$.

In general the answer to this question is negative. In § 1 we give some method for constructing non-geometric decompositions. We prove there that for any $g \geqslant 2$ there is infinitely many non-geometric and not automorphic equivalent decompositions of kind $\pi_1(T_g, x) = F_2 *_{F_2} F_{2g-1}$ where F_n denotes a free group of rank n.

However, our main theorem 4.8 asserts that in some sense there is a positive answer. To understand this theorem one need to read definitions in the subsection B. Now we will formulate this theorem rather informally.

Theorem 4.8′. *Let T be a closed surface. Then any decomposition of $\pi_1(T, x)$ into amalgamated product (or more generally into the fundamental group of a finite graph of groups) with finitely generated edge group(s) is almost geometric. This means that there is a subgroup H of a finite index in $\pi_1(T, x)$ such that the induced decomposition of H is geometric in the corresponding covering of T.*

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In § 2 and § 3 the following two auxiliary theorems are proved.

Theorem 2.6. *Let T be a closed surface with a basepoint x and let $\pi_1(T,x) = G_1 \underset{G_3}{*} G_2$ be a decomposition of its fundamental group into an amalgamated product. If G_3 is realized by an incompressible subsurface in T then this decomposition is geometric.*

Theorem 3.1. *Let T be a closed surface with a basepoint x and let $\pi_1(T,x) = G \underset{H_1=t^{-1}H_2t}{*}$ be a decomposition of its fundamental group into an HNN-extension. If H_1 is realized by an incompressible subsurface in T, then G is also realized by an incompressible subsurface in a 2-fold covering of T. Moreover, G is the fundamental group of a graph of groups with cyclic edge groups and with two distinguished vertex groups H_1 and H_2.*

In § 5 we define a new notion *the edge rigidity*. Informally, a group G has the edge rigidity property if for any finite set of its finitely generated subgroups $G_1,\dots,G_n$ there is only a finite number of variants for vertex subgroups in the set of all decompositions of G into the fundamental group of graph of groups with the edge subgroups $G_1,\dots,G_n$.

Theorem 5.1. *The fundamental group of any closed surface different from the Klein bottle has the edge rigidity property.*

B. First we remind definitions of a graph of groups and its fundamental group according to [Se], and then define some new notions needed to understand Theorem 4.8.

Let X be a connected graph. Denote the set of its vertexes by X^0 and the set of its edges by X^1. The initial vertex of an edge e will be denoted by $\alpha(e)$ and the terminal one by $\omega(e)$. The opposite edge to e will be denoted by $\bar{e}$. The rank of the fundamental group of X will be denoted by $rk(X)$. The usual topological realization of X will be denoted by X also.

A graph of groups $(\mathbb{G},X)$ is a tuple consisting of the graph X, a set of vertex groups G_u $(u\in X^0)$, a set of edge groups G_e $(e\in X^1)$, and a set of embeddings $\alpha_e: G_e\to G_{\alpha(e)}$ and $\omega_e: G_e\to G_{\omega(e)}$ $(e\in X^1)$. It is assumed that $G_e=G_{\bar{e}},\ \omega_e=\alpha_{\bar{e}}$.

Let v be a fixed vertex of X.

The fundamental group $\pi_1(\mathbb{G},X,v)$ is a group consisting of all sequences of the form $g_1e_1g_2e_2\dots e_ng_{n+1}$ where $e_1e_2\dots e_n$ is a closed path in X with initial vertex v, $g_i\in G_{\alpha(e_i)}$ for $1\leqslant i\leqslant n$, and $g_{n+1}\in G_v$. The multiplication in $\pi_1(\mathbb{G},X,v)$ is given as in a free group (by concatenation) with additional relations in each G_u $(u\in X^0)$, the relations $e\bar{e}=1$ and $\alpha_e(g)=e\omega_e(g)\bar{e}$ where $e\in X^1, g\in G_e$.

This notion generalizes the notions of an amalgamated product and an HNN-extension, and is needed to describe subgroups of amalgamated products and HNN-extensions according to Bass – Serre theory of groups acting on trees [Se]. The following two definitions are needed to generalize Definition 0.1 to an arbitrary decomposition of $\pi_1(T,x)$ into the fundamental group of a graph of groups.

Definition 0.2. Let T be a compact surface with a basepoint x. Let $(\mathbb{G},X)$ be a finite graph of groups. We say that $(\mathbb{G},X)$ *is geometrically realized in* T, if the following four conditions hold.

(1) There is a fixed immersion of X into T. For any vertex u and for any edge e of X denote by u_* and by e_* their images in T.

(2) Each vertex group G_u is identified with $\pi_1(S_u, u_*)$ where S_u is an incompressible subsurface of T containing u_*.

(3) Each edge group G_e is identified with $\pi_1(S_e, u_*)$ where u is the initial vertex of e and S_e is an incompressible subsurface of S_u containing u_*. After the identification the inclusion $\alpha_e : G_e \to G_u$ corresponds to the canonical inclusion $\pi_1(S_e, u_*) \to \pi_1(S_u, u_*)$.

(4) Let e be an edge of X with initial and terminal vertices u and v respectively, and let g be an element of G_e. In accordance with (2) the element $\alpha_e(g)$ corresponds to a homotopy class $[l]$ in $\pi_1(S_u, u_*)$ and the element $\omega_e(g)$ corresponds to a homotopy class $[l']$ in $\pi_1(S_v, v_*)$. The relation $\alpha_e(g) = e\omega_e(g)\overline{e}$ is valid in $\pi_1(\mathbb{G}, X, u)$. The corresponding equality $[l] = [e_* l' \overline{e_*}]$ must be valid in $\pi_1(T, u_*)$.

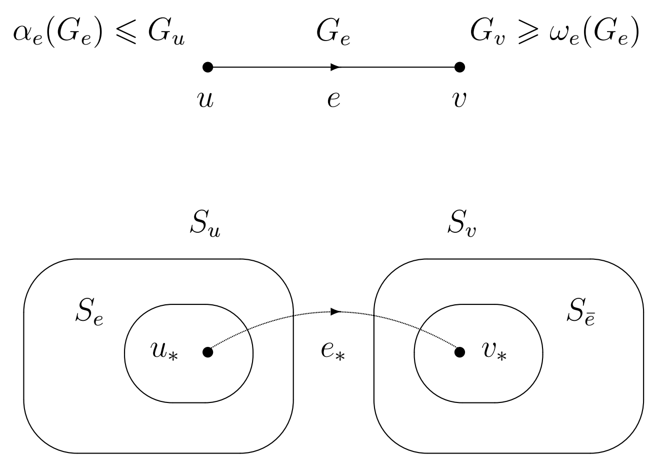

Figure 0

Now suppose that the graph of groups $(\mathbb{G}, X)$ is geometrically realized in T. For any vertex $u \in X^0$ and any element $g \in G_u$ choose a closed path $\widetilde{g}$ in S_u that originates at u_* and whose homotopy class is identified with g. Fix a vertex $v \in X^0$. Then we can define a homomorphism $\theta : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, v_*)$ by the following rule: an element $g_1 e_1 g_2 \dots e_n g_{n+1}$ of the group $\pi_1(\mathbb{G}, X, v)$ is mapped to the homotopy class of the path $\widetilde{g}_1 e_{1*} \widetilde{g}_2 \dots e_{n*} \widetilde{g}_{n+1}$.

Definition 0.3. Let $(\mathbb{G}, X)$ be a graph of groups, $v \in X^0$ and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be a homomorphism. We say that φ is *geometric* if there is a geometric realization of $(\mathbb{G}, X)$ for which $x = v_*$ and the homomorphism θ constructed above coincides with φ. We say that *the decomposition* $\pi_1(\mathbb{G}, X, v)$ *is geometrically realized in* T (with respect to φ).

Remark 0.4. Let T be a closed surface with a basepoint x and let $\varphi : G_1 \underset{G_3}{*} G_2 \to \pi_1(T, x)$ be a geometric isomorphism. Then the decomposition $\pi_1(T, x) = \varphi(G_1) \underset{\varphi(G_3)}{*} \varphi(G_2)$ is geometric in the sense of Definition 0.1. This follows from Theorem 2.6, because $\varphi(G_3)$ is realized by an incompressible subsurface in T.

Remark 0.5. Let $H \leqslant \pi_1(\mathbb{G}, X, v)$. By the Bass – Serre theory there is the induced decomposition of H: $H = \pi_1(\mathbb{H}, Y, w)$. If an isomorphism $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ is geometric and $p : (\widetilde{T}, \widetilde{x}) \to (T, x)$ is a covering corresponding to the subgroup $\varphi(H)$, then the isomorphism $p_*^{-1} \circ \varphi|_H : \pi_1(\mathbb{H}, Y, w) \to \pi_1(\widetilde{T}, \widetilde{x})$ is also geometric. If $\{S_u\}, \{S_e\}$ and $\{e_*\}$ are sets of subsurfaces and paths in T corresponding to a realization of the graph of groups $(\mathbb{G}, X)$, then the connected components of $p^{-1}(S_u), p^{-1}(S_e)$ and $p^{-1}(e_*)$ are the sets of subsurfaces and paths in $\widetilde{T}$ corresponding to a realization of the graph of groups $(\mathbb{H}, Y)$.

Let $(\mathbb{G}, X, v)$ be a graph of groups with a fixed vertex v and let Γ be a maximal subtree in X. For an arbitrary vertex $u \in X^0$ let p_u be the reduced path in Γ from v to u. The subgroups $p_u G_u p_u^{-1} = \{p_u g p_u^{-1} \,|\, g \in G_u\}$ where $u \in X^0$ are called *the vertex subgroups*, the subgroups $p_{\alpha(e)} \alpha_e(G_e) p_{\alpha(e)}^{-1}$ where $e \in X^1$ are called *the edge subgroups* of the group $\pi_1(\mathbb{G}, X, v)$ with respect to Γ. We will denote these subgroups by G_u and G_e if there will not be a confusion. The conjugacy classes of the vertex and edge subgroups do not depend on the choice of Γ. Note, if a subgroup $H \leqslant \pi_1(T, x)$ is realized in T, then any its conjugate is realized in T also. Therefore one can speak on the realizability of vertex and edge subgroups without mentioning the chosen maximal tree.

The following theorem is a generalization of Theorems 2.6 and 3.1, and is needed to prove Theorem 4.8.

Theorem 4.7. *Let T be a closed surface, let $(\mathbb{G}, X)$ be a finite graph of groups, and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism such that the images of edge subgroups of $\pi_1(\mathbb{G}, X, v)$ are realized in T. Then there is a subgroup H of index $2^{rk(X)}$ in $\pi_1(\mathbb{G}, X, v)$ such that for its induced decomposition $\pi_1(\mathbb{H}, Y, w)$ and for the covering $p : (\overline{T}, \overline{x}) \to (T, x)$, corresponding to H, the isomorphism $p_*^{-1} \circ \varphi|_H : \pi_1(\mathbb{H}, Y, w) \to \pi_1(\overline{T}, \overline{x})$ is geometric.*

Let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism. Choose a maximal subtree Γ in X and define the edge subgroups G_e of the group $\pi_1(\mathbb{G}, X, v)$ with respect to Γ . Fix a generating set $\mathcal{G}$ of $\pi_1(T, x)$ and fix a generating set $\mathcal{G}_e$ of G_e for each $e \in X^1$. Let s_e be the sum of lengths of elements of $\mathcal{G}_e$ with respect to $\mathcal{G}$.

Now we are able to formulate our main theorem.

Theorem 4.8. *Let T be a closed surface, let $(\mathbb{G}, X)$ be a finite graph of groups with finitely generated edge groups, and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism. Then there is a subgroup H of a finite index n in $\pi_1(\mathbb{G}, X, v)$ such that for its induced decomposition $\pi_1(\mathbb{H}, Y, w)$ and for the covering $p : (\widetilde{T}, \widetilde{x}) \to (T, x)$, corresponding to H, the isomorphism $p_*^{-1} \circ \varphi|_H : \pi_1(\mathbb{H}, Y, w) \to \pi_1(\widetilde{T}, \widetilde{x})$ is geometric.*

There is a recursive function f such that $n \leqslant f(s)$ where $s = \sum_{e \in X^1} s_e$.

§ 1. Non-geometric decompositions of $\pi_1(T_g, x)$ into an amalgamated product

There is a simple method for constructing new decompositions from known one: if there is a decomposition $G = G_1 *_{G_3} G_2$ and there is an element $u \in G_1$ such that $\langle G_3, u \rangle = G_3 * \langle u \rangle$, then there is the decomposition $G = G_1 *_{(G_3 * \langle u \rangle)} (G_2 * \langle u \rangle)$.

Let T_g be a closed orientable surface of genus $g \geqslant 2$ with a basepoint x. Using this method and Lemma 1.1, we can construct a non-trivial decomposition $\pi_1(T_g, x) = G_1 \underset{G_3}{*} G_2$ with an arbitrary large $rk(G_2)$. But if S is a proper incompressible subsurface in T_g, then

$\pi_1(S,x)$ is a free group of rank at most $2g-1$. Therefore this decomposition will be non-geometric when $rk(G_2) > 2g-1$.

Lemma 1.1. *Let F_n be a free group of finite rank n and let $H \neq \{1\}$ be a finitely generated subgroup of infinite index in F_n. Then in F_n there is a subgroup L of infinite rank such that $\langle H, L\rangle = H * L$.*

Proof. By M. Hall property [H] there is a subgroup H_1 of finite index in F_n and there is a subgroup $M \leqslant H_1$ such that $H_1 = H * M$. Since H is a subgroup of infinite index in F_n, we have $M \neq \{1\}$. Take $x \in H \setminus \{1\}$ and $y \in M \setminus \{1\}$. Then we can set $L = \langle y^{-i}xy^i \,|\, i \geqslant 1\rangle$.

Of special interest are decompositions $\pi_1(T_g,x) = G_1 *_{G_3} G_2$ with $rk(G_i) \leqslant 2g-1$, $i = 1,2,3$. We will prove that among them there are non-geometric decompositions also.

For any set X let $F(X)$ denote the free group with the basis X. For any word $u \in F(X)$ let $||u||$ denote the length of u with respect to X. Set $[x,y] = x^{-1}y^{-1}xy$. For any group G and an element $g \in G$ let $\widehat{g}$ denote the conjugation by g: $\widehat{g}(x) = g^{-1}xg$, $x \in G$.

Theorem 1.2. *Let T_g be a closed orientable surface of genus $g \geqslant 2$ with a basepoint x. Consider the following presentation of its fundamental group $\pi_1(T_g,x)$:*

$$\langle a_1, b_1, \dots, a_g, b_g | \prod_{i=1}^{g} [a_i, b_i]\rangle.$$

Let u be an element of $F(a_1,b_1)$ which is not a power of $[a_1,b_1]$. Then the decomposition

$$\pi_1(T_g,x) = \langle a_1, b_1\rangle \underset{\langle [b_1,a_1],u\rangle = \langle \prod_{i=2}^{g}[a_i,b_i],u\rangle}{*} \langle a_2, b_2, \dots, a_g, b_g, u\rangle \tag{1}$$

is geometric if and only if $u = \alpha(a_1)[a_1,b_1]^k$, where α is an automorphism of $F(a_1,b_1)$ which fixes or inverts $[a_1,b_1]$, $k \in \mathbb{Z}$.

Let G be a group. We say that two decompositions $G = A_1 *_{A_3} A_2$ and $G = B_1 *_{B_3} B_2$ are *automorphic equivalent* if there is an automorphism φ of G such that $\varphi(A_i) = B_i$, $i = 1,2,3$.

Corollary 1.3. 1) *There is an algorithm which, given an element u of $F(a_1,b_1)$, decides whether the decomposition* (1) *is geometric.*

2) *For each $g \geqslant 2$ there is infinitely many non-geometric and pairwise not automorphic equivalent decompositions of $\pi_1(T_g,x)$ of kind $F_2 *_{F_2} F_{2g-1}$.*

We will use the following two lemmas.

Lemma 1.4 [CMZ]. *Let $a^{k_1}b^{l_1}\dots a^{k_s}b^{l_s}$ be a primitive element of $F(a,b)$, where $s \geqslant 1$ and all of the indicated exponents are non-zero. Then, modulo trivial changes of notations (the possible replacement of a by a^{-1} or b by b^{-1}, or a by b and b by a throughout), there is an integer $n > 0$ such that $k_1 = \dots = k_s = 1$ and $\{l_1, l_2, \dots, l_s\} \subseteq \{n, n+1\}$.*

Lemma 1.5. *Let H be a finitely generated subgroup in $\pi_1(T_g,x)$, $g \geqslant 2$ and let S_1, S_2 be two incompressible subsurfaces in T_g realizing H. Then there is an isotopy of T_g which maps (S_1,x) onto (S_2,x) and induces the identity on $\pi_1(T_g,x)$.*

Proof. In [B, Lemma 4.6] it was proved the existence of an isotopy i which maps S_1 onto S_2. Since $x \in int(S_k)$, $k = 1, 2$, we may assume that $i(x) = x$. Then i_* is an inner automorphism of $\pi_1(T_g, x)$ such that $i_*(H) = H$. Split the surface T_g into subsurfaces by cutting it along ∂S_1. This gives a presentation of the group $\pi_1(T_g, x)$ as the fundamental group of a graph of groups. One of the vertex groups coincides with H. Analysing this graph of groups, conclude that the normalizer of H coincides with H. Therefore there is an element $h \in H$ such that $i_* = \widehat{h}$. Since $h \in H$, there is an isotopy j' of the surface S_1 such that $j'(x) = x$ and $j'_* = \widehat{h}|_H$. Extend j' to an isotopy j of T_g. Then $j^{-1}i$ is the desired isotopy.

Proof of Theorem 1.2. Suppose that the decomposition (1) is geometric. Let S_1, S_2 and S_3 be incompressible subsurfaces corresponding to this decomposition. Here S_1 realizes $G_1 = \langle a_1, b_1 \rangle$, S_2 realizes the second factor, S_3 realizes $G_3 = \langle [a_1, b_1], u \rangle$. According to Lemma 1.5 we may assume that S_1 coincides with the subsurface depicted in Figure 1. We see that in S_1 there is an incompressible subsurface S realizing $G = \langle [a_1, b_1], a_1 \rangle$ with the property $\partial S_1 \subset S$.

Note that up to homeomorphisms fixing ∂S_1 there is only one incompressible subsurface Δ in S_1 with the following properties:

1) $x \in int(\Delta)$,
2) $\partial S_1 \subset \Delta$,
3) $\pi_1(\Delta, x)$ is a free group of rank 2,
4) $\pi_1(\Delta, x)$ is a proper subgroup of G_1.

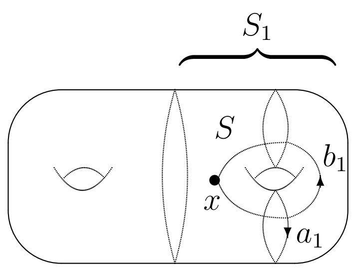

Figure 1

Therefore there is a homeomorphism h of the subsurface S_1 such that $h(S_3) = S$. Since $x \in int(S)$ and $x \in int(S_3)$, we may assume that h fixes x. Let h_* denote the automorphism of $\pi_1(S_1, x) = F(a_1, b_1)$ induced by h. Then $h_*(G_3) = G$. It is known that any automorphism of $F(a_1, b_1)$ stabilizes or inverts the commutator $[a_1, b_1]$ up to conjugacy. Then there are $v \in F(a_1, b_1)$ and $\varphi \in \mathrm{Aut}(F(a_1, b_1))$ such that $h_* = \widehat{v} \circ \varphi$ and φ stabilizes or inverts $[a_1, b_1]$. Then $v^{-1}[a_1, b_1]v \in G = \langle a_1, b_1^{-1}a_1b_1 \rangle$. Write $v^{-1}[a_1, b_1]v$ as $g^{-1}wg$ where $g \in G$ and $w = a_1^{\epsilon_1} \dots b_1^{-1}a_1^{\epsilon_2}b_1$ is a reduced word in a_1 and $b_1^{-1}a_1b_1$. Then $w = [a_1, b_1]$ and vg^{-1} is a power of $[a_1, b_1]$. Hence $v \in G$.

This implies that $\varphi(G_3) = G$, that is $\langle [a_1, b_1], a_1 \rangle = \langle [a_1, b_1], \varphi(u) \rangle$. Therefore $\varphi(u) = [a_1, b_1]^p a_1^{\varepsilon} [a_1, b_1]^q$ for some $p, q \in \mathbb{Z}, \varepsilon \in \{-1, 1\}$.

Let ψ be an automorphism of $F(a_1, b_1)$ such that $\psi(a_1) = a_1^{-1}, \psi(b_1) = b_1a_1$. Set $\varphi_1 = \varphi^{-1}$ if $\varepsilon = 1$, and $\varphi_1 = \varphi^{-1} \circ \psi$ if $\varepsilon = -1$. Then $u = [a_1, b_1]^{p_1} \varphi_1(a_1) [a_1, b_1]^{q_1}$ where

$p_1=p, q_1=q$ if φ stabilizes $[a_1,b_1]$, and $p_1=-p, q_1=-q$ if φ inverts $[a_1,b_1]$. It remains to set $\alpha=\widehat{[a_1,b_1]}^{-p_1}\circ\varphi_1,\ k=p_1+q_1$.

Conversely, suppose that $u=\alpha(a_1)[a_1,b_1]^k$, where α is an automorphism of $F(a_1,b_1)$ which fixes or inverts $[a_1,b_1],\ k\in\mathbb{Z}$. Then the decomposition (1) is automorphic equivalent to the analogous one with $u=a_1$. Hence it is geometric.

Proof of Corollary 1.3. 1) It is clear that if u is a power of $[a_1,b_1]$, then the decomposition (1) is geometric. Therefore suppose that u is not a power of $[a_1,b_1]$. Suppose that $u=\alpha(a_1)[a_1,b_1]^k$ where α is an automorphism of $F(a_1,b_1)$, fixing or inverting $[a_1,b_1]$. Then $|k|\leqslant ||u||/4+1$, otherwise the word $u[a_1,b_1]^{-k}$ after cyclic reducing contains a letter $z\in\{a_1,b_1\}$ together with z^{-1}. This contradicts to Lemma 1.4. So, it is sufficient to answer the following question:

Given $w\in F(a_1,b_1)$, is there an automorphism α of $F(a_1,b_1)$ such that $\alpha(a_1)=w$ and α fixes or inverts $[a_1,b_1]$?

This can be done by Whitehead's algorithm (see [LS]).

2) By Theorem 1.2 and Lemma 1.4, the decomposition (1) is non-geometric for any $u=a^i$, $i=2,3,\dots$. Using the abelianization of $F(a_1,b_1)$, we can deduce that these decompositions are pairwise not automorphic equivalent.

Conjecture. *Any non-trivial decomposition $\pi_1(T_g,x)=G_1\underset{G_3}{*}G_2$, where G_3 is finitely generated, can be obtained from a decomposition of $\pi_1(T_g,x)$ over Z by the method described at the beginning of this section.*

§ 2. Criterion for geometricity of decomposition of $\pi_1(T,x)$ into an amalgamated product

Recall some definitions from [O]. Let S be a surface and let $\mathcal{U}$ be an alphabet. *A diagram on S over the alphabet $\mathcal{U}$* is a cellular subdivision Δ of S whose edges e are labeled by letters $\varphi(e)\in\mathcal{U}\cup\mathcal{U}^{-1}\cup\{1\}$ so that $\varphi(e^{-1})=(\varphi(e))^{-1}$. The *label of a path* $p=e_1\dots e_n$ in the 1-skeleton of Δ is the word $\varphi(p)=\varphi(e_1)\dots\varphi(e_n)$. Let G be a group and let $\langle\mathcal{U}\,|\,\mathcal{R}\rangle$ be a presentation of G. A 2-cell in Δ is called *$\mathcal{R}$-cell* if the label of its boundary path is graphically equal, up to a cyclic permutation and inversion, to a word $R\in\mathcal{R}$. A 2-cell in Δ is called *$\mathcal{O}$-cell* if the label of its boundary path $e_1\dots e_n$ is graphically equal to $\varphi(e_1)\dots\varphi(e_n)$ where either $\varphi(e_i)\equiv 1$ for all i, or there are indexes $i\neq j$ such that $\varphi(e_i)\equiv a\in\mathcal{U},\ \varphi(e_j)\equiv a^{-1}$ and $\varphi(e_k)\equiv 1$ for $k\neq i,j$. *A diagram on S over the presentation $\langle\mathcal{U}\,|\,\mathcal{R}\rangle$* is a diagram on S over the alphabet $\mathcal{U}$ such that each of its 2-cells is an $\mathcal{R}$-cell or an $\mathcal{O}$-cell. The following lemma is called van Kampen's lemma.

Lemma 2.1. *Let $\langle\mathcal{U}\,|\,\mathcal{R}\rangle$ be a presentation of a group G. Let W be a non-empty word in the alphabet $\mathcal{U}\cup\mathcal{U}^{-1}$. Then $W=1$ in G iff there is a diagram on a disk over this presentation such that the label of a boundary loop of this disk is graphically equal to W.*

Lemma 2.2. *Let G be the fundamental group of a finite graph of groups. If G and all edge groups are finitely generated, then all vertex groups are also finitely generated.*

The proof follows by induction by the number of edges in the graph. Therefore it is sufficient to analyze the cases of an amalgamated product and an HNN-extension.

Let $G = G_1 \underset{G_3}{*} G_2$, $G = \langle g_1, \dots, g_n \rangle$, and $G_3 = \langle c_1, \dots, c_k \rangle$. Write each g_i as $g_i = a_{i1}b_{i1} \dots a_{i,s_i}b_{i,s_i}$ where $a_{ij} \in G_1$, $b_{ij} \in G_2$. Then G_1 is generated by the set consisting of all c_i and a_{ij}, and G_2 is generated by the set consisting of all c_i and b_{ij}. The case where G is an HNN-extension can be considered in a similar way.

In the following lemma we use the notion of the fundamental group of graph of groups with respect to a maximal subtree [Se]. This lemma can be proved using normal forms.

Lemma 2.3. *Let $(\mathbb{G}, X)$ be a graph of groups, let G_v and G_u be two its vertex groups, and let Δ be a maximal subtree in X. Suppose that $G_v^g \leqslant G_u$ in $\pi_1(\mathbb{G}, X, \Delta)$. Then $g = g_1e_1 \dots g_ne_ng_{n+1}$ where $e_1 \dots e_n$ is a path in X from v to u, $g_j \in G_{\alpha(e_j)}$ for $1 \leqslant j \leqslant n$, $g_{n+1} \in G_u$, and $G_v^{g_1e_1 \dots g_ie_i} \leqslant \omega_{e_i}(G_{e_i})$, $G_v^{g_1e_1 \dots e_ig_{i+1}} \leqslant \alpha_{e_{i+1}}(G_{e_{i+1}})$ for all i.*

Lemma 2.4. *Let S_u and S_v be two disjoint incompressible subsurfaces in a closed surface T. If the group $\pi_1(S_v)$ is conjugate to a subgroup of $\pi_1(S_u)$, then S_v is an annulus. Moreover, there is a component of $\overline{T \setminus (S_u \cup S_v)}$ which is an annulus with one boundary component lying in S_u and other one lying in S_v.*

Proof. The subdivision of T induced by the subsurfaces S_u and S_v gives a presentation of $\pi_1(T, x)$ as the fundamental group of a graph of groups $(\mathbb{G}, X)$. The set X^0 consists of the subsurfaces S_u, S_v and of the components of $\overline{T \setminus (S_u \cup S_v)}$. The set X^1 consists of the boundary components of subsurfaces from X^0. It follows from Lemma 2.3 that there are subsurfaces $S_v = C_1, \dots, C_{n+1} = S_u$ from X^0 and circles $Z_1, \dots, Z_n$ from X^1 such that Z_i is one of the common boundary components of C_i and C_{i+1}. Moreover, the inclusion of Z_1 into C_1 induces the isomorphism of their fundamental groups, and Z_i is freely homotopic to Z_{i+1} in C_{i+1}. The first assertion implies that C_1 is an annulus, the second implies that $Z_i = Z_{i+1}$ or that C_{i+1} is an annulus. If n is the minimal possible number, then $Z_i \neq Z_{i+1}$. Hence $C_2, \dots, C_n$ are annulii. The union of these annulii is an annulus also.

The following lemma can be proved in a similar way.

Lemma 2.5. *Let T be a compact surface, let S be an incompressible subsurface in T, $x \in S$. Let $1 \neq a \in \pi_1(S, x)$, $g \in \pi_1(T, x) \setminus \pi_1(S, x)$ and $a^g \in \pi_1(S, x)$. Then one of the following holds:*

1) *a and a^g are powers of homotopy classes of loops which are freely homotopic in S to two different boundary components of S. These components divide T into two parts – the part containing S and the part which is an annulus.*

2) *a and a^g are powers of homotopy classes of loops which are freely homotopic in S to the same boundary component of S. This component divides T into two parts – the part containing S and the part which is a Möbius band.*

Theorem 2.6. *Let T be a closed surface with a basepoint x and let $\pi_1(T, x) = G_1 \underset{G_3}{*} G_2$ be a decomposition of its fundamental group into an amalgamated product. If G_3 is realized by an incompressible subsurface in T, then this decomposition is geometric.*

Proof. We may assume that the decomposition $\pi_1(T, x) = G_1 \underset{G_3}{*} G_2$ is non-trivial, that is $G_3 \neq G_1$ and $G_3 \neq G_2$. Then T is not a torus. First consider the case where T is a Klein bottle. Then $\pi_1(T, x)$ has the presentation $\langle a, b \mid b^{-1}ab = a^{-1} \rangle$. The decomposition from lemma 2.7 is geometric (see Figure 2). All other decompositions are conjugate to it. Hence they are geometric also, because any conjugation is induced by an isotopy by Baer's theorem (see [ZVC]).

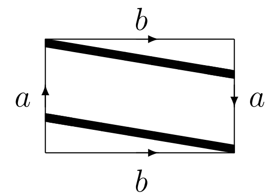

Figure 2

Lemma 2.7. *The group $G = \langle a, b \,|\, b^{-1}ab = a^{-1}\rangle$ has the unique (up to conjugacy and permuting of factors) decomposition into a non-trivial amalgamated product: $G = \langle b\rangle \underset{\langle b^2\rangle}{*} \langle ba\rangle$.*

Proof. Let $G = G_1 \underset{G_3}{*} G_2$ be a decomposition of G into a non-trivial amalgamated product. Since $\langle b^2\rangle$ is the center of G, we have $\langle b^2\rangle \leqslant G_3$. Set $G' = G/\langle b^2\rangle$, $G'_i = G_i/\langle b^2\rangle$, $i = 1, 2, 3$. Then $Z_2 * Z_2 \cong G' = G'_1 \underset{G'_3}{*} G'_2$. Since $Z_2 * Z_2$ does not contain a free group of rank 2, $|G'_1 : G'_3| = |G'_2 : G'_3| = 2$. Then $G'/G'_3 \cong Z_2 * Z_2$. Since $Z_2 * Z_2$ is a Hopfian group, $G'_3 = \{1\}$, hence $G_3 = \langle b^2\rangle$. Since $|G_1 : G_3| = |G_2 : G_3| = 2$ and G is a torsion free group, G_1 and G_2 are infinite cyclic groups. Simple calculations show that (up to conjugacy and permuting of factors) G_1 is generated by the element ba^k, and G_2 is generated by the element ba^{k+1} for some k. Conjugating by $a^{-k/2}$ for even k and by $ba^{(k+1)/2}$ for odd k, we get the decomposition from Lemma 2.7.

Now consider the case where T is not a Klein bottle. Let T be a closed surface of genus g. Then the group $\pi_1(T, x)$ has the presentation

$$\langle t_1, u_1, \ldots, t_g, u_g | \prod_{i=1}^{g} [t_i, u_i]\rangle$$

if T is orientable, and the presentation

$$\langle v_1, \ldots, v_g | \, v_1^2 \cdots v_g^2\rangle$$

if T is not orientable. For simultaneous consideration of these cases we write these presentations as

$$\langle a_1, \ldots, a_k | \textstyle\prod_*\rangle.$$

Let $\mathcal{D}$ be a disk whose boundary is divided into orientable intervals, labeled by elements from the set $\{a_1, a_1^{-1}, \ldots, a_k, a_k^{-1}\}$ so that the label of the boundary of the disk is cyclically equal to $\prod_*$. The surface T can be obtained from $\mathcal{D}$ by gluing the edges with the same labels. Let $p : \mathcal{D} \to T$ be the corresponding morphism of complexes.

Now we will construct a complex K corresponding to the decomposition $\pi_1(T, x) = G_1 \underset{G_3}{*} G_2$. Since the index of G_i in $\pi_1(T, x)$ is infinite, G_i is a free group $(i = 1, 2, 3)$. Since G_3 is realized by an incompressible subsurface in T, G_3 is finitely generated. By Lemma

2.2 each group G_i is finitely generated. Let $\mathcal{S}_i$ be a basis of G_i. Write each element $s \in \mathcal{S}_3$ as a word $U_{i,s}$ in elements from $\mathcal{S}_i \cup \mathcal{S}_i^{-1}$ $(i=1,2)$. It is clear that the group $\pi_1(T,x)$ has the presentation $\langle \mathcal{S} \,|\, \mathcal{R} \rangle$ where $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \mathcal{S}_3$ and $\mathcal{R} = \{s^{-1}U_{i,s} \,|\, s \in \mathcal{S}_3,\ i=1,2\}$.

Let (R_i, x_i) be a graph with the unique vertex x_i and with the set of positively oriented edges $\{\widetilde{s} \,|\, s \in \mathcal{S}_i\}$, $i=1,2,3$. Let Γ be a graph consisting of the graphs R_1, R_2, R_3 and of two oriented edges e_i $(i=1,2)$ which connect vertexes x_3 and x_i. For each $s \in \mathcal{S}_3$ glue two 2-cells $D_{1,s}$ and $D_{2,s}$ to Γ in accordance with relations $s^{-1}U_{i,s}$: if $U_{i,s} = u_1 \dots u_r$ where all $u_j \in \mathcal{S}_i \cup \mathcal{S}_i^{-1}$, then set $\partial(D_{i,s}) = (\widetilde{s})^{-1} e_i \widetilde{u}_1 e_i^{-1} \dots e_i \widetilde{u}_r e_i^{-1}$. For each $s \in S_i$ $(i=1,2)$ glue 2-cell $\mathcal{O}_s$ to Γ by identifying the boundary of $\mathcal{O}_s$ with the path $e_i \widetilde{s} e_i^{-1} e_i \widetilde{s}^{-1} e_i^{-1}$. Denote the complex we have constructed by K (Figure 3). It is clear that $\pi_1(T,x) \cong \pi_1(K, x_3)$.

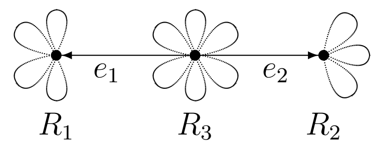

Figure 3

Below we will construct a subdivision of T and a continious map $f : (T,x) \to (K, x_3)$ inducing an isomorphism of fundamental groups. Write each generator a_i as a word $w_i(g_1, \dots, g_n)$ where each $g_j \in \mathcal{S}_1 \cup \mathcal{S}_2$. If an edge from the boundary of disk $\mathcal{D}$ has a label $a_i^{\pm 1}$, then we subdivide it into small edges labeled by letters from the set $\{g_1, g_1^{-1}, \dots, g_n, g_n^{-1}\}$ so that the word reading along this edge is equal to $w_i^{\pm 1}(g_1, \dots, g_n)$. So, we obtain a disk $\mathcal{D}_1$ whose boundary label $W(g_1, \dots, g_n)$ is equal to 1 in $\pi_1(T,x)$. By van Kampen's lemma we may assume that $\mathcal{D}_1$ is a diagram on a disk over the presentation $\langle \mathcal{S} \,|\, \mathcal{R} \rangle$. Using the projection p, we can get a diagram Δ on T over the presentation $\langle \mathcal{S} \,|\, \mathcal{R} \rangle$. Subdivide each edge labeled by $s \in \mathcal{S}_i$ $(i=1,2)$ into three edges with labels $e_i, \widetilde{s}, e_i^{-1}$. We do not subdivide edges labeled by $s \in \mathcal{S}_3$ but change their labels from s to $\widetilde{s}$. Denote the new diagram on T over the alphabet $\{\widetilde{s} \,|\, s \in \mathcal{S}\} \cup \{e_1, e_2\}$ by T again. Now, define a continues map $f : T \to K$, sending the edges labeled by $e_i, \widetilde{s}, e_i^{-1}$ to the edges $e_i, \widetilde{s}, e_i^{-1}$, the edges labeled by 1 to the vertex x_3, and extending this map onto 2-cells obviously. Denote the initial vertex of the path $p(a_1)$ by x. Then f induces the epimorphism $f_* : \pi_1(T,x) \to \pi_1(K, x_3)$. This epimorphism is an isomorphism since the group $\pi_1(T,x)$ is Hopfian.

Describe briefly a plan of the proof of the theorem. The preimage $f^{-1}(R_3)$ consists of a finite number of subcomplexes of T. Subdividing the complexes T and K, and redefining the map f in a neighborhood of $f^{-1}(R_3)$, we may assume that each component C of $f^{-1}(R_3)$ is a subsurface in T. Moreover, we may assume that C is an incompressible subsurface. Indeed, if some component of the complement of C is a disk, then we can redefine f on this disk so that not only the boundary of this disk, but the whole disk is mapped into R_3. We may achieve also that any component of $f^{-1}(R_3)$ is not a disk. Let S_3 be an incompressible subsurface in T realizing the subgroup G_3. By obvious identifications we have $f_*(\pi_1(S_3, x)) = G_3 = \pi_1(R_3, x_3)$. The difficulty is that initially $f(S_3)$ not necessarily lie in R_3. By Claim 1 below we may assume that one of the components of

$f^{-1}(R_3)$ realizes G_3. Denote this component by S. By Claim 2 each other component of $f^{-1}(R_3)$ is a ring parallel to a boundary component of S. If the surface T is orientable, then redefining f, it is possible to "adjoin" these rings to S and to achieve the coincidence of $f^{-1}(R_3)$ with S. If T is non-orientable, we adjoin all these rings to S except some of them, which lie in the components of $T \setminus S$ homeomorphic to a Möbius band. After that it will be proved that G_1 is the fundamental group of the union of S and some components of the complement of S; G_2 is the fundamental group of the union of S and the remaining components of the complement of S.

We will use the following transformations of the surface T and the map f.

Transformation $D(l)$. Let l be a simple (possible closed) arc in T with ends y and z. Suppose that $f(y) = f(z) = x_3$ and that the loop $f(l)$ is homotopic to a loop in R_3. Cut the surface T along $int(l)$ and glue a disc D by identifying its boundary with the boundary of this cut. Choose in D a simple path l' from y to z which divides D into two disks so that each of these disks contains exactly one edge of this cut. Subdivide D into cells and continue f on D so that the loop $f(l')$ lies in R_3.

Claim 1. *Using a finite number of transformations of kind $D(l)$ it is possible to get that one of the components of $f^{-1}(R_3)$ is an incompressible subsurface realizing G_3.*

Proof. Let S_3 be a subsurface realizing G_3. Suppose that S_3 has r boundary components. Let $\gamma_1, \dots, \gamma_{r+t}$ be a system of simple closed curves in S_3 based at x such that the following hold:

(1) $\gamma_i \cap \gamma_j = \{x\}$ for $i \neq j$ and γ_k is freely homotopic in S_3 to the k-th boundary component of S_3 $(1 \leqslant k \leqslant r)$,

(2) cutting S_3 along $\gamma_1, \dots, \gamma_{r+t}$, we get r rings and a disk P.

We consider these rings and the disk as embedded in S_3. Using transformations of kind $D(\gamma_i)$, we get that all loops $f(\gamma_i')$ lie in R_3. Therefore, we may assume at the beginning that all loops $f(\gamma_i)$ lie in R_3. Since the boundary of the disk P is mapped to R_3, we can perform new subdivisions of P and redefine f so that $f|\partial P$ remains unchanged and $f(P) \subset R_3$. Similarly, one can redefine f in a regular neighborhood of each ring so that $f(S_3) \subseteq R_3$ and the claim will be satisfied.

Denote the component from Claim 1 by S.

Claim 2. *If S_1 is a component of $f^{-1}(R_3)$ different from S, then S_1 is a ring. Moreover, the ring S_1 is parallel to some boundary component of S.*

Proof. Redefining f in a neighborhood of S_1, we may assume that there is a point $y \in S_1 \cap f^{-1}(x_3)$. Let l be a simple path in T from x to y. Let H be the subgroup consisting of homotopy classes of loops lsl^{-1} where s goes over all loops in S_1 based at y. Note that $f(l)$ is a loop in K based at x_3, and $f(s)$ is a loop in R_3 based at x_3. Hence the subgroup $f_*(H)$ is conjugate to a subgroup of $\pi_1(R_3, x_3) = f_*(\pi_1(S, x))$ by the element $[f(l)]$. Since f_* is an isomorphism, H is conjugate to a subgroup of $\pi_1(S, x)$ in $\pi_1(T, x)$, and the claim follows from Lemma 2.4.

Let S_1 be a componenent of $f^{-1}(R_3)$ different from S. Redefining f, it may be assumed that there is a point $y \in S_1 \cup f^{-1}(x_3)$. Let s be an arbitrary loop in S_1 based at y whose homotopy class generates $\pi_1(S_1, y)$. Consider three cases.

Case 1. The component of $\overline{T \setminus S}$ containing S_1 is neither a ring nor a Möbius band.

By Claim 2 the closure $\overline{T \setminus (S \cup S_1)}$ contains the unique component C which is a ring with one boundary component in S and the other one in S_1. Let l be a simple curve in $S \cup C \cup S_1$ from x to y.

CLAIM 3. *The loop $f(l)$ is homotopic to a loop from R_3.*

Proof. Denote $z = [f(l)]$. Since S is a retract of $S \cup C \cup S_1$, $[lsl^{-1}] \in G_3$. The element $[f(lsl^{-1})] \in \pi_1(R_3, x_3)$ is conjugate to $[f(s)] \in \pi_1(R_3, x_3)$ by z. Hence the element $[lsl^{-1}] \in \pi_1(S, x)$ is conjugate to $f_*^{-1}([f(s)]) \in \pi_1(S, x)$ by $f_*^{-1}(z)$. It follows from Lemma 2.5 that $f_*^{-1}(z) \in \pi_1(S, x)$, therefore $z \in \pi_1(R_3, x_3)$.

Making the transformation $D(l)$, we may assume that $f(l) \subset R_3$. If we cut the ring C along l, we obtain a disk whose boundary is mapped by f in R_3. This enable us to redefine f on C so that $f(C) \subseteq R_3$. After that the number of components of $f^{-1}(R_3)$ is reduced by one.

Case 2. The component of $\overline{T \setminus S}$ containing S_1 is a ring.

Let C be this ring, let C_1, C_2 be two components of $\overline{S_1 \setminus C}$, and let l_i be a simple curve from x to y in $S \cup C_i \cup S_1$, $i = 1, 2$. Denote $t = [l_1 l_2^{-1}]$, $z_i = [f(l_i)]$, $t_i = f_*^{-1}(z_i)$. Then $t = t_1 t_2^{-1}$. Set $A_i = \langle [l_i s l_i^{-1}] \rangle$.

The group $\pi_1(S \cup C, x)$ is an HNN-extension with the base $\pi_1(S, x)$, the stable latter t, and associated subgroups A_1 and A_2.

Arguing as in case 1, we get $A_1^{t_1} \leqslant \pi_1(S, x)$. Since T is not a torus, $t_1 \in \pi_1(S, x)$ or $t_1 \in t\pi_1(S, x)$. If $t_1 \in t\pi_1(S, x)$, then $t_2 = t^{-1} t_1 \in \pi_1(S, x)$. Hence z_1 or z_2 belongs to $\pi_1(R_3, x_3)$. This enable us to redefine f on C_1 or on C_2 and to reduce the number of components of $f^{-1}(R_3)$.

Case 3. The component $\overline{T \setminus S}$ containing S_1 is a Möbius band.

Let M be this Möbius band. Then $\overline{T \setminus (S \cup S_1)}$ contains the unique component C which is a ring with one boundary component in S and the other one in S_1. Assume that S_1 is a component of $f^{-1}(R_3)$ which is the nearest to S among those which lie in M, that is $int(C) \cap f^{-1}(R_3) = \emptyset$. Let l be a simple curve in $S \cup C \cup S_1$ from x to y, $z = [f(l)]$. We have

$$\pi_1(S \cup M, x) = \pi_1(S, x) \underset{\langle a^2 \rangle}{*} \langle a \rangle,$$

where $a^2 = [lsl^{-1}]$. Arguing as in the proof of Claim 3 and recalling that T is not a Klein bottle, we get $f_*^{-1}(z) \in \pi_1(S, x) \cup a\pi_1(S, x)$. If $f_*^{-1}(z) \in \pi_1(S, x)$, then the loop $f(l)$ is homotopic to a loop in R_3. So, we can redefine f and adjoin S_1 to S as in Case 1.

Let $f_*^{-1}(z) \in a\pi_1(S, x)$. Since l intersects only one component of $\overline{T \setminus (S \cup S_1)}$, $z \in G_1$ or $z \in G_2$. Hence $f_*(a) \in G_1$ or $f_*(a) \in G_2$. We will call M by Möbius band of kind 1 or kind 2 respectively. In this subcase we does not adjoin S_1 to S.

After a finite number of such changes, we get that $f^{-1}(R_3)$ will have the unique component outside the union of Möbius bands of kinds 1 and 2. This component realizes G_3. Denote it by S_3, and denote the union of Möbius bands of kind i by M_i. The subcomplex K_3 divides the complex K into two components. Let R_i' denote the component containing R_i $(i = 1, 2)$. Then a part of components of $T \setminus (S_3 \cup M_1 \cup M_2)$ lies in the preimage of R_1', another part lies in the preimage of R_2'. Let S_i denote the union of S_3,

M_i and those components of $T\setminus(S_3\cup M_1\cup M_2)$ which lie in the preimage of R_i' $(i=1,2)$. Set $G_i'=\pi_1(S_i,x)$.

Then all S_i are incompressible subsurfaces, $T=S_1\cup S_2$, $S_1\cap S_2=S_3$, hence $\pi_1(T,x)=G_1'\underset{G_3}{*}G_2'$. Since $\pi_1(T,x)=G_1\underset{G_3}{*}G_2$, $G_1'\leqslant G_1$, $G_2'\leqslant G_2$, it follows from the normal form of an element in the amalgamated product that $G_1=G_1'$, $G_2=G_2'$. Theorem 2.6 is proved.

§ 3. Criterion for geometricity of decomposition of $\pi_1(T,x)$ into an HNN-extensions

The following example shows how to construct a nontrivial HNN-extensions isomorphic to $\pi_1(T_g,x)$.

Example. Let N be a subsurface in T_g such that $\overline{T_g\setminus N}$ is a ring, $x\in N$. Then $\pi_1(T_g,x)=\langle H,t\,|\,Z_1=t^{-1}Z_2t\rangle$ where $H=\pi_1(N,x)$, Z_1 and Z_2 are subgroups of H corresponding to the boundary components of N, t is a stable letter corresponding to the handle $\overline{T_g\setminus N}$. Introduce new generators $\overline{h}$ and new relations $\overline{h}=t^{-1}ht$ $(h\in H)$. For any subgroup $K\leqslant H$ let $\overline{K}$ denote the group $\{\overline{k}\,|\,k\in K\}$. Then we can rewrite the presentation of $\pi_1(T_g,x)$ as

$$\langle H,\overline{H},t\,|\,Z_1=\overline{Z}_2,\,\overline{H}=t^{-1}Ht\rangle=\langle H\underset{Z_1=\overline{Z}_2}{*}\overline{H},\,t\,|\,\overline{H}=t^{-1}Ht\rangle.$$

The base $H\underset{Z_1=\overline{Z}_2}{*}\overline{H}$ of this HNN-extension is not realized in T_g, however it is realised in a 2-fold covering $\widetilde{T}_g$ (Figure 4).

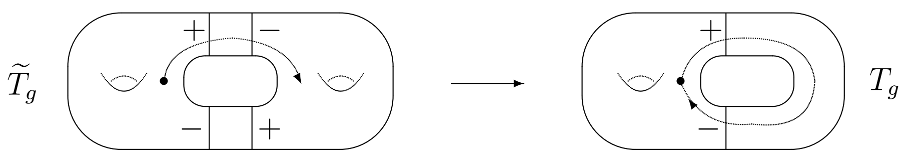

Figure 4

Theorem 3.1. *Let T be a closed surface with a basepoint x and let $\pi_1(T,x)=G\underset{H_1=t^{-1}H_2t}{*}$ be a decomposition of its fundamental group into an HNN-extension. If H_1 is realized by an incompressible subsurface in T, then G is also realized by an incompressible subsurface in a 2-fold covering of T. Moreover, G is the fundamental group of a graph of groups with cyclic edge groups and with two distinguished vertex groups H_1 and H_2.*

Proof. The theorem is clear if T is a torus. If T is a Klein bottle, then the theorem follows from Lemma 3.2 and Nielsen's theorem that for any closed surface each automorphism of its fundamental group is induced by a homeomorphism of this surface (see [ZVC]).

Lemma 3.2. *The group $A=\langle a,b\,|\,b^{-1}ab=a^{-1}\rangle$ has the unique (up to automorphisms) presentation as an HNN-extension.*

Proof. Let $A = \langle G, t \,|\, H_1 = t^{-1}H_2t\rangle$. Since every nontrivial subgroup of infinite index in A is isomorphic to Z, $G \cong Z$. Easy computations show that $G = H_1$ and t inverts a generator of G.

Now, suppose that T is neither a torus, nor a Klein bottle.

Let (R_0, v_0), (R_1, v_1) be two roses, whose fundamental groups are identified with G and H_1. Let Γ denote the graph consisting of these roses and two oriented edges e_1 and e_2 joining the vertexes v_1 and v_0. Glue 2-cells to Γ so that the fundamental group of the resulting complex K with respect to v_1 is naturally identified with the group $G \underset{H_1=t^{-1}H_2t}{*}$ (Figure 5).

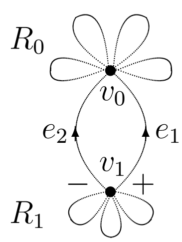

Figure 5

The elements of G correspond to loops in $\{v_1\} \cup \{e_1, \bar{e}_1\} \cup R_0$ based at v_1, and the stable letter t corresponds to the loop $e_1e_2^{-1}$.

More precisely, let $\varphi : H_1 \to H_2$ be an isomorphism such that $\varphi(h) = tht^{-1}$ for $h \in H_1$. Let $\{h_1, \dots, h_n\}$ be a basis of H_1, let $\{g_1, \dots, g_m\}$ be a basis of G, and let $h_i = u_i(g_1, \dots, g_m)$, $\varphi(h_i) = w_i(g_1, \dots, g_m)$. For each h_i glue 2-cells to Γ along the paths $\widetilde{h}_ie_1\widetilde{u}_i^{-1}e_1^{-1}$ and $\widetilde{h}_ie_2\widetilde{w}_i^{-1}e_2^{-1}$ where $\widetilde{h}_i$ is the simple loop in R_1 corresponding to the element h_i; $\widetilde{u}_i$ and $\widetilde{w}_i$ are the loops in R_0 corresponding to the words u_i and w_i. Let K denote the resulting complex.

As in the proof of Theorem 2.6 it is possible to construct a continues map $f : (T, x) \to (K, v_1)$ which induces an isomorphism of fundamental groups. We will identify $\pi_1(T, x)$ and $\pi_1(K, v_1)$ using f_*.

First consider the case where the surface T is orientable. In this case we can get as in § 2 that the preimage $f^{-1}(R_1)$ is an incompressible subsurface S in T realizing the subgroup H_1. A boundary component of S will be called *positive* (*negative*) if it has a regular neighborhood which is mapped into e_1 (into e_2) by f_*.

Let $M_1, \dots, M_r$ be all components of $T \setminus int(S)$ ordered so that for some $p \leqslant r$ each of the components $M_1, \dots, M_p$ has at least one positive boundary component, and each of the components $M_{p+1}, \dots, M_r$ has only negative boundary components. Note that there is M_i which has both positive and negative boundary components. Otherwise, considering the map $f : T \to K$, we get that the group $\pi_1(T, x)$ is generated by G and $t^{-1}Gt$, that is impossible. So, assume that M_1 is one of these components.

Write a presentation of $\pi_1(T, x)$ using subdivision of T into S and $M_1, \dots, M_r$. The positive boundary components of S lying in M_i will be denoted by $a_{i1}, \dots, a_{in_i}$, the negative by $b_{i1}, \dots, b_{im_i}$. In each such component a_{ij}, b_{ij} choose a point, an orientation, and consider a_{ij} and b_{ij} as loops.

In S choose a basepoint x and simple paths P_{ij}, and Q_{ij} from x to the initial points of a_{ij}, and b_{ij}, respectively. In each M_i choose a basepoint x_i and simple paths p_{ij}, and q_{ij} from x_i to the initial points of a_{ij} and b_{ij}, respectively. Assume that no pair of these paths has common interior point (Figure 6).

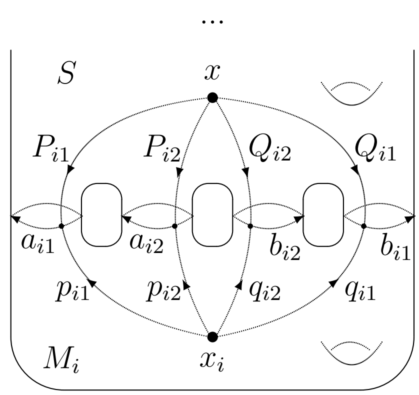

Figure 6

For $i \leqslant p$ denote

$$x_{ij} = P_{ij} a_{ij} P_{ij}^{-1}, \quad y_{ij} = Q_{ij} b_{ij} Q_{ij}^{-1}, \quad x'_{ij} = P_{i1} p_{i1}^{-1} p_{ij} a_{ij} p_{ij}^{-1} p_{i1} P_{i1}^{-1},$$

$$y'_{ij} = P_{i1} p_{i1}^{-1} q_{ij} b_{ij} q_{ij}^{-1} p_{i1} P_{i1}^{-1}, \quad t_{ij} = P_{i1} p_{i1}^{-1} p_{ij} P_{ij}^{-1}, \quad l_{ij} = P_{i1} p_{i1}^{-1} q_{ij} Q_{ij}^{-1}.$$

For $i > p$ denote

$$u_{ij} = Q_{ij} b_{ij} Q_{ij}^{-1}, \quad v_{ij} = Q_{i1} q_{i1}^{-1} q_{ij} b_{ij} q_{ij}^{-1} q_{i1} Q_{i1}^{-1}, \quad d_{ij} = Q_{i1} q_{i1}^{-1} q_{ij} Q_{ij}^{-1}.$$

Embed the group $\pi_1(M_i, x_i)$ into the group $\pi_1(T, x)$, using the map $[l] \mapsto [P_{i1} p_{i1}^{-1} l p_{i1} P_{i1}^{-1}]$ for $i \leqslant p$, and the map $[l] \mapsto [Q_{i1} q_{i1}^{-1} l q_{i1} Q_{i1}^{-1}]$ for $i > p$. Denote the image of this embedding by $\pi_1(M_i)$. For convenience we will denote loops and their homotopy classes by the same latters. Below pairs of indexes i, k and i, j, and the pair of indexes k, j are going over the sets $\bigcup\limits_{1 \leqslant s \leqslant p} (\{s\} \times \{1, \dots, n_s\})$, $\bigcup\limits_{1 \leqslant s \leqslant p} (\{s\} \times \{1, \dots, m_s\})$, and $\bigcup\limits_{p+1 \leqslant s \leqslant r} (\{s\} \times \{1, \dots, m_s\})$, respectively. Let F be the free group with the basis $\{t_{ik}, l_{ij}, d_{kj}\}$. Then the group $\pi_1(T, x)$ has the presentation

$$\langle \pi_1(S, x) * \pi_1(M_1) * \dots * \pi_1(M_r) * F \mid$$

$$t_{i1} = 1,\ t_{ik} x_{ik} t_{ik}^{-1} = x'_{ik},\ l_{ij} y_{ij} l_{ij}^{-1} = y'_{ij},\ d_{k1} = 1,\ d_{kj} u_{kj} d_{kj}^{-1} = v_{kj} \rangle. \qquad (2)$$

Introduce new generators L_{ij}, y'' $(y \in \pi_1(S, x))$, D_{kj}, v'' $(v \in \cup_{s=p+1}^{r} \pi_1(M_s))$ and new relations $L_{ij} = l_{ij} l_{11}^{-1}$, $y'' = l_{11} y l_{11}^{-1}$, $D_{kj} = l_{11} d_{kj} l_{11}^{-1}$, $v'' = l_{11} v l_{11}^{-1}$. Let $\pi_1(S'')$ denote the isomorphic copy of the group $\pi_1(S, x)$ consisting of the elements y'' where y goes over $\pi_1(S, x)$. Let $\pi_1(M''_i)$ denote the isomorphic copy of the group $\pi_1(M_i)$ consisting of the elements v'' where v goes over $\pi_1(M_i)$. In the subsequent these groups will correspond to subsurfaces S'' and M_i'' in a 2-fold covering $\widetilde{T}$ of T. Let $\mathcal{F}$ denote the free group with the basis $\{t_{ik}, L_{ij}, D_{kj}\}$. Then, using Tietze transformations, we can rewrite the presentstion (2) as

$$\langle \pi_1(S,x) * \pi_1(M_1) * \cdots * \pi_1(M_p) * \pi_1(M''_{p+1}) * \cdots * \pi_1(M''_r) * \pi_1(S'') * \mathcal{F},\ l_{11} \mid$$

$$t_{i1} = 1,\ t_{ik}x_{ik}t_{ik}^{-1} = x'_{ik},\ L_{11} = 1,\ L_{ij}y''_{ij}L_{ij}^{-1} = y'_{ij},\ D_{k1} = 1,\ D_{kj}u''_{kj}D_{kj}^{-1} = v''_{kj},$$
$$l_{11}yl_{11}^{-1} = y''\ (y \in \pi_1(S,x))\rangle. \tag{3}$$

Now, it is clear that $\pi_1(T,x)$ is an HNN-extension with the base $G' = \langle \pi_1(S,x),\ \pi_1(M_1), \ldots, \pi_1(M_p),\ \pi_1(M''_{p+1}), \ldots, \pi_1(M''_r),\ \pi_1(S''),\ \mathcal{F}\rangle$, the stable letter l_{11}, and the associated subgroups $\pi_1(S,x)$ and $\pi_1(S'')$:

$$\pi_1(T,x) = \langle G',\ l_{11} \mid \pi_1(S'') = l_{11}\pi_1(S,x)l_{11}^{-1}\rangle. \tag{4}$$

Lemma 3.3. *Let l be a loop in T based at x.*

1) *If l intersects only positive boundary components of S, then $[l] \in G$.*

2) *If l intersects only negative boundary components of S, then $[l] \in t^{-1}Gt$.*

3) *If l intersects (transversely) exactly two boundary components of S, first positive, and then negative, then $[l] = gt$ for some $g \in G$.*

Proof. Consider the loop $f(l)$ in K. In the first case $f(l)$ is homotopic to the loop from the subcomplex $\{v_1\} \cup \{e_1, \bar{e}_1\} \cup R_0$ whose fundamental group is identified with G. In the second case $f(l)$ is homotopic to a loop from the subcomplex $\{v_1\} \cup \{e_2, \bar{e}_2\} \cup R_0$ whose fundamental group is identified with $t^{-1}Gt$.

In the third case $f(l)$ is homotopic to a loop $h_1e_1ue_2^{-1}h_2$ where h_1, h_2 are loops in R_1 and u is a loop in R_0. Hence $f(l)$ is homotopic to the loop $h_1e_1ue_1^{-1} \cdot e_1e_2^{-1}h_2e_2e_1^{-1} \cdot e_1e_2^{-1}$. Since $[h_1e_1ue_1^{-1}] \in G$, $[e_1e_2^{-1}] = t$ and $tH_1t^{-1} = H_2 \leqslant G$, we get $[l] = gt$ where $g \in G$.

Lemma 3.4. *$G' = G$.*

Proof. Lemma 3.3 implies that $\pi_1(M_i) \leqslant G$ for $i = 1, \ldots, p$, $\pi_1(M_i) \leqslant t^{-1}Gt$ for $i = p+1, \ldots, r$, $t_{ik} \in G$, $l_{ij} = g_{ij}t$ where $g_{ij} \in G$, and $d_{kj} = t^{-1}f_{kj}t$ where $f_{kj} \in G$. Also $\pi_1(S'') = l_{11}\pi_1(S,x)l_{11}^{-1} = g_{11}tH_1t^{-1}g_{11}^{-1} = g_{11}H_2g_{11}^{-1} \leqslant G$. Hence $G' \leqslant G$. Replace the stable letter t in the initial presentation $\langle G,\ t \mid H_2 = tH_1t^{-1}\rangle$ by the stable letter l_{11}. We get the new presentation $\langle G,\ l_{11} \mid g_{11}H_2g_{11}^{-1} = l_{11}H_1l_{11}^{-1}\rangle = \langle G,\ l_{11} \mid \pi_1(S'') = l_{11}\pi_1(S,x)l_{11}^{-1}\rangle$. From the normal form of an element in the HNN-extension and from (4) we get $G = G'$.

Now we will prove that G is realized in a 2-fold covering $\widetilde{T}$ of the surface T which can be constructed in the following way. Cut the surface T along all curves b_{sj}, $s \in \{i, k\}$. We get a surface (probably disconnected) with boundary components $\dot{b}_{sj}$ and $\ddot{b}_{sj}$. Take two copies T' and T'' of this surface and glue boundary components $\dot{b}_{sj}$ and $\ddot{b}_{sj}$ of the first copy to the boundary components $\ddot{b}_{sj}$ and $\dot{b}_{sj}$ of the second copy. Let $\widetilde{T}$ denote the surface we have obtained and let $\rho : \widetilde{T} \to T$ be the corresponding covering (Figure 7). We regard the surfaces T' and T'' as embedded in $\widetilde{T}$. Let S' and S'' be the components of $\rho^{-1}(S)$ lying in T' and in T'', respectively. Those boundary components of S' and S''

which are mapped by ρ to positive (negative) boundary components of S will be called positive (negative).

Denote the subsurface $T' \cup S''$ by M', and the subsurface $T'' \cup S'$ by M''. Obviously, M' is one of the components, which appear by cutting $\widetilde{T}$ along the negative boundary components of S' and along the positive boundary components of S''. The subsurface M' is colored in Figure 7. Let x' and x'' be the lifts of x in S' and in S''. Take x' as a basepoint of $\widetilde{T}$. It is clear that $\pi_1(M', x') = G' = G$. The last assertion of Theorem 3.1 follows from a consideration of the surface M' or from the presentation (3) using equalities $\pi_1(S, x) = H_1$ and $\pi_1(S'') = g_{11} H_2 g_{11}^{-1}$.

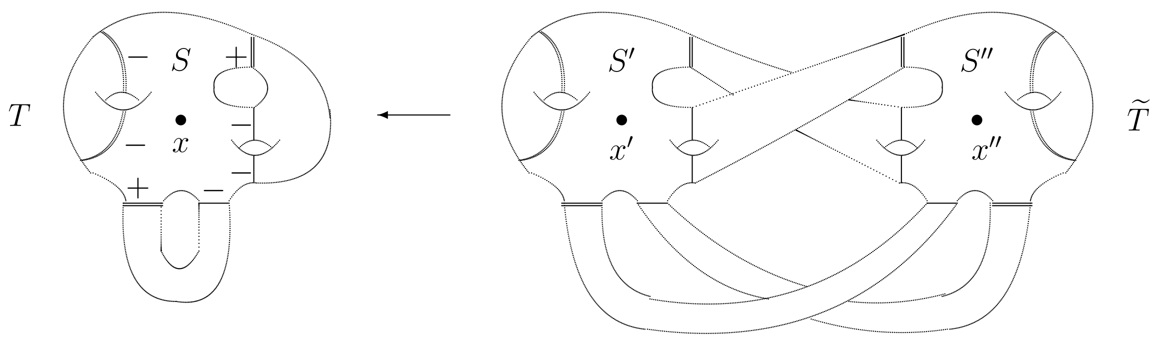

Figure 7

Now, suppose that the surface T is non-orientable. In this case we can not achieve the situation where the preimage $f^{-1}(R_1)$ is connected and coincides with an incompressible subsurface S in T, realizing the subgroup H_1. The obstacle is the subcase of Case 3 from § 2 where $f_*^{-1}(z) \in a\pi_1(S, x)$ (here and below we use notations from the analysis of this subcase). Then $f_*(a) = zh$ for some $h \in H_1$. Moreover, in this case the inclusion $a^2 \in \pi_1(S, x)$ holds, hence $f_*(a^2) \in H_1$. Consider 4 variants.

1) The boundary components $S \cap C$ and $C \cap S_1$ are positive. Then $z \in G$, hence $f_*(a) \in G$.

2) The boundary components $S \cap C$ and $C \cap S_1$ are negative. Then $z \in t^{-1}Gt$, hence $f_*(a) \in t^{-1}Gt$.

3) The boundary component $S \cap C$ is positive and the boundary component $C \cap S_1$ is negative. Then $z = gt$ for some $g \in G$, that contradicts to the inclusion $(zh)^2 \in H_1$.

4) The boundary component $S \cap C$ is negative and the boundary component $C \cap S_1$ is positive. Then $z = t^{-1}g$ for some $g \in G$ and this variant is also impossible.

If the first (the second) variant holds, we say that the Möbius band M, which is considered in Case 3 from § 2, is of kind 1 (of kind 2). Arguing as in § 2, we can achieve the situation where $f^{-1}(R_1)$ will have the unique component outside the union of Möbius bands of kinds 1 and 2. Denote this component by S. Now, the proof can be completed as in the case where T is orientable.

§ 4. Virtual geometricity of decompositions of $\pi_1(T,x)$ into the fundamental group of graph of groups

In this section T is a closed surface with a basepoint x. We prove Theorems 4.7 and 4.8 using the following technical lemmas.

Lemma 4.1. *Let $A_1 \leqslant A_2 \leqslant \pi_1(T,x)$. If S_1 and S_2 are incompressible subsurfaces in T realizing A_1 and A_2, then there is an isotopy i of T such that the subsurface $i(S_1)$ lies in S_2 and realizes A_1.*

The proof of this lemma is analogous to the proof of [B, Lemma 4.6].

Lemma 4.2. *Let $A_1 \leqslant A_2 \leqslant \pi_1(T,x)$. If A_1 is realized in T, then A_1 is realized in the covering of T which corresponds to A_2.*

The following lemma is more general.

Lemma 4.3. *Let $A_1, A_2 \leqslant \pi_1(T,x)$. If A_1 is realized in T, then $A_1 \cap A_2$ is realized in the covering of T which corresponds to A_2.*

Proof. Let S be an incompressible subsurface in T, $x \in S$ and $\pi_1(S,x) = A_1$. Let $p : (\widetilde{T}, \widetilde{x}) \to (T,x)$ be the covering corresponding to A_2. Then the component of $p^{-1}(S)$, containing $\widetilde{x}$, realizes $A_1 \cap A_2$ in $\widetilde{T}$.

Let S be an incompressible subsurface in T and let q be a path from x to some point $u \in S$. We will say that the subgroup $H \leqslant \pi_1(T,x)$ is realized by the pair (q,S) if $H = \{[qlq^{-1}] \mid [l] \in \pi_1(S,u)\}$.

Lemma 4.4. *Let $A_1 \leqslant A_2 \leqslant \pi_1(T,x)$. If A_1 and A_2 are realized by pairs (q_1,S_1) and (q_2,S_2), then A_1 is realized by a pair (q_2,S_2') where $S_2' \subset S_2$.*

Proof. We may change the base point and assume that q_2 is the trivial path based at x. Let q_1' be a simple path from x to the terminal point of q_1 such that q_1' and ∂S_1 have at most one common point. Let S_1' be the union of S_1 and a small regular neighborhood of the curve q_1'. Then $\pi_1(S_1',x)$ is conjugate to A_1 by the element $[q_1' q_1^{-1}]$. Let i be an isotopy inducing the conjugation by this element. Then $i(S_1')$ is a subsurface realizing A_1. By Lemma 4.1 the subgroup A_1 is realized by a subsurface S_2' in S_2.

Corollary 4.5. *If a subgroup $A \leqslant \pi_1(T,x)$ is realized by a pair (q,S), then it is realized by an incompressible subsurface in T.*

Denote this subsurface by $\{q,S\}$.

Lemma 4.6. *Let $\pi_1(\mathbb{G},X,v) = \pi_1(T,x)$ and let the edge subrgroups of $\pi_1(\mathbb{G},X,v)$ are realized in T. Let H be a subgroup of a finite index in $\pi_1(T,x)$ and let $p : (\widetilde{T},\widetilde{x}) \to (T,x)$ be the covering corresponding to H. Suppose that all vertex subgroups of the induced decomposition $\pi_1(\mathbb{H},Y,w)$ of H are realized in $\widetilde{T}$. Then the natural identification of $\pi_1(\mathbb{H},Y,w)$ with $\pi_1(\widetilde{T},\widetilde{x})$ is geometric.*

Proof. Denote $G = \pi_1(\mathbb{G},X,v)$. There is n-fold covering of graphs $\rho : (Y,w) \to (X,v)$ where $n = |G:H|$. Let Γ be a maximal subtree in X. We can choose a maximal subtree Δ in Y so that it contains all n lifts of Γ. We may assume that all vertex and edge subgroups of $\pi_1(\mathbb{G},X,v)$ (of $\pi_1(\mathbb{H},Y,w)$) are defined with respect to Γ (with respect to

Δ). Let $v_1, \dots, v_n$ be all lifts of v in T, let l_i be the reduced path in Δ from w to v_i, and let g_i be the element of G, corresponding to the homotopy class of the path $\rho(l_i)$. It is clear that $L = \{g_1, \dots, g_n\}$ is a right transversal of H in G.

Each vertex subgroup H has the form $gVg^{-1} \cap H$ where $g \in L$ and V is a vertex subgroup of G. If E is an edge subgroup in V, then $gEg^{-1} \cap H$ is an edge subgroup in $gVg^{-1} \cap H$. It follows from the condition of lemma that the group $gVg^{-1} \cap H$ is realized by a subsurface $S_{g,V}$ in $(\widetilde{T}, \widetilde{x})$. Let l be a loop in T, whose homotopy class is equal to g, and let $\widetilde{l}$ be its lift in $\widetilde{T}$ which originates at $\widetilde{x}$. Let S_E be a subsurface in T realizing E, and let $S_{g,E}$ be its lift in $\widetilde{T}$, containing the terminal point of $\widetilde{l}$. Then the subgroup $gEg^{-1} \cap H$ is realized in $\widetilde{T}$ by the pair $(\widetilde{l}, S_{g,E})$. By Corollary 4.5 and Lemma 4.1 it is realized by a subsurface in $S_{g,V}$.

For each vertex $u \in Y^0$ set $u_* = \widetilde{x}$. Let e be an edge in Y with initial vertex u_1 and with terminal vertex u_2. Let E_1, and E_2 be the edge subgroups of H corresponding to e and $\bar{e}$. Denote the reduced path in Δ from w to u_i by p_i. Then $t^{-1}E_1t = E_2$ where $t = p_1 e p_2^{-1}$. Let e_* be a loop in $(\widetilde{T}, \widetilde{x})$, whose homotopy class is equal to t. This gives a realization of graph of groups $(\mathbb{H}, Y)$ in $\widetilde{T}$ which induces a geometric isomorphism of groups $\pi_1(\mathbb{H}, Y, w)$ and $\pi_1(\widetilde{T}, \widetilde{x})$.

Theorem 4.7. *Let T be a closed surface, let $(\mathbb{G}, X)$ be a finite graph of groups, and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism such that the images of edge subgroups of $\pi_1(\mathbb{G}, X, v)$ are realized in T. Then there is a subgroup H of index $2^{rk(X)}$ in $\pi_1(\mathbb{G}, X, v)$ such that for its induced decomposition $\pi_1(\mathbb{H}, Y, w)$ and for the covering $p : (\overline{T}, \overline{x}) \to (T, x)$, corresponding to H, the isomorphism $p_*^{-1} \circ \varphi|_H : \pi_1(\mathbb{H}, Y, w) \to \pi_1(\overline{T}, \overline{x})$ is geometric.*

Proof. We will identify the groups $\pi_1(\mathbb{G}, X, v)$ and $\pi_1(T, x)$ using φ, and the groups $\pi_1(\mathbb{H}, Y, w)$ and $\pi_1(\overline{T}, \overline{x})$ using $p_*^{-1} \circ \varphi|_H$. If T is a torus or a Klein bottle, then the proof is direct (in the case of an HNN-extension the subgroup H is defined below). So, suppose that T is not a torus and is not a Klein bottle.

Choose a maximal tree Γ in X and an orientation X^1_+. For $u \in X^0$ denote the reduced path in Γ from v to u by p_u. Let $X^1_+ \setminus \Gamma^1 = \{e_1, \dots, e_k\}$. Let t_i denote the element $p_{\alpha(e_i)} e_i p^{-1}_{\omega(e_i)}$ of $\pi_1(\mathbb{G}, X, v)$. Let H be the kernel of the epimorphism $\pi_1(\mathbb{G}, X, v) \to \prod_{i=1}^k \langle t_i \,|\, t_i^2 = 1 \rangle$ which sends each vertex subgroup to 1 and each t_i to t_i. We will prove the theorem by induction by k.

Suppose that $k = 0$. Then X is a tree. By Lemma 4.6 it is sufficient to prove that each vertex group G_v is realized in T. Let $E = \{e_1, \dots, e_m\}$ be the set of all edges in X emanating from v. For $e \in E$ the graph $X \setminus \{e, \bar{e}\}$ consists of two connected components. Denote the component which contains v by $X_{1,e}$, and the other one by $X_{2,e}$. This splitting induces the decomposition $\pi_1(\mathbb{G}, X, v) = G_{1,e} \underset{G_e}{*} G_{2,e}$. By Theorem 2.6 this decomposition is geometric, that is there are incompressible subsurfaces $S_{1,e}, S_{2,e}$ and S_e realizing $G_{1,e}$, $G_{2,e}$ and G_e, moreover $T = S_{1,e} \cup S_{2,e}$ and $S_e = S_{1,e} \cap S_{2,e}$. We will construct a chain of incompressible subsurfaces $S_{1,v} \supseteq S_{2,v} \supseteq \dots \supseteq S_{m,v}$ such that $\pi_1(S_{i,v}, x) = G_{1,e_1} \cap \dots \cap G_{1,e_i}$, $i = 1, \dots, m$. Then $S_{m,v}$ will realize G_v. Set $S_{1,v} = S_{1,e}$. Suppose that the subsurface $S_{i,v}$ is defined. Since $G_{e_{i+1}} \leqslant G_{1,e_1} \cap \dots \cap G_{1,e_i}$, we may assume by Lemma 4.1 that $S_{e_{i+1}} \subseteq S_{i,v}$. Set $S_{i+1,v} = S_{i,v} \cap S_{1,e_{i+1}}$.

So, for $k = 0$ the theorem is proved. Suppose that $k = n \geqslant 1$ and the theorem is proved for $k = n-1$. Let G be the fundamental group of graph of groups $(\mathbb{G}_1, X_1)$ which is obtained from $(\mathbb{G}, X)$ by deleting the edges $e_n, \bar{e}_n$ and the groups $G_{e_n}, G_{\bar{e}_n}$. Then $\pi_1(\mathbb{G}, X, v) = \langle G, t \,|\, H_1 = t^{-1}H_2t\rangle$ where $t = t_n$, H_1 and H_2 are associated subgroups corresponding to the embeddings α_{e_n} and $\alpha_{\bar{e}_n}$ into the vertex groups, which we denote by V_1 and V_2.

Let K be the kernel of the homomorphism $\langle G, t \,|\, H_1 = t^{-1}H_2t\rangle \to \langle t \,|\, t^2 = 1\rangle$, which sends G to 1 and t to t. The induced decomposition of K is the fundamental group of graph of groups $\mathcal{K}$, which is depicted in Figure 8 on the left. The groups H_1 and H_2 are embedded into the top group G identically and into the bottom group G by the maps $h_1 \mapsto th_1t^{-1}$ $(h_1 \in H_1)$ and $h_2 \mapsto t^{-1}h_2t$ $(h_2 \in H_2)$.

Figure 8

Let N be the kernel of the epimorphism $G = \pi_1(\mathbb{G}_1, X_1, v) \to \prod_{i=1}^{n-1}\langle t_i \,|\, t_i^2 = 1\rangle$, which sends each vertex group to 1 and each stable letter t_i to t_i. Let $\mathcal{N}$ be the graph of groups corresponding to the decomposition of N with respect to $\pi_1(\mathbb{G}_1, X_1, v)$. Then $\mathcal{N}$ has 2^{n-1} vertexes with vertex groups V_1 and 2^{n-1} vertexes with vertex groups V_2.

The graph of groups corresponding to the decomposition of H with respect to $\pi_1(\mathbb{G}, X, v)$ is depicted in Figure 8 on the right. It can be obtained from two copies of $\mathcal{N}$ (the top and the bottom one) by connecting them by 2^{n-1} edges with edge groups H_1, and by 2^{n-1} edges with edge groups H_2. Each H_1-edge connects a V_1-vertex of the top copy with the corresponding V_2-vertex of the bottom copy. Each H_2-edge connects a V_2-vertex of the top copy with the corresponding V_1-vertex of the bottom copy.

The covering $\widetilde{T}$ from the proof of Theorem 3.1 corresponds to the group K. Informally, the subsurfaces M', M'' and S', S'' in $\widetilde{T}$ correspond to the vertex groups G, G and to the edge groups H_1, H_2 of the graph of groups $\mathcal{K}$. Note that $\widetilde{T} = M' \cup M''$, $M' \cap M'' = S' \cup S''$ and $S' \cap S'' = \emptyset$ (see Figure 7).

Now, describe formally the realization of $\mathcal{K}$ in $\widetilde{T}$. Let l be a loop in T, whose homotopy class is equal to t. Let l' be the lift of this loop into M' with the origin x' and the end x''. Let l'' be the lift of this loop in M'' with the origin x'' and the end x'. The points x' and x'' correspond to the vertices of the graph $\mathcal{K}$, the paths l', l'' and their inverses correspond to the edges of this graph.

The subsurfaces M' and M'' with base points x' and x'' correspond to the vertex groups. The subsurfaces S', $\{l', S''\}$ in M', and the subsurfaces $\{l'', S'\}$, S'' in M'' correspond to the edge groups.

By Lemmas 4.1 and 4.2 the edge subgroups of G, which correspond to the edges $e_1, \dots, e_{n-1}$, are realized in M'. By induction the graph of groups $\mathcal{N}$ is geometrically realized in the 2^{n-1}-fold covering $\overline{M'}$ of M', corresponding to the subgroup N of G. This covering is regular, $H_1 \leqslant N$, and H_1 is realized by the subsurface S' in M'. Hence, there are exactly 2^{n-1} lifts of S' into $\overline{M'}$. By analogy, there are exactly 2^{n-1} lifts of S'' into $\overline{M'}$. Symmetrically we can construct 2^{n-1}-fold covering $\overline{M''}$ of the surface M''.

Glue $\overline{M'}$ to $\overline{M''}$ by identifying the corresponding lifts of S', S''. As a result we obtain 2^n-fold covering $\overline{T}$ corresponding to H. All vertex groups of H are realized in $\overline{T}$. By Lemma 4.6 the natural identification $\pi_1(\mathbb{H}, Y, w)$ with $\pi_1(\overline{T}, \overline{x})$ is geometric. Theorem 4.7 is proved.

In the proof of Theorem 4.8 we will use the following theorem of P. Scott.

Theorem [Sc]. *Let Σ be a compact surface with a basepoint x. For any finitely generated subgroup H of $\pi_1(\Sigma, x)$ there is a finite covering $p : (\Sigma_1, x_1) \to (\Sigma, x)$ and an incompressible subsurface $S \subseteq \Sigma_1$ such that $x_1 \in int(S)$ and $p_*(\pi_1(S, x_1)) = H$.*

Remind the definition of numbers s_e, which will be used in Theorem 4.8. Let $\mathcal{A}$ be a fixed system of generators of $\pi_1(T, x)$, and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism. Choose a maximal subtree in X and define edge subgroups G_e of the group $\pi_1(\mathbb{G}, X, v)$ with respect to this tree. Suppose that all the groups G_e are finitely generated. Choose in $\varphi(G_e)$ a finite set of generators and denote the sum of length of its elements with respect to $\mathcal{A}$ by s_e.

Theorem 4.8. *Let T be a closed surface, let $(\mathbb{G}, X)$ be a finite graph of groups with finitely generated edge groups, and let $\varphi : \pi_1(\mathbb{G}, X, v) \to \pi_1(T, x)$ be an isomorphism. Then there is a subgroup H of a finite index n in $\pi_1(\mathbb{G}, X, v)$ such that for its induced decomposition $\pi_1(\mathbb{H}, Y, w)$ and the covering $p : (\widetilde{T}, \widetilde{x}) \to (T, x)$, corresponding to H, the isomorphism $p_*^{-1} \circ \varphi|_H : \pi_1(\mathbb{H}, Y, w) \to \pi_1(\widetilde{T}, \widetilde{x})$ is geometric.*

There is a recursive function f such that $n \leqslant f(s)$ where $s = \sum_{e \in X^1} s_e$.

Proof. Identify the groups $\pi_1(\mathbb{G}, X, v)$ and $\pi_1(T, x)$ using the isomorphism φ. By the theorem of P. Scott, for each edge subgroup G_e there is a subgroup of finite index $G'_e \leqslant \pi_1(T, x)$ such that G_e is realized in the finite covering corresponding to G'_e. For any $g \in \pi_1(T, x)$ the subgroup G_e^g is realized in the finite covering corresponding to $(G'_e)^g$. Set $N = \bigcap_{e \in X^1} (\bigcap_{g \in \pi_1(T,x)} (G'_e)^g)$. It is clear that N is a subgroup of finite index in $\pi_1(T, x)$. Let $\overline{T}$ be the covering of T corresponding to N. The edge groups from the induced decomposition $N = \pi_1(\mathbb{N}, Z, u)$ have the form $G_e^g \cap N$. By Lemma 4.3 they are realized by incompressible subsurfaces in $\overline{T}$. By theorem 4.7 there is a subgroup H of index $2^{rk(Z)}$ in N whose induced decomposition $\pi_1(\mathbb{H}, Y, w)$ is realized in the corresponding $2^{rk(Z)}$-fold covering $\widetilde{T}$ of the surface $\overline{T}$.

Now estimate the number n. Since the group $\pi_1(T, x)$ is not a non-trivial free product, we may assume that all the groups G_e are non-trivial, hence $|X^1| \leqslant s$. In [Sc] a procedure for constructing the covering corresponding to G'_e is given. This procedure is effective, because it uses a core of the covering corresponding to G_e, and this core can be constructed effectively by Proposition 3.3 in [B]. Analyzing the proofs in [Sc] and [B], we can deduce that there is a monotone recursive function f such that $|\pi_1(T, x) : G'_e| \leqslant f(s_e)$. Then $m = |\pi_1(T, x) : N| \leqslant \prod_{e \in X^1} (f(s_e))! \leqslant ((f(s))!)^s$. By Reidemeister – Shreier method the group N is generated by at most $(|\mathcal{A}| - 1)m + 1$ elements. Since $\pi_1(Z, u)$ is a factor group of N, $rk(Z) \leqslant (|\mathcal{A}| - 1)m + 1$. This implies the desired estimation of n.

§ 5. The edge rigidity property

In this section we investigate decompositions of groups, concentrating on the following question: in what extent the edge groups in these decompositions determine vertex groups? Define some notions.

The *extended graph of groups* is the ordered set $(\mathbb{G}, X, x, \Gamma)$ where $(\mathbb{G}, X)$ is a graph of groups, $x \in X^0$ and Γ is a maximal tree in X. Set $\pi_1(\mathbb{G}, X, x, \Gamma) = \pi_1(\mathbb{G}, X, x)$. For an arbitrary vertex $v \in X^0$ denote the reduced path in Γ from x to v by p_v. The subgroups $p_u G_u p_u^{-1} = \{p_u g p_u^{-1} \mid g \in G_u\}$, where $u \in X^0$, are called the vertex subgroups, the subgroups $p_{\alpha(e)}\alpha_e(G_e)p_{\alpha(e)}^{-1}$, where $e \in X^1$, are called the edge subgroups of the group $\pi_1(\mathbb{G}, X, v)$ with respect to Γ.

We will say that the group G has *the edge rigidity property with respect to a finite set of its subgroups* $G_1, \dots, G_n$ if there is only a finite number of variants for sets of vertex subgroups under identifications of G with the fundamental groups of extended graph of groups with the edge subgroups $G_1, \dots, G_n$.

We will say that G has *the edge rigidity property* if G has the edge rigidity property with respect to any finite set of its finitely generated subgroups.

Theorem 5.1. *The fundamental group of a closed surface different from the Klein bottle has the edge rigidity property.*

Proof. We will use the terminology and notations from the proof of Theorems 2.6 and 3.1. Let T be a closed surface different from the Klein bottle and the torus (for the torus the theorem is obvious). Since $\pi_1(T, x)$ is freely indecomposable, it is sufficient to prove that $\pi_1(T, x)$ has the edge rigidity property with respect to any finite set of non-trivial finitely generated subgroups.

Suppose that $\pi_1(T, x) = G_1 \underset{G_3}{*} G_2$ and G_3 is realized by an incompressible subsurface S in T. It follows from the proof of Theorem 2.6 that G_1 is the fundamental group of the union of S and some components of the complement of S in T, and G_2 is the fundamental group of the union of S and the remaining components of this complement. So, if G_3 is fixed, then there is only a finite number of variants for G_1 and G_2.

Suppose that the group $\pi_1(T, x)$ is identified with an HNN-extension $\langle G, t \mid H_1 = t^{-1}H_2 t\rangle$ and H_1 is realized by an incompressible subsurface S in T. It follows from the proof of Theorem 3.1 that $G = \pi_1(M', x')$ where M' is a subsurface in a 2-fold covering $\widetilde{T}$ of T. The subsurface M' contains the subsurfaces S' and S'' (see § 3), the boundary of M' is the union of all negative boundary components of S' and all positive boundary components of S''. If H_1 is a fixed group, then there is only a finite number of variants for a marking of boundary components of S by plus and minus. The same holds for S' and S''. Hence there is only a finite number of variants for G.

Suppose that the group $\pi_1(T, x)$ is identified with the fundamental group $\pi_1(\mathbb{G}, X, v, \Gamma)$ whose edge subgroups are realized by incompressible subsurfaces in T. Fix a vertex u in X. Let $E(u)$ denote the set of all edges of X, emanating from u. For $e \in E$ let $X_{e,u}$ denote the component of $X \setminus \{e, \bar{e}\}$ containing u. Let $G_{e,u}$ be the subgroup of $\pi_1(\mathbb{G}, X, v, \Gamma)$ corresponding to $X_{e,u}$. If e separates X, then the group $\pi_1(\mathbb{G}, X, v, \Gamma)$ can be expressed as an amalgamated product such that $G_{e,u}$ is one of its factors. If e does not separate X, then the group can be expressed as an HNN-extension with the base $G_{e,u}$. So, there is only a finite number of variants for $G_{e,u}$. Then the vertex subgroup $p_u G_u p_u^{-1} = \bigcap_{e\in E(u)} G_{e,u}$ is defined up to a finite number of variants also.

Consider the general case, assuming that the edge subgroups of the group $\pi_1(\mathbb{G}, X, v, \Gamma)$ are finitely generated. By Theorem 4.8 there is a subgroup H of a finite index in $\pi_1(\mathbb{G}, X, v)$ whose induced decomposition $\pi_1(\mathbb{H}, Y, w)$ is geometrically realized in the covering $(\widetilde{T}, \widetilde{x})$ corresponding to H. This index depends on the edge subgroups G_e ($e \in X^1$) only. So, if we fix the subgroups G_e, then there is only a finite number of variants for vertex subgroups of $\pi_1(\mathbb{H}, Y, w)$. There are groups $G_u \cap H$ among these subgroups, where G_u goes over the set of vertex subgroups of $\pi_1(\mathbb{G}, X, v, \Gamma)$. By Lemma 5.2 there is a finite number of variants for the groups G_u also.

The following lemma follows from [G, Theorem 6].

Lemma 5.2. *Let T be a compact surface different from the Klein bottle, and let x be a basepoint of T. For any nontrivial finitely generated subgroup $H \leqslant \pi_1(T, x)$ there is a largest subgroup G with the property $|G : H| < \infty$.*

It is clear that the free group of rank $n \geqslant 2$ does not have the edge rigidity property with respect to the trivial subgroup.

The fundamental group of the Klein bottle $G = \langle a, b \,|\, b^{-1}ab = a^{-1}\rangle$ does not have the edge rigidity property with respect to the subgroup $\langle b^2\rangle$, because there is the decomposition $G = \langle ba^{2k}\rangle \underset{\langle b^2\rangle}{*} \langle ba^{2k+1}\rangle$ for any integer k.

Describe some unusial examples of groups G without the edge rigidity property (see [BW]). In these examples $G = A *_C B_i$, where the subgroups A are C fixed, $B_i \not\cong B_j$ for $i \neq j$. In the first example each of the groups B_i is represented as an amalgamated product, in the second example as an HNN-extension.

1. Let $C < A$, $\{C_i\}_{i\geqslant 1}$ be proper subgroups of C, $\tau_i : C_1 \to C_i$ be isomorphisms, and $\{a_i\}_{i\geqslant 1}$ be elements of A such that $a_1 = 1$, and $\tau_i(c_1) = a_i c_1 a_i^{-1}$ for any $c_1 \in C_1$, $i \geqslant 1$. Let $D_1 < D$, $\varphi : C_1 \to D_1$ be an isomorphism. Set $\varphi_i = \varphi \circ \tau_i^{-1}$, $B_i = C *_{C_i \overset{\varphi_i}{=} D_1} D$, and $G_i = A *_C B_i$. It is clear that for each $i \geqslant 1$ there is an isomorphism $\psi_i : G_1 \to G_i$ such that $\psi_i(a) = a$ for $a \in A$ and $\psi_i(d) = a_i^{-1} d a_i$ for $d \in D$. Show that there are finitely presented groups A, B_i and C with the above properties and such that $B_i \not\cong B_j$ for $i \neq j$.

Set $C = \langle x \,|\, -\rangle$, $C_i = \langle x^{2^i}\rangle$, $A = \langle C, t \,|\, txt^{-1} = x^2\rangle$, $a_i = t^{i-1}, D = \langle y\rangle$, $D_1 = \langle y^2\rangle$, $\varphi(x^2) = y^2$. Then $B_i = \langle x, y \,|\, x^{2^i} = y^2\rangle$.

2. Let groups A, C, C_i and elements a_i be as in the first example, $B_i = \langle C, \widetilde{t}_i \,|\, \widetilde{t}_i c_1 \widetilde{t}_i^{-1} = \tau_i(c_1)\, (c_1 \in C_1)\rangle$, $G_i = A *_C B_i$. Then for any $i \geqslant 1$ there is the isomorphism $\psi_i : G_1 \to G_i$ such that $\psi_i(a) = a$ for $a \in A$ and $\psi_i(\widetilde{t}_1) = a_i^{-1}\widetilde{t}_i$.

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