On embeddings of $Out(F_n)$, the outer automorphism group of the free group of rank n, into the group $Out(F_m)$ for $m > n$

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Let F_n be the free group of finite rank n, $Aut(F_n)$, $Out(F_n)$, and $Inn(F_n)$ be the groups of all, outer, and inner automorphisms of F_n respectively. Clearly, the group $Aut(F_n)$ can be embedded into the group $Aut(F_{n+1})$. In the article of Magnus and Tretkoff [1] it is mentioned, that the group $Out(F_n)$ can be embedded into the group $Out(F_{n+1})$ for $n \geqslant 2$. This assertion was disproved by D. G. Khramtsov [2].

In this article we prove the following theorem.

Theorem. *For each $n \geqslant 1$ the group $Out(F_n)$ can be embedded into the group $Out(F_m)$ where $m = 1 + (n-1)k^n$, and k is an arbitrary natural number co-prime to $n - 1$.*

Proof. For $n = 1$ the theorem is obvious. So, furthermore we assume that $n \geqslant 2$. Therefore $Inn(F_n) \simeq F_n$, and we identify the element $x \in F_n$ with the automorphism $\widehat{x} \in Inn(F_n)$, acting by the rule: $g \mapsto x^{-1}gx,\ g \in F_n$.

Let k be a fixed natural number co-prime to $n-1$, and let H be the verbal subgroup generated by the commutator subgroup F_n' and by all words of the form w^k. Then $F_n/H \simeq \underbrace{Z_k \times \cdots \times Z_k}_{n}$, and the rank of H is $1 + (n-1)k^n$. The group $Aut(F_n)$ can be embedded into the group $Aut(H)$ using the map $\alpha \mapsto \alpha|_H$, where $\alpha \in Aut(F_n)$. Indeed, if $\alpha \in Aut(F_n)$ and the restriction of α on H is the identical automorphism, then $\alpha(x^{k^n}) = x^{k^n}$ for each $x \in F_n$. Since the root extraction in the group F_n is unique, $\alpha(x) = x$. This embedding induces an embedding of the group $Aut(F_n)/H$ into the group $Aut(H)/H \simeq Out(H) \simeq Out(F_m)$. We have $(Aut(F_n)/H)/(F_n/H) \simeq Out(F_n)$. Further we will prove that the corresponding extension of the group F_n/H by the group $Out(F_n)$ is splittable. Then the group $Out(F_n)$ is embeddable into $Aut(F_n)/H$, and hence into $Out(F_m)$ also.

Further we will use the following composition rule for automorphisms: if $\phi, \psi \in Aut(F_n)$, then $\phi\psi(x) = \psi(\phi(x))$ for $x \in F_n$. Denote $[x,y] = xyx^{-1}y^{-1}$, $x^y = y^{-1}xy$. Let $X = \{x_1, \dots, x_n\}$ be a base of F_n. Nielsen proved (see for example [3]), that the group $Aut(F_n)$ is generated by the set of automorphisms $\{E_{xy} \,|\, x, y \in X^{\pm 1},\ y \neq x, x^{-1}\}$, of the form $E_{xy} : x \mapsto xy,\ z \mapsto z$ for $z \in X^{\pm 1} \setminus \{x, x^{-1}\}$, together with the automorphism τ, which inverts x_1 and fixes the remaining generators from X. Gersten showed in [4] that $Aut(F_n)$ has the presentation with this generator set and the following set of defining relations:

($\mathcal{R}1$) $E_{xy}^{-1} = E_{xy^{-1}}$,

($\mathcal{R}2$) $[E_{xy}, E_{zt}] = 1$ for $z \neq x, y, y^{-1},\ t \neq x, x^{-1}$,

($\mathcal{R}3$) $[E_{xy}, E_{yz}] = E_{xz}$ for $z \neq x, x^{-1}$,

($\mathcal{R}4$) $B_{yx} = B_{y^{-1}x^{-1}}$, where $B_{yx} = E_{xy}E_{y^{-1}x}E_{x^{-1}y}^{-1}$,

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($\mathcal{R}5$) $B_{yx}^4 = 1$,

($\mathcal{R}6$) $\tau^2 = 1$,

($\mathcal{R}7$) $\tau E_{xy} \tau = E_{\tau(x)\tau(y)}$.

Write shortly the presentation of $Aut(F_n)$ as

$$Aut(F_n) = \langle \tau, E_{xy} \,|\, r_1(E_{xy}, \tau) = \cdots = r_l(E_{xy}, \tau) = 1 \rangle,$$

where $r_1, \ldots, r_l$ are words, corresponding to the relations ($\mathcal{R}1$) – ($\mathcal{R}7$). Then

$$\begin{aligned} Aut(F_n)/H = \langle \tau, E_{xy}, x_1, \ldots, x_n \mid r_1(E_{xy}, \tau) = \cdots = r_l(E_{xy}, \tau) = 1, \\ x_1 = r_{l+1}(E_{xy}), \ldots, x_n = r_{l+n}(E_{xy}) \rangle / H, \end{aligned} \tag{1}$$

where the words $r_{l+1}, \ldots, r_{l+n}$ express automorphisms $\widehat{x}_1, \ldots, \widehat{x}_n$ in generators E_{xy}:

$$r_{l+i}(E_{xy}) = E_{x_1 x_i} E_{x_1^{-1} x_i} \ldots E_{x_{i-1} x_i} E_{x_{i-1}^{-1} x_i} E_{x_{i+1} x_i} E_{x_{i+1}^{-1} x_i} \ldots E_{x_n x_i} E_{x_n^{-1} x_i}.$$

We have $Out(F_n) = \langle \tau, E_{xy} \mid r_1(E_{xy}, \tau) = \cdots = r_{l+n}(E_{xy}, \tau) = 1 \rangle$.

For each E_{xy} find such a word w_{xy} from F_n (call this word by "multiplying"), that using notations $E'_{xy} = E_{xy} w_{xy}$ we have the following inclusions:

$$\begin{aligned} &r_i(E'_{xy}, \tau) \in F'_n \ (i = 1, \ldots, l). \\ &r_i(E'_{xy}, \tau) \in H \ (i = l+1, \ldots, l+n). \end{aligned} \tag{2}$$

Then $Aut(F_n)/H = \langle \tau, E'_{xy}, x_1, \ldots, x_n \,|\, r_i(E'_{xy}, \tau) = 1 \ (i = 1, \ldots, l+n),\ x_j^{E'_{xy}} = E'_{xy}(x_j),\ x_j^\tau = \tau(x_j) \ (j = 1, \ldots, n),\ h = 1 \ (h \in H) \rangle \simeq (F_n/H) \rtimes Out(F_n)$, and the splitting will be proved.

The theorem will be proved if the system of equations for multiplying words, coming from (2), is decidable. Let us make computations for $n \geqslant 4$. For $n = 2$ and 3 computations are similar, moreover some cases are excluded, and the system will be simpler.

There is a natural epimorphism from $Aut(F_n)$ to $GL_n(\mathbb{Z}) \simeq Aut(F_n/F'_n)$. Matrices which are images of automorphisms E_{xy} and B_{xy} denote by $\mathbf{E_{xy}}$ and $\mathbf{B_{xy}}$. For $w \in F_n$ and $x \in X^{\pm 1}$ denote by $l_x(w)$ the exponent by x in the abelianization of the word w, by $l(w)$ the vector $(l_{x_1}(w), \ldots, l_{x_n}(w))$, and by $\mathbf{0}$ the vector $l(1) = (0, \ldots, 0)$. Note that $l_{x_i^{-1}}(w) = -l_{x_i}(w)$, $l(w^{-1}) = -l(w)$. For $g, h \in F_n$ write $g = h \pmod{F'_n}$ if $gh^{-1} \in F'_n$.

Consider relation ($\mathcal{R}1$). From the condition (2) for multiplying words we have $w_{xy}^{-1} E_{xy}^{-1} = E_{xy^{-1}} w_{xy^{-1}} \pmod{F'_n}$. Then it is necessary that the equality $(w_{xy}^{-1})^{E_{xy}^{-1}} = w_{xy^{-1}} \pmod{F'_n}$ is satisfied. This equality is equivalent to the vector equation $-l(w_{xy}) \cdot \mathbf{E_{xy}^{-1}} = l(w_{xy^{-1}})$. From this we get the following equations for multiplying words:

$$\begin{aligned} &-l_v(w_{xy}) = l_v(w_{xy^{-1}}) \ \text{ for } \ v \neq y, y^{-1},\ v \in X^{\pm 1}, \\ &-l_y(w_{xy}) + l_x(w_{xy}) = l_y(w_{xy^{-1}}). \end{aligned}$$

Consider relation ($\mathcal{R}2$). By ($\mathcal{R}1$) it may be assumed that $y, t \in X$. Then the relation is equivalent to the system of relations (a)–(f):

(a) $[E_{xy}, E_{zt}] = 1$, where $t \neq y, y^{-1}$; $z \neq x^{-1}$,

(b) $[E_{xy}, E_{x^{-1}y}] = 1$,
(c) $[E_{xy}, E_{x^{-1}y^{-1}}] = 1$,
(d) $[E_{xy}, E_{x^{-1}t}] = 1$, where $t \neq y, y^{-1}$,
(e) $[E_{xy}, E_{zy}] = 1$, where $z \neq x^{-1}$,
(f) $[E_{xy}, E_{zy^{-1}}] = 1$, where $z \neq x^{-1}$.

Relations (c) and (f) can be deduced from relations (b) and (e) using relations $(\mathcal{R}1)$.

Consider the case (a): $E_{xy}E_{zt}E_{xy}^{-1}E_{zt}^{-1} = 1$, where $t \neq y, y^{-1}$; $z \neq x^{-1}$. By (2) we have $E_{xy}w_{xy}E_{zt}w_{zt}w_{xy}^{-1}E_{xy}^{-1}w_{zt}^{-1}E_{zt}^{-1} = 1 \pmod{F_n'}$. Then it is necessary that the following inclusion is satisfied:

$$w_{xy}^{E_{zt}E_{xy}^{-1}E_{zt}^{-1}}(w_{zt}w_{xy}^{-1})^{E_{xy}^{-1}E_{zt}^{-1}}(w_{zt}^{-1})^{E_{zt}^{-1}} \in F_n'.$$

This inclusion is equivalent to the vector equality

$$l(w_{xy}) \cdot \mathbf{E_{xy}^{-1}} + (l(w_{zt}) - l(w_{xy})) \cdot \mathbf{E_{xy}^{-1}E_{zt}^{-1}} + (-l(w_{zt})) \cdot \mathbf{E_{zt}^{-1}} = \mathbf{0}.$$

Herefrom we get the following equations for multiplying words:
$l_y(w_{xy}) - l_x(w_{xy}) + l_y(w_{zt}) - l_y(w_{xy}) - l_x(w_{zt}) + l_x(w_{xy}) - l_y(w_{zt}) = 0$,
$l_t(w_{xy}) + l_t(w_{zt}) - l_t(w_{xy}) - l_z(w_{zt}) + l_z(w_{xy}) - l_t(w_{zt}) + l_z(w_{zt}) = 0$.

All other cases can be considered in a similar way and they lead to the following equations:
(a) $-l_x(w_{zt}) = 0,\ l_z(w_{xy}) = 0\ for\ t \neq x, x^{-1}, y, y^{-1};\ z \neq x, x^{-1}, y, y^{-1}$
(b) $-l_x(w_{x^{-1}y}) - l_x(w_{xy}) = 0$,
(d) $-l_x(w_{x^{-1}t}) = 0,\ -l_x(w_{xy}) = 0$
(e) $-l_x(w_{zy}) + l_z(w_{xy}) = 0\ for\ z \neq x, x^{-1}$.

Now consider only relation $(\mathcal{R}5)$. For all other relations of the given presentation of $Aut(F_n)$ write only equations for multiplying words.
Relation $(\mathcal{R}5)$: $B_{yx}^4 = 1$, where $B_{yx} = E_{xy}E_{y^{-1}x}E_{x^{-1}y}^{-1}$.
An easy computation gives that $l(w) \cdot \mathbf{B_{yx}} = l'(w)$, where $l_x'(w) = l_y(w)$, $l_y'(w) = -l_x(w)$, $l_z'(w) = l_z(w)$ for $z \neq x, x^{-1}, y, y^{-1}$:

	$l_x(w)$	$l_y(w)$
$\mathbf{B_{yx}}$	$-l_y(w)$	$l_x(w)$
$\mathbf{B_{yx}^2}$	$-l_x(w)$	$-l_y(w)$
$\mathbf{B_{yx}^3}$	$l_y(w)$	$-l_x(w)$
$\mathbf{B_{yx}^4}$	$l_x(w)$	$l_y(w)$

By $(\mathcal{R}2)$ it follows that $(E_{xy}w_{xy}E_{y^{-1}x}w_{y^{-1}x}w_{x^{-1}y}^{-1}E_{x^{-1}y}^{-1})^4 = 1 \pmod{F_n'}$.
Then it is necessary that the following inclusion is satisfied:

$$w_{xy}^{E_{y^{-1}x}E_{x^{-1}y}^{-1}B_{yx}^3}(w_{y^{-1}x}w_{x^{-1}y}^{-1})^{E_{x^{-1}y}^{-1}B_{yx}^3} \ldots w_{xy}^{E_{y^{-1}x}E_{x^{-1}y}^{-1}}(w_{y^{-1}x}w_{x^{-1}y}^{-1})^{E_{x^{-1}y}^{-1}} \in F_n'.$$

This inclusion is equivalent to the vector equation

$$l(w_{xy}) \cdot \mathbf{E_{y^{-1}x}E_{x^{-1}y}^{-1}B_{yx}^3} + (l(w_{y^{-1}x}) - l(w_{x^{-1}y})) \cdot \mathbf{E_{x^{-1}y}^{-1}B_{yx}^3} + \ldots +$$

$$+l(w_{xy})\cdot\mathbf{E_{y^{-1}x}E^{-1}_{x^{-1}y}}+(l(w_{y^{-1}x})-l(w_{x^{-1}y}))\cdot\mathbf{E^{-1}_{x^{-1}y}}=\mathbf{0}.$$

Taking into consideration actions of powers of the matrix $\mathbf{B_{yx}}$, we get the following equation:

$4(l_z(w_{xy})+l_z(w_{y^{-1}x})-l_z(w_{x^{-1}y}))=0$ for $z\neq x,x^{-1},y,y^{-1}$.

Thus, from the conditions $r_i(E'_{xy},\tau)\in F'_n$ for $i=1,\dots,l$ we obtain the following system of equations:

1. $l_y(w_{xy})=-l_y(w_{xy^{-1}})+l_x(w_{xy})$,
   $l_v(w_{xy})=-l_v(w_{xy^{-1}})\ for\ v\neq y,y^{-1}$,
2. $l_x(w_{yz})=0\ for\ z\neq x,x^{-1}$,
3. $-l_x(w_{yz})-l_y(w_{xz})=0$,
   $l_y(w_{xy})+l_x(w_{yz})+l_x(w_{xz})-l_z(w_{xz})=0\ for\ x\neq z,z^{-1}$,
4. $-l_y(w_{yx})+l_x(w_{y^{-1}x})-l_x(w_{x^{-1}y})-l_y(w_{x^{-1}y})+l_x(w_{yx})-l_y(w_{yx})=0$,
   $l_x(w_{xy})+l_y(w_{y^{-1}x})-l_y(w_{x^{-1}y})+l_x(w_{y^{-1}x})+l_x(w_{yx})-l_y(w_{xy})$,
   $l_v(w_{y^{-1}x})+l_v(w_{yx})\ for\ v\neq x,x^{-1},y,y^{-1},,$
5. $l_z(w_{xy})+l_z(w_{y^{-1}x})-l_z(w_{x^{-1}y})=0\ for\ z\neq x,x^{-1},y,y^{-1}$.
6. –
7. $l_{x_1}(w_{xy})=-l_{x_1}(w_{xy})\ for\ x_1\neq x,x^{-1},y,y^{-1}$,
   $l_{x_1}(w_{x_1y})=-l_{x_1}(w_{x_1^{-1}y}),\ l_v(w_{x_1y})=l_v(w_{x_1^{-1}y})\ for\ v\neq x_1,x_1^{-1}$,
   $l_v(w_{xx_1})=-l_v(w_{xx_1})\ for\ v\neq x_1,x_1^{-1}$.

Further we will write equations, which can be deduced from conditions $r_{l+i}(E'_{xy})\in H$, $i=1,\dots,n$. Recall that

$$r_{l+i}(E_{xy})=E_{x_1x_i}E_{x_1^{-1}x_i}\dots E_{x_{i-1}x_i}E_{x_{i-1}^{-1}x_i}E_{x_{i+1}x_i}E_{x_{i+1}^{-1}x_i}\dots E_{x_nx_i}E_{x_n^{-1}x_i}.$$

So, we have

$$E_{x_1x_i}w_{x_1x_i}E_{x_1^{-1}x_i}w_{x_1^{-1}x_i}\dots E_{x_{i-1}x_i}w_{x_{i-1}x_i}E_{x_{i-1}^{-1}x_i}w_{x_{i-1}^{-1}x_i}\cdot$$
$$\cdot E_{x_{i+1}x_i}w_{x_{i+1}x_i}E_{x_{i+1}^{-1}x_i}w_{x_{i+1}^{-1}x_i}\dots E_{x_nx_i}w_{x_nx_i}E_{x_n^{-1}x_i}w_{x_n^{-1}x_i}=1\ (\mathrm{mod}\ H).$$

Then $w_{x_1x_i}^{E^{-1}_{x_1x_i}r_{l+i}(E_{xy})\dots E_{x_n^{-1}x_i}}\dots w_{x_n^{-1}x_i}\in x_i^{-1}H$. Since the product of matrices $\mathbf{E_{x_kx_i}}$ and $\mathbf{E_{x_k^{-1}x_i}}$ is the identity matrix, the corresponding vector equation is

$$\sum_{\substack{j=1\\j\neq i}}^{n}(l(w_{x_jx_i})\cdot\mathbf{E_{x_j^{-1}x_i}}+l(w_{x_j^{-1}x_i}))=e_i,$$

where e_i is a vector whose components are multiple of k, except the i-th component where $i\equiv-1\ (\mathrm{mod}\ k)$. Herefrom we get the following equations:

$$8.\ \sum_{\substack{j=1\\j\neq i}}^{n}(l_{x_i}(w_{x_jx_i})-l_{x_1}(w_{x_jx_i})+l_{x_i}(w_{x_j^{-1}x_i}))\equiv-1\ (\mathrm{mod}\ k),\ i=1,\dots,n.$$

Simplifying the system of equation 1 – 8, we get the final system:

$l_z(w_{xy})=0$ for all $z\neq y,y^{-1}$,

$$l_y(w_{xy}) = -l_y(w_{xy^{-1}}),$$
$$l_y(w_{xy}) = l_z(w_{xz}),$$
$$l_x(w_{yx}) + l_x(w_{y^{-1}x}) = l_y(w_{xy}) + l_y(w_{x^{-1}y}) \text{ for } x, y, z \in X^{\pm 1},$$
$$\sum_{\substack{j=1 \\ j \neq i}}^{n} (l_{x_i}(w_{x_j x_i}) + l_{x_i}(w_{x_j^{-1} x_i})) \equiv -1 \pmod{k}.$$

Since the congruence $a \cdot (n-1) \equiv -1 \pmod{k}$ is decidable if $n-1$ and k are co-prime, the final system is solvable also.

Conjecture. The group $Out(F_n)$ can be embedded into the group $Out(F_m)$ for $m = 2n$.

Reference

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