Algebra and Logic, Vol. 40, No. 1, 2001

THE AUTOMORPHIC CONJUGACY PROBLEM FOR SUBGROUPS OF FUNDAMENTAL GROUPS OF COMPACT SURFACES

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Let Σ be a compact connected surface with basepoint x and H_1 and H_2 be two finitely generated subgroups of $\pi_1(\Sigma, x)$ on finite sets of generators. It is proved that there exists an algorithm which decides whether there is an automorphism $\alpha \in \mathrm{Aut}(\pi_1(\Sigma, x))$ for which $\alpha(H_1) = H_2$, and if so, it finds such.

For the case where G is a free group of finite rank, Whitehead in [1] described an algorithm which solves the following problem:

Let $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_n\}$ be finite sets of elements of G. Does there exist an automorphism of G taking a_i to b_i for all i, and if so, which?

In [2], a similar problem was solved for finite sets of finitely generated subgroups of a free group. In [3, 4], Whitehead's method was extended to free products of finitely many freely indecomposable groups, on the assumption that the above-envisaged problem is solvable for each of these. Note that a free group of finite rank is isomorphic to the fundamental group of an appropriate compact connected surface with a non-empty boundary, and vice versa. In [5], the problem was resolved for fundamental groups of compact surfaces with no boundary.

Here, we prove the following analog of Gersten's theorem.

THEOREM 5.2. Let Σ be a compact connected surface with a basepoint x. Let H_1 and H_2 be two finitely generated subgroups of $\pi_1(\Sigma, x)$ given by finite sets of generators. There is an algorithm which decides whether there is an automorphism $\alpha \in \mathrm{Aut}(\pi_1(\Sigma, x))$ such that $\alpha(H_1) = H_2$, and if so, it finds one.

We specify that the group $\pi_1(\Sigma, x)$ is given by the presentation $\langle g_1, \ldots, g_n \mid \prod_* \rangle$ which coincides with presentations (1) or (2) under Sec. 1, depending on whether Σ is orientable or not. We say that its subgroup H *is given by a finite set of generators* if there exists a finite set of words in the alphabet $\{g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}\}$ whose images in $\pi_1(\Sigma, x)$ generate H.

Theorem 5.2 is proved using a realization of subgroups H_1 and H_2 by incompressible subsurfaces in appropriate finite covers of Σ (see Sec. 4). If such covers coincide, then the question (for the case where Σ is closed) can be reduced to the following: Is there an homeomorphism of the cover which is a lift of some homeomorphism of Σ and which takes the boundary of one subsurface to the boundary of the other?

Below, when we speak about the ways of constructing covers, subsurfaces, and curves, we mean constructions in the simplicial category. For an arbitrary group G and its element g, $\widehat{g}$ denotes conjugation by g, written $\widehat{g}(h) = g^{-1}hg = h^g$, where $h \in G$.

*Supported by RFFR grant 99-01-00576.

Translated from *Algebra i Logika*, Vol. 40, No. 1, pp. 30-59, January-February, 2001. Original article submitted June 17, 1999.

1. SOME CLASSICAL THEOREMS ON HOMEOMORPHISMS OF SURFACES AND AUTOMORPHISMS OF THEIR FUNDAMENTAL GROUPS

Hereinafter we assume that all surfaces are connected. A surface is said to be *closed* if it is compact and its boundary is empty.

Let Σ be a compact surface of genus g with m boundary components and basepoint x. Then $\pi_1(\Sigma, x)$ is represented as

$$\left\langle s_1, \dots, s_m, t_1, u_1, \dots, t_g, u_g \;\middle|\; s_1 \cdots s_m \prod_{i=1}^{g} [t_i, u_i] \right\rangle, \tag{1}$$

if Σ is orientable, and as

$$\langle s_1, \dots, s_m, v_1, \dots, v_g \mid s_1 \cdots s_m v_1^2 \cdots v_g^2 \rangle \tag{2}$$

if Σ is non-orientable. In order to treat the two cases simultaneously, we write these presentations in the form

$$\left\langle g_1, \dots, g_n \;\middle|\; \prod_* \right\rangle. \tag{3}$$

Denote by $\mathrm{Homeo}(\Sigma)$ a group of all homeomorphisms of Σ onto itself, and by $\mathrm{Isot}(\Sigma)$ a group of all homeomorphisms of Σ isotopic to the identity homeomorphism. Let N be an arbitrary subset of Σ. Write $\mathrm{Homeo}(\Sigma, N)$ and $\mathrm{Isot}(\Sigma, N)$ for subgroups of $\mathrm{Homeo}(\Sigma)$ and of $\mathrm{Isot}(\Sigma)$ consisting of all homeomorphisms which fix N pointwise. Let $\mathrm{Isot}_x(\Sigma)$ be a subgroup of $\mathrm{Isot}(\Sigma, x)$ consisting of those homeomorphisms which are isotopic, relative to x, to the identity one. By Baer's theorem, an homeomorphism of Σ is isotopic (relative to x) to the identity homeomorphism if and only if the former is homotopic (relative to x) to the latter (see [6, Secs. 5.14.1 and 5.16.7]). A group $\mathrm{Homeo}(\Sigma)/\mathrm{Isot}(\Sigma)$ is called the *mapping class* group of Σ and is denoted by $\mathrm{Homeot}(\Sigma)$.

Consider the above presentation $\langle g_1, \dots, g_n \mid \prod_* \rangle$ of the group $\pi_1(\Sigma, x)$. Let F be a free group with basis $G_1, \dots, G_n$ and let $\rho : F \to \pi_1(\Sigma, x)$ be a standard homomorphism such that every upper case letter is mapped to a corresponding lower case letter. For $L \in F$, set $w(L)$ equal to $+1$ or -1, depending on whether the number of V_i's in L is even or odd. For $l \in \pi_1(\Sigma, x)$, put $w(l) = w(L)$, where $l = \rho(L)$. This definition is correct.

Every homeomorphism of $\mathrm{Homeo}(\Sigma, x)$ induces some automorphism of the fundamental group $\pi_1(\Sigma, x)$. Every homeomorphism of $\mathrm{Isot}(\Sigma, x)$ induces some inner automorphism of $\pi_1(\Sigma, x)$. Denote a group of all inner automorphisms of the group $\pi_1(\Sigma, x)$ by $\mathrm{Inn}(\pi_1(\Sigma, x))$. We are then faced up to the natural isomorphism $\mathrm{Isot}(\Sigma, x)/\mathrm{Isot}_x(\Sigma) \cong \mathrm{Inn}(\pi_1(\Sigma, x))$ (see [6, Sec. 5.13.2]).

Denote by $\mathrm{Aut}_*(\pi_1(\Sigma, x))$ a group of all automorphisms of $\pi_1(\Sigma, x)$ which are induced by the homeomorphisms of Σ fixing x. A consequence of the Epstein and Dehn–Nielsen theorems (cf. [6, Thms. 5.15.3, 5.7.1, and 5.7.2]) is the following:

THEOREM 1.1. Let Σ be a compact surface which is not a disk or sphere and let $\pi_1(\Sigma, x)$ have presentation (1) or (2). Then the following hold:

(1) The natural homomorphism

$$\varphi : \mathrm{Homeot}(\Sigma) \to \mathrm{Aut}_*(\pi_1(\Sigma, x))/\mathrm{Inn}(\pi_1(\Sigma, x))$$

is an isomorphism.

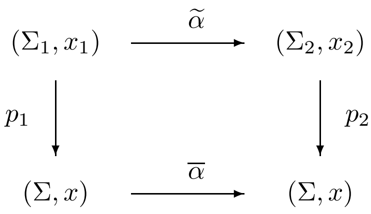

Fig. 1

(2) $\mathrm{Aut}_*(\pi_1(\Sigma, x))$ consists of those automorphisms α of the group $\pi_1(\Sigma, x)$ for which $\alpha(s_i) = l_i^{-1} s_{\sigma(i)}^{\varepsilon_i} l_i$, $i = 1, \ldots, m$, where $l_i \in \pi_1(\Sigma, x)$, $\varepsilon_i = \pm 1, \sigma$ is a permutation of symbols $1, \ldots, m$, and $w(l_i)\varepsilon_i$ is a constant not depending on i.

(3) There is an epimorphism $\mathrm{Aut}_*(F) \to \mathrm{Aut}_*(\pi_1(\Sigma, x))$, where $\mathrm{Aut}_*(F)$ consists of those automorphisms $\boldsymbol{\alpha}$ of F for which $\boldsymbol{\alpha}(S_i) = L_i^{-1} S_{\sigma(i)}^{\varepsilon_i} L_i$, $i = 1, \ldots, m$, $\boldsymbol{\alpha}(\prod_*(G_1, \ldots, G_n)) = L^{-1}(\prod_*(G_1, \ldots, G_n))^{\varepsilon} L$, where $L_i, L \in F$, $\varepsilon, \varepsilon_i = \pm 1$, σ is a permutation of symbols $1, \ldots, m$, and $w(L_i)\varepsilon_i = w(L)\varepsilon$.

By McCool's theorem in [7], the group $\mathrm{Aut}_*(F)$ (and hence $\mathrm{Homeot}(\Sigma)$) is finitely generated (see also [8]).

THEOREM 1.2. Let Σ be a closed surface with a basepoint x.

(1) Each automorphism $\alpha \in \pi_1(\Sigma, x)$ is induced by an homeomorphism $\overline{\alpha} \in \mathrm{Homeo}(\Sigma, x)$.

(2) If G_1 and G_2 are subgroups of $\pi_1(\Sigma, x)$ such that $\alpha(G_1) = G_2$, then $\overline{\alpha}$ lifts to an homeomorphism $\widetilde{\alpha}$ between the covering spaces (Σ_1, x_1) and (Σ_2, x_2) corresponding to G_1 and G_2. Thus the diagram depicted in Fig. 1 is commutative.

Proof. The first assertion is Nielsen's theorem in [9]. The second one can be deduced from the theorem on lifts for covering maps (see [10, Thm. 2.4.5]). Here, we sketch its proof. Let y be a point in Σ_1 and $f : [0,1] \to \Sigma_1$ be a path such that $f(0) = x_1$ and $f(1) = y$. Let $\widetilde{f}$ be a unique lift of the path $\overline{\alpha} \circ p_1 \circ f$ in Σ_2 such that $\tilde{f}(0) = x_2$. Set $\widetilde{\alpha}(y) = \widetilde{f}(1)$. In view of the condition that $\alpha(G_1) = G_2$, this definition does not depend on the choice of f. It is easy to prove that $\widetilde{\alpha} : \Sigma_1 \to \Sigma_2$ is an homeomorphism.

THEOREM 1.3 [5]. Let Σ be a compact surface with a basepoint x. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be elements of $\pi_1(\Sigma, x)$. There is an algorithm which decides whether there is an automorphism φ of $\pi_1(\Sigma, x)$ taking a_i to a conjugate of b_i for all $i = 1, \ldots, n$, and if so, it finds one.

2. MINIMAL REPRESENTATIVES OF CLOSED CURVES ON SURFACES

Let Σ be a surface. We say that two closed curves (subsurfaces) in Σ are *isotopic* if there is an isotopy of Σ sending the first curve (subsurface) to the second. Let γ and γ' be two closed curves in Σ. Write $\gamma \sim \gamma'$, if γ is isotopic to γ', and $\gamma \simeq \gamma'$ if γ is freely homotopic to γ'. A closed curve γ in Σ is said to be *primitive* if there is no closed curve γ_1 such that $\gamma \simeq \gamma_1^n$ for some $n > 1$.

A closed curve γ in Σ is said to be in the *general position* if it is immersed in Σ and intersects itself transversely, without triple points. In [11], some combinatorial tricks (moves) are described which work in such a way as to not change the free homotopy class of γ. They are

1. Elimination of a monogon bounding a disk in $\Sigma \setminus \gamma$.
2. Elimination of a bigon bounding a disk in $\Sigma \setminus \gamma$.

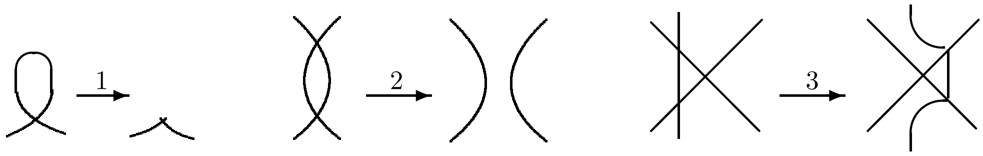

Fig. 2

3. Homotopy of an edge of a triangle in $\Sigma \setminus \gamma$ across the opposite vertex (see Fig. 2).

The first two moves decrease the number of self-intersection points of the curve. The third one leaves the self-intersection number unchanged, and we call it a *triangle move*.

A closed curve γ in the general position in Σ is called a *minimal representative* if it has the minimal possible number of self-intersections among all curves which are in the general position and are freely homotopic to γ. The definitions given and the moves described can well be extended to finite n-tuples of closed curves.

THEOREM 2.1 [11]. Let Σ be a compact surface.

(1) A closed curve γ in the general position in Σ can be sent to a minimal representative by using a finite number of moves like 1, 2, and 3 above.

(2) Let γ and β be two homotopic minimal representatives. Then there exist a curve γ', obtained from γ by finitely many triangle moves, and an isotopy of Σ sending γ' to β.

(3) Clauses (1) and (2) hold for n-tuples of closed curves provided that the curves in each n-tuple are primitive and pairwise non freely homotopic.

Up to homeomorphism, there are not more than finitely many closed curves in Σ which are in the general position and have a given number of self-intersection points. Therefore, there exists an algorithm for finding minimal representatives in the simplicial category.

3. AN EFFECTIVE CONSTRUCTION OF THE CORES OF COVERS CORRESPONDING TO FINITELY GENERATED SUBGROUPS

Definition 3.1. Let Σ_1 be a surface with a basepoint x_1. A compact subsurface C in Σ_1 is called a *core* of Σ_1 if $x_1 \in C$ and the inclusion $C \subseteq \Sigma_1$ induces an isomorphism $\pi_1(C, x_1) \to \pi_1(\Sigma_1, x_1)$.

Remark 3.2. Let Σ be a closed surface and H be a non-trivial finitely generated subgroup of $\pi_1(\Sigma, x)$. Assume that Σ_H is the covering space corresponding to H and suppose that C is a core of Σ_H. Then the following hold:

(1) $C = \Sigma_H$ if and only if H is a subgroup of finite index in $\pi_1(\Sigma, x)$.

(2) If H is a subgroup of infinite index in $\pi_1(\Sigma, x)$ then $\Sigma_H \setminus \mathrm{int}(C)$ consists of a finite number of non-compact cylinders.

Now we define a standard simplicial surface. Let G be a group having one of the following presentations:

$$\left\langle a_1, b_1, \ldots, a_l, b_l \;\middle|\; \prod_{i=1}^{l} [a_i, b_i] \right\rangle, \tag{4}$$

$$\langle v_1, \ldots, v_l \mid v_1^2 \cdots v_l^2 \rangle. \tag{5}$$

As above, we write $G = \langle g_1, \ldots, g_n \mid \prod_* \rangle$ to consider the two cases simultaneously. Let K' be an n-gon with oriented edges labelled by the letters $g_1, \ldots, g_n$, so that the word $\prod_*$ is read from along its boundary. Let Σ' be a surface obtained from K' by gluing edges with same labels. Denote the unique vertex of Σ' by x. Let K be a barycentric subdivision of K'. In this case each oriented edge e of K' is divided into two edges, e^- and e^+, so that the beginning of e coincides with the beginning of e^- and the end of e coincides with the end of e^+. If e is labelled by g_i then we label e^- by g_i^- and e^+ by g_i^+. The other $2n$ oriented edges in K which commence at a vertex corresponding to the 2-cell of K' are labelled by $f_1, \ldots, f_{2n}$. This induces a subdivision Σ of the complex Σ' and labelling of its edges. We call such subdivision the *standard simplicial surface associated with* G.

The following proposition shows how to construct cores for some covers.

Proposition 3.3. Let G be a group having a presentation by (4) or (5).

(1) Let $h_1, \ldots, h_k$ and g be elements of G given by words in the generators $a_1, b_1, \ldots, a_l, b_l$ (resp., in $v_1, \ldots, v_l$). Then there is an algorithm which decides whether g belongs to the subgroup H generated by $h_1, \ldots, h_k$.

(2) Let Σ be the standard simplicial surface associated with G and let $p : (\Sigma_H, x_1) \to (\Sigma, x)$ be a simplicial covering corresponding to the subgroup H given above. There then exists an algorithm which constructs a core of the surface Σ_H.

Proof. The proposition is obvious for the cases where Σ is a sphere, a torus, a Klein bottle, or a projective plane. We so consider the other cases.

The covering p induces labelling of the edges of Σ_H by letters in the set $X \cup X^{-1}$, where $X = \{g_1^-, g_1^+, \ldots, g_n^-, g_n^+, f_1, \ldots, f_{2n}\}$. Each path in the 1-skeleton of Σ_H is labelled by a product of labels of the edges which that path consists of. We identify the word $g_i^- g_i^+$ with g_i, $i = 1, \ldots, n$. For arbitrary vertices x and y in Σ_H, define a distance $d(x, y)$ between them as the minimal number of edges needed to form a path connecting x and y.

An hyperbolic plane $\mathbb{H}^2$, if treated as the universal covering space for Σ, can be thought of as tiled by the copies of K. Therefore, there is a constant δ with the following property. Let D be a one-connected part of $\mathbb{H}^2$, which is the union of some copies of K. Let e be the number of boundary edges of D and S the number of 2-simplices contained in D. Then $S \leqslant \delta e$.

A core of Σ_H can be constructed starting with K by applying two operations defined for an arbitrary simplicial complex N whose edges are labelled by the letters of $X \cup X^{-1}$. They are

— Let e be an edge in N incident to just one 2-simplex and e' be an edge of some 2-simplex s of K having the same label as has e. Then we glue s to N, identifying e with e'.

— Let e and e' be two edges in N with same labels, incident to just one 2-simplex each. Then we identify e with e' in N.

These operations will be applied only if the star (cf. [6]) of every vertex of the resulting complex is mapped, in a one-to-one manner, to a star of K, with the labelling of edges preserved. This condition is necessary to rule out multiple coverings of stars.

Now we do some analysis. Denote by $|f|$ the length of a word f in the alphabet $\{g_1, g_1^{-1}, \ldots, g_n, g_n^{-1}\}$. Let $C \subseteq \Sigma_H$ be a core corresponding to H whose boundary components (if any) are at a distance not less than d apart from the basepoint x_1, where $d = 2\max\{|g|, |h_1|, \ldots, |h_k|\}$. Then C contains loops $l_1, \ldots, l_k$ which commence at x_1 and are labelled by $h_1, \ldots, h_k$, and C contains a path which begins in x_1 and is labelled by g. That path is a loop if and only if $g \in H$.

Without loss of generality, we may assume that every vertex on the boundary of C is d apart from x_1; otherwise, some 2-simplex adjacent to the boundary could be removed. Indeed, let v be a vertex on a boundary component B of C such that $d(x_1, v) \geqslant d + 1$. Let Δ be a 2-simplex with vertices v, u, and w, where $u \in B$. Instead of C, we can then take the closure of a component of $C \setminus \overline{\Delta}$ containing x_1.

The number of vertices which are 1 apart from each given vertex of C is at most $2n$; so, the number of vertices on the boundary of C is at most L, and $L = (2n)^d$. In particular, C has at most L boundary components.

Cut the surface C along $l_1, \ldots, l_k$. By Van Kampen's theorem, each connected component of $C \setminus \{l_1, \ldots, l_k\}$ is homeomorphic to a disk or annulus; the closure of each annulus contains a boundary component of C. The sum of perimeters of disk components is at most $4(|h_1| + \cdots + |h_k|)$. Hence they contain at most $4\delta(|h_1| + \cdots + |h_k|)$ 2-simplices.

Let A be the closure of one of the annulii in question. One boundary component of A coincides with some boundary component of C and the other lies in the union of paths l_i. Since x_1 also lies in the union of l_i's, there is a simple path α of length at most d which connects the boundary components of A. Then $A \setminus \alpha$ is homeomorphic to a disk of perimeter at most $2(|h_1| + \cdots + |h_k|) + L + 2d$. Hence it contains at most $\delta(2(|h_1| + \cdots + |h_k|) + L + 2d)$ 2-simplices. Since the number of annulii does not exceed the number of boundary components of C, the C contains at most $m = \delta(L(2(|h_1| + \cdots + |h_k|) + L + 2d) + 4(|h_1| + \cdots + |h_k|))$ 2-simplices.

We show how to construct C. Denote by x_1 a vertex of K which originates at the vertex x_1 of K'. Starting with K and using the two operations given above, we construct all 2-complexes $C_1, \ldots, C_t$ such that every C_i has the following properties:

(a) C_i consists of at most m 2-simplices;

(b) C_i contains loops which commence at x_1 and are labelled by $h_1, \ldots, h_k$;

(c) every vertex on the boundary of C_i lies at a distance d away from x_1.

Every complex C_i is part of some covering corresponding to a subgroup which contains H. Among C_i's, there are cores of coverings corresponding to the subgroups H and $\langle H, g\rangle$. Therefore, if $g \in H$ then each C_i contains a loop which commences at x_1 and is labelled by g. If $g \notin H$, then there is at least one C_i such that the path which begins in x_1 and is labelled by g is not closed. We can thus decide whether or not g belongs to H.

At this point we explain how to chose the core of Σ_H among these C_i. For every C_i, consider a finite set of loops which commence at x_1 and whose homotopy classes generate $\pi_1(C_i, x_1)$. C_i is the core of Σ_H if and only if labels of the loops belong to H and labels of the boundary components of C_i are non-trivial elements of G. This completes the proof of the proposition.

4. A REALIZATION OF FINITELY GENERATED SUBGROUPS OF $\pi_1(\Sigma, x)$ BY INCOMPRESSIBLE SUBSURFACES

Definition 4.1. Let Σ be a surface. A compact subsurface S in Σ is said to be *incompressible* if no component of the closure of $\Sigma \setminus S$ is a 2-disc whose boundary is in ∂S.

If S is incompressible and $x \in S$, then the natural homomorphism $\pi_1(S, x) \to \pi_1(\Sigma, x)$ is injective. We so identify the group $\pi_1(S, x)$ with its image in $\pi_1(\Sigma, x)$. The following theorem holds.

THEOREM 4.2 [12]. Let Σ be a closed surface with a basepoint x. For any finitely generated subgroup H of $\pi_1(\Sigma, x)$, there are a finite covering $p : (\Sigma_1, x_1) \to (\Sigma, x)$ and an incompressible subsurface $S \subseteq \Sigma_1$ such that $x_1 \in \text{int}(S)$ and $p_*(\pi_1(S, x_1)) = H$.

Remark 4.3. Let the subgroup H be given by a finite set of generators. A procedure for constructing the covering p and the incompressible subsurface S in Σ_1 is proposed in [12] where it is based on the idea of using the core of a covering corresponding to H. By Proposition 3.3, the procedure is effective in the simplicial category. The set of generators for $p_*(\pi_1(\Sigma_1, x_1))$ can also be found effectively.

Definition 4.4. In the situation of Theorem 4.2, we say that H *is realized by an incompressible subsurface* S of Σ_1.

Proposition 4.5. Let Σ be a closed surface, H be a finitely generated subgroup in $\pi_1(\Sigma, x)$, and S_1 and S_2 be two incompressible subsurfaces in Σ realizing H. Then there is an isotopy of Σ sending S_1 to S_2.

Proof. For the cases where Σ is a sphere, a torus, a Klein bottle, or a projective plane, the proposition is obvious. Therefore we assume that Σ is a surface other than these four. We may also suppose that $H \neq \pi_1(\Sigma, x)$. Since H is realized by an incompressible subsurface in Σ, H is a subgroup of infinite index in $\pi_1(\Sigma, x)$. Let Γ_i be a set of boundary components of S_i, $i = 1, 2$.

Claim 1. For proper choices of orientations of the components in Γ_1 and in Γ_2, there exists a bijection $j : \Gamma_1 \to \Gamma_2$ such that γ is isotopic to $j(\gamma)$ for any $\gamma \in \Gamma_1$.

Proof. By Theorem 2.1, it is sufficient to prove a slightly modified version of the claim in which the word "isotopic" is replaced by the phrase "freely homotopic." Let $p : (\Sigma_1, x_1) \to (\Sigma, x)$ be the covering corresponding to H and $\widetilde{S_1}$ and $\widetilde{S_2}$ be the connected components of $p^{-1}(S_1)$ and $p^{-1}(S_2)$ containing x_1. Denote a set of boundary components of $\widetilde{S}_i$ by $\widetilde{\Gamma}_i$, $i = 1, 2$. Then $p|\widetilde{S}_i$ is an homeomorphism, and each component of $\Sigma_1 \setminus \mathrm{int}(\widetilde{S}_i)$ is a non-compact cylinder. There is a one-to-one correspondence between the set of these components and the set of ends in Σ_1.

Therefore each oriented component of $\partial\widetilde{S_1}$ can be homotoped to "infinity," and to some oriented component of $\partial\widetilde{S_2}$ then. This supplies us with proper orientations of the curves in $\widetilde{\Gamma}_1$ and in $\widetilde{\Gamma}_2$ and yields a bijection $\widetilde{j} : \widetilde{\Gamma}_1 \to \widetilde{\Gamma}_2$ such that $\widetilde{\gamma}$ is freely homotopic to $\widetilde{j}(\widetilde{\gamma})$ for $\widetilde{\gamma} \in \widetilde{\Gamma}_1$. Since p maps freely homotopic curves to freely homotopic, $\widetilde{j}$ induces the desired bijection j.

Claim 2. Let $\Gamma_1 = \{\alpha_1, \beta_1, \dots, \alpha_s, \beta_s, \gamma_1, \dots, \gamma_t\}$ be a set of pairwise disjoint, simple, closed, and non-contractable curves in Σ such that $\alpha_k \sim \beta_k$, $k = 1, \dots, s$, and curves in the set $\{\alpha_1, \dots, \alpha_s, \gamma_1, \dots, \gamma_t\}$ are pairwise non isotopic. Let $\Gamma_2 = \{\alpha'_1, \beta'_1, \dots, \alpha'_s, \beta'_s, \gamma'_1, \dots, \gamma'_t\}$ be a set of pairwise disjoint, simple, closed curves in Σ such that $\alpha_k \sim \alpha'_k$, $\beta_k \sim \beta'_k$, and $\gamma_l \sim \gamma'_l$ for all k and l. Then there is an isotopy i of Σ such that $i(\gamma_l) = \gamma'_l$, $l = 1, \dots, t$, and for every k, we have $i(\alpha_k) = \alpha'_k$, $i(\beta_k) = \beta'_k$, or $i(\alpha_k) = \beta'_k$, $i(\beta_k) = \alpha'_k$.

Proof. For every k, denote by $T_{\alpha_k\beta_k}$ the closure of a component of $\Sigma \setminus (\alpha_k \cup \beta_k)$ which is a cylinder with the boundary $\alpha_k \cup \beta_k$. Choose a simple closed curve δ_k in $\mathrm{int}(T_{\alpha_k\beta_k})$ isotopic to α_k. Similarly we define $T_{\alpha'_k\beta'_k}$ and δ'_k. By Theorem 2.1, there is an isotopy i of Σ sending δ_k to δ'_k and γ_l to γ'_l for all k and l. We may assume that $i(T_{\alpha_k\beta_k}) = (T_{\alpha'_k\beta'_k})$. This completes the proof of Claim 2.

From the two claims, it follows that there exists an isotopy i of Σ such that $i(\partial S_1) = \partial S_2$. Hence $i(S_1) = S_2$, or $i(S_1) = \overline{\Sigma \setminus S_2}$ if $\overline{\Sigma \setminus S_2}$ is connected.

Suppose that $\overline{\Sigma \setminus S_2}$ is connected and $i(S_1) = \overline{\Sigma \setminus S_2}$. Let D be a disk in S_2 which contains x in its interior and is such that $D \cap \partial S_2$ is an arc in ∂D. Set $S'_2 = (\overline{\Sigma \setminus S_2}) \cup D$. Then there is an isotopy i' such that $i'(S_1) = S'_2$ and $i'(x) = x$. This implies that the subgroup $\pi_1(S'_2, x)$ is conjugate to a subgroup $\pi_1(S_1, x) = \pi_1(S_2, x)$. Analyzing the graph of groups decomposition of $\pi_1(\Sigma, x)$ induced by the pair (S_2, Σ), we conclude that this case is impossible. This completes the proof of the proposition.

The following two lemmas enable us to verify whether two incompressible subsurfaces in Σ are isotopic, based on the mere fact that the conjugacy problem for elements in $\pi_1(\Sigma, x)$ and the equality problem for two finitely generated subgroups in this group are decidable by Proposition 3.3.

LEMMA 4.6. Let Σ be a closed surface with a basepoint x. Let X and Y be two incompressible subsurfaces in Σ such that $x \in \text{int}(X)$ and $x \in \text{int}(Y)$. Then the subgroups $\pi_1(X,x)$ and $\pi_1(Y,x)$ are conjugate in $\pi_1(\Sigma,x)$ if and only if there exists an isotopy i of Σ such that $i(Y) = X$.

Proof. Suppose that $\pi_1(X,x) = (\pi_1(Y,x))^g$ for some element $g \in \pi_1(\Sigma,x)$. There is an isotopy j of Σ such that $j(x) = x$ and $j_* = \hat{g}$. Then $\pi_1(X,x) = \pi_1(j(Y),x)$. By Proposition 4.5, the subsurfaces X and $j(Y)$ are isotopic. Hence the desired isotopy i exists. The converse is evident.

Remark. Let Σ be a closed surface other than a torus or Klein bottle. Let b be an element of $\pi_1(\Sigma,x)$ such that its conjugacy class is defined by a simple closed curve in Σ. If that curve is the boundary of an embedded Möbius band in Σ, then there is an element b_1 such that $b_1^2 = b$. Denote b_1 by $\sqrt{b}$. In this case the centralizer of b in $\pi_1(\Sigma,x)$ is equal to $\langle\sqrt{b}\rangle$. In the other cases that centralizer is equal to $\langle b\rangle$.

LEMMA 4.7. Let Σ be a closed surface other than a Klein bottle and x be a basepoint in Σ. Suppose that X and Y are two incompressible subsurfaces in Σ such that $x \in \text{int}(X)$, $x \in \text{int}(Y)$, and $i(X) = Y$ for some isotopy i of Σ. Assume that C is an oriented boundary component of X, l is a path in X which begins with x and ends in C, and a is an homotopy class of the path α which goes through l, C, and l^{-1} consequently. Let s be a path in Y which begins with x and ends in $i(C)$ and b be an homotopy class of the path β which goes through s, $i(C)$, and s^{-1} consequently. Then a and b are conjugate in $\pi_1(\Sigma,x)$, and the equality $a^g = b$ implies $(\pi_1(X,x))^g = \pi_1(Y,x)$ or $(\pi_1(X,x))^{g\sqrt{b}} = \pi_1(Y,x)$.

Proof. If Σ is a sphere, a torus, or a projective plane, there is nothing to prove. Therefore we assume that Σ is a surface different from these three, and from a Klein bottle too. We may also assume that $i(x) = x$. Let f be an element of $\pi_1(\Sigma,x)$ such that $\hat{f} = i_*$. Since $i(\alpha)$ and β are freely homotopic in Y, we have $a^f = b^h$ for some $h \in \pi_1(Y,x)$. Let g be an arbitrary element of $\pi_1(\Sigma,x)$ such that $a^g = b$. The remark before Lemma 4.7 implies that $g = fh^{-1}b^t$ or $g = fh^{-1}b^t\sqrt{b}$ for some $t \in \mathbb{Z}$. Now the lemma follows from the equalities $(\pi_1(X,x))^f = i_*(\pi_1(X,x)) = \pi_1(Y,x)$.

Convention. Sometimes, when we consider a covering $p : (\Sigma',x') \to (\Sigma,x)$, we identify the group $\pi_1(\Sigma',x')$ with a subgroup $p_*(\pi_1(\Sigma',x'))$ of $\pi_1(\Sigma,x)$.

LEMMA 4.8. Let Σ be a closed surface with a basepoint x and H be a finitely generated subgroup in $\pi_1(\Sigma,x)$ given by a finite set of generators. There then exists an algorithm deciding whether H is realized by an incompressible subsurface in Σ. If such subsurface exists, we can construct it in Σ by passing to the simplicial category.

Proof. The cases where Σ is a sphere, a torus, or a Klein bottle are obvious. We so assume that Σ is a surface other than these three. By Theorem 4.2, there is a finite covering $p : (\Sigma_1,x_1) \to (\Sigma,x)$ such that H is realized by an incompressible subsurface S_1 in Σ_1. Let Γ_1 be a set of all boundary components of S_1. Set $p(\Gamma_1) = \{p(a) \mid a \in \Gamma_1\}$ and $G = p_*(\pi_1(\Sigma_1,x_1))$.

Convention. Let Ω be a compact surface and Δ its incompressible subsurface. Furthermore, the orientation of boundary components of Δ is chosen so that two boundary components are isotopic as oriented curves whenever they are isotopic.

Claim 1. Suppose that H is realized by an incompressible subsurface S in Σ. Let $\Gamma = \{\alpha_1,\beta_1,\dots,\alpha_s,\beta_s,\gamma_1,\dots,\gamma_t\}$ be a set of oriented boundary components of S such that $\alpha_k \sim \beta_k$, $k = 1,\dots,s$, and $\{\alpha_1,\dots,\alpha_s,\gamma_1,\dots,\gamma_t\}$ consists of pairwise non isotopic curves. Then there is an orientation of components in Γ_1 such that $p(\Gamma_1) = \{\alpha''_1,\beta''_1,\dots,\alpha''_s,\beta''_s,\gamma''_1,\dots,\gamma''_t\}$, where $\alpha''_k \simeq \beta''_k \simeq \alpha_k$, $k = 1,\dots,s$, and $\gamma''_l \simeq \gamma_l$, $l = 1,\dots,t$. In particular, $\{\alpha''_1,\dots,\alpha''_s,\gamma''_1,\dots,\gamma''_t\}$ is a set of primitive and pairwise non freely homotopic curves. This set has a minimal representative which consists of simple non-intersecting curves.

Proof. Denote by S_2 a connected component of the preimage $p^{-1}(S)$ which contains x_1. Then H is realized in Σ_1 by the subsurfaces S_1 and S_2. By Proposition 4.5, there is an isotopy of Σ_1 sending S_1 to S_2. In particular, the boundary components of S_1 and S_2 can be oriented so as to be freely homotopic. It remains to observe that p sends freely homotopic curves to freely homotopic.

Remark. If H is realized by an incompressible subsurface S in Σ, for any orientation of boundary components of S_1, then, orientations of the boundary components of S can be chosen as is agreed upon in Claim 1.

We thus choose some orientation of components in Γ_1. First we need to verify that the set $p(\Gamma_1)$ does not contain three pairwise freely homotopic curves. If so, let $p(\Gamma_1) = \{\alpha_1'', \beta_1'', \dots, \alpha_s'', \beta_s'', \gamma_1'', \dots, \gamma_t''\}$, where $\alpha_k'' \simeq \beta_k''$, $k = 1, \dots, s$, and every two curves in the set $\{\alpha_1'', \dots, \alpha_s'', \gamma_1'', \dots, \gamma_t''\}$ are not freely homotopic. Also assume that these curves are all primitive. Find, then, a minimal representative $\{\alpha_1', \dots, \alpha_s', \gamma_1', \dots, \gamma_t'\}$ of the latter set and verify whether the last conclusion of Claim 1 holds. If so, for each α_j', choose a closed curve β_j' which is isotopic to α_j' and is such that curves in the set $\Gamma' = \{\alpha_1', \beta_1', \dots, \alpha_s', \beta_s', \gamma_1', \dots, \gamma_t'\}$ are pairwise disjoint. Let $X_1', \dots, X_n'$ be closures of those components in $\Sigma \setminus \bigcup \Gamma'$ whose boundary coincides with $\bigcup \Gamma'$. Then $n \leqslant 2$. If the conditions of Claim 1 are satisfied then $\alpha_k' \sim \alpha_k$, $\beta_k' \sim \beta_k$, and $\gamma_l' \sim \gamma_l$ for all k and l and for proper choices of orientations of the components in Γ. In this case there exists an isotopy i of Σ such that $i(\partial S) = \bigcup \Gamma'$, which follows from Claim 2 in Proposition 4.5. Then $i(S) = X_j'$ for some j. For each $j = 1, \dots, n$, we choose a subsurface X_j which is isotopic to X_j' and contains x in its interior. The subsurface S is isotopic to a subsurface X_j for some j. Therefore, if $n = 0$ then H cannot be realized by an incompressible subsurface in Σ. We thus let n be equal to 1 or 2.

Claim 2. A group H is realized by an incompressible subsurface in Σ if and only if there is $g \in \pi_1(\Sigma, x)$ such that $\pi_1(X_j, x) = H^g$ for some j.

Proof. If $\pi_1(X_j, x) = H^g$, then H is realized by the incompressible subsurface $i^{-1}(X_j)$, where i is an isotopy such that $i_* = \hat{g}$. The converse follows from Lemma 4.6 and the fact that the incompressible subsurface S which realizes H in Σ is isotopic to some subsurface X_j.

Let $\{g_1, \dots, g_l\}$ be a complete system of representatives of right cosets of $\pi_1(\Sigma, x)$ in G and let $i_1, \dots, i_l$ be isotopies of Σ such that $(i_s)_* = \hat{g}_s$ for $s = 1, \dots, l$. Set

$$\mathcal{A} = \{i_s^{-1}(X_j) \mid s = 1, \dots, l;\ j = 1, \dots, n\}.$$

Claim 2 requires that we answer the following question:

Is there a $g \in G$ such that $\pi_1(X, x) = H^g$ for some $X \in \mathcal{A}$?

Since $H \leq G$, such X should lie in

$$\mathcal{B} = \{X \in \mathcal{A} \mid \pi_1(X, x) \leq G\}.$$

This set may be found by appealing to Proposition 3.3 and Remark 4.3. Each subsurface $X \in \mathcal{B}$ can be lifted to an homeomorphic incompressible subsurface $\widetilde{X}$ in Σ_1 containing x_1. Using the embedding p_*, we word the above question as follows:

Is there $\widetilde{g} \in \pi_1(\Sigma_1, x_1)$ such that $\pi_1(\widetilde{X}, x_1) = (\pi_1(S_1, x_1))^{\widetilde{g}}$ for some $X \in \mathcal{B}$?

This can be answered by appealing to Lemmas 4.6 and 4.7 and making use of the fact that the conjugacy problem for two elements in the group $\pi_1(\Sigma_1, x_1)$ and the equality problem for two finitely generated subgroups in that group are decidable by Proposition 3.3. Suppose that $\pi_1(\widetilde{X}, x_1) = (\pi_1(S_1, x_1))^{\widetilde{g}}$ for some $\widetilde{g} \in \pi_1(\Sigma_1, x_1)$ and $X \in \mathcal{B}$. Then $\pi_1(X, x) = H^g$, where $g = p_*(\widetilde{g})$. This implies that H is realized by the incompressible subsurface $i^{-1}(X)$ in Σ, where i is an isotopy of Σ such that $i_* = \hat{g}$. The lemma is proved.

5. PROOF OF THE MAIN THEOREM

LEMMA 5.1. Let G be an m-generated group and H a subgroup of index k in G. Let M be a finitely generated subgroup of $\mathrm{Aut}(G)$. If M is defined on a finite set of generators, then we can find a finite subset $A \subseteq M$ such that

$$\{\alpha(H) \mid \alpha \in M\} = \{\alpha(H) \mid \alpha \in A\}.$$

Moreover, if the image of H under the action of any automorphism of M is computable, and if the equality problem for two given subgroups of index k in G is decidable, then we can find a complete system of representatives of right (left) cosets of M in $\mathrm{St}_M(H) = \{\alpha \in M \mid \alpha(H) = H\}$.

Proof. Let B be a finite set of generators for M. On the Reidemeister–Schreier method, there exists a computable function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that if G is an m-generated group then the number of its subgroups of index k is at most $f(k, m)$. We define the graph Γ. Its vertices are all subgroups of index k in G. Two vertices H_i and H_j are connected by an oriented edge labelled b if $b(H_i) = H_j$ and $b \in B$. If K is a subgroup of G, then $K = \alpha(H)$ for some $\alpha \in M$ if and only if K and H lie in a same component of Γ and α is the label of a path going from H to K. Since the diameter of each component of Γ is at most $f(k, m)$, we can take $A = \{b_1 \cdots b_s \mid b_i \in B \cup B^{-1}, i = 1, \ldots, s, s \leqslant f(k, m)\}$.

If the last condition of Lemma 5.1 is satisfied, then the graph Γ can be constructed algorithmically, and the proof of the last conclusion of the present lemma will be an easy exercise.

THEOREM 5.2. Let Σ be a compact connected surface with a basepoint x. Let H_1 and H_2 be two finitely generated subgroups of $\pi_1(\Sigma, x)$ given by finite sets of generators. There then exists an algorithm which decides whether there is an automorphism $\alpha \in \mathrm{Aut}(\pi_1(\Sigma, x))$ such that $\alpha(H_1) = H_2$, and if so, it finds one.

Proof. If Σ is a surface with a boundary, then $\pi_1(\Sigma, x)$ is a free group, and the theorem then follows from [2]. Therefore we assume that Σ is a closed surface other than a sphere, a torus, a Klein bottle, or a projective plane. If H_1 and H_2 are cyclic groups, then the problem reduces to their generators and so is solvable by [5]. Therefore we assume that H_1 and H_2 are not cyclic.

Let $p_1 : (\Sigma_1, x_1) \to (\Sigma, x)$ be a finite covering such that the group H_1 is realized by an incompressible subsurface S_1 in (Σ_1, x_1). Write H_1^* to denote a subgroup of finite index in $\pi_1(\Sigma, x)$ corresponding to the given covering (see Remark 4.3). Set $\mathrm{St}(H_1^*) = \{\alpha \in \mathrm{Aut}(\pi_1(\Sigma, x)) \mid \alpha(H_1^*) = H_1^*\}$. By Lemma 5.1 and Proposition 3.3, we can find a finite set A of representatives of right cosets of $\mathrm{Aut}(\pi_1(\Sigma, x))$ in $\mathrm{St}(H_1^*)$. Conjugating H_2 by elements of A^{-1} reduces our problem to the following (for the new H_2):

(A) Is there an $\alpha \in \mathrm{St}(H_1^*)$ such that $H_2 = \alpha(H_1)$?

If 'yes,' then H_2 is realized by the incompressible subsurface $\widetilde{\alpha}(S_1)$ in (Σ_2, x_2), where $p_2 : (\Sigma_2, x_2) \to (\Sigma, x)$ is the covering corresponding to a subgroup $\alpha(H_1^*)$ (see Thm. 1.2). Since $\alpha(H_1^*) = H_1^*$, it follows that $(\Sigma_2, x_2) = (\Sigma_1, x_1)$ and $p_2 = p_1$.

Lemma 4.8 gives us the possibility to decide whether H_2 is realized by an incompressible subsurface in (Σ_1, x_1), and if so, to construct one. If it is not, then the answer to (A) is no. We so assume that an incompressible subsurface in (Σ_1, x_1) realizing H_2 exists. Denote it by S_2. Using Proposition 4.5 and Theorem 1.2, we arrive at an equivalent geometric formulation of (A) which reads thus:

(G) Is there an homeomorphism $\overline{\alpha} \in \mathrm{Homeo}(\Sigma, x)$ which lifts to an homeomorphism $\widetilde{\alpha} \in \mathrm{Homeo}(\Sigma_1, x_1)$ sending the subsurface S_1 to a subsurface isotopic to S_2?

The subsurface S_2 is "almost" defined by its boundary in the sense that at most one component of $\overline{\Sigma_1 \setminus S_2}$ has the boundary ∂S_2. The most difficult instance is one where $\overline{\Sigma_1 \setminus S_2}$ is homeomorphic to S_2,

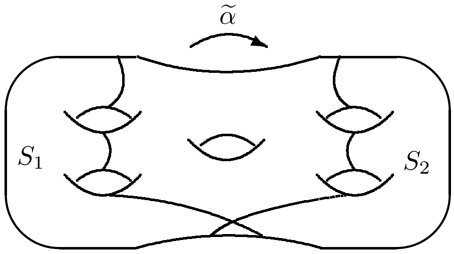

Fig. 3

in which case we cannot assert that $\widetilde{\alpha}(\partial S_1) = \partial S_2$ implies $\widetilde{\alpha}(S_1) = S_2$. Preparatory to answering (G), we raise a slightly easier question inquiring:

(G$_1$) Is there an homeomorphism $\overline{\alpha} \in \mathrm{Homeo}(\Sigma, x)$ which lifts to an homeomorphism $\widetilde{\alpha} \in \mathrm{Homeo}(\Sigma_1, x_1)$ sending ∂S_1 to $i(\partial S_2)$ for some isotopy i of Σ_1?

Let $C_1, \ldots, C_s, C'_1, \ldots, C'_s, C_{s+1}, \ldots, C_k$ all be boundary components of S_1 ordered so that C_l is isotopic to C'_l for $l = 1, \ldots, s$ and C_l is not isotopic to C_j for $l \neq j$. For every $j = 1, \ldots, k$, choose an orientation of C_j and a point $u_j \in C_j$. Let G_j be the corresponding boundary loop. Choose a path q_j in S_1 going from x_1 to u_j, letting $\overline{G_j}$ be a path which goes through q_j, G_j, and q_j^{-1} consequently. Denote by g_j an element of H_1^* equal to the homotopy class of the curve $p_1(\overline{G_j})$.

Let $D_1, \ldots, D_s, D'_1, \ldots, D'_s, D_{s+1}, \ldots, D_k$ all be boundary components of S_2 ordered so that D_l is isotopic to D'_l for $l = 1, \ldots, s$ and D_l is not isotopic to D_j for $l \neq j$. (We assume that the numbers s and k for the subsurfaces S_1 and S_2 coincide, since otherwise the answer to question (G$_1$) is no.) Once we have oriented the components $D_1, \ldots, D_k$ we can similarly define the boundary loops $F_1, \ldots, F_k$ and the elements $f_1, \ldots, f_k$ of the group H_1^*.

For each $l = 1, \ldots, s$, choose an isotopy sending C_l to C'_l and then use it to orient C'_l and define a boundary loop G'_l. Similarly we can orient D'_l and define a boundary loop F'_l. Now question (G$_1$) can be reduced to the following:

(A$_1$) Is it true that the set

$$\mathcal{O} = \{\gamma \in \mathrm{St}(H_1^*) \mid \gamma(g_j) \text{ is conjugate to } f_j \text{ in } H_1^*,\ j = 1, \ldots, k\}$$

is non-empty? More exactly, we can answer question (G$_1$) if we reply to question (A$_1$) for all orientation choices of the boundary components $D_1, \ldots, D_k$ of S_2 and for all numerations of these components by the symbols $1, \ldots, k$. This reduction follows from Theorem 1.2 and the following two remarks.

(1) Two closed curves in Σ_1 are freely homotopic if and only if they define a same conjugacy class in $\pi_1(\Sigma_1, x_1)$. From this and Theorem 2.1, it follows that every two simple closed curves in Σ_1 are isotopic if and only if they define a same conjugacy class in $\pi_1(\Sigma_1, x_1)$.

(2) By Claim 2 in the proof of Proposition 4.5, the curves $\widetilde{\alpha}(G_j)$ and F_j are isotopic for $j = 1, \ldots, k$ if and only if there is a common isotopy i of Σ_1 such that $\{\widetilde{\alpha}(G_j), \widetilde{\alpha}(G'_j)\} = \{i(F_j), i(F'_j)\}$, for $j = 1, \ldots, s$, and $\widetilde{\alpha}(G_j) = i(F_j)$ for $j = s+1, \ldots, k$.

Let

$$K = \{\gamma \in \mathrm{Aut}(\pi_1(\Sigma, x)) \mid \gamma(g_j) \text{ is conjugate to } f_j \text{ in } \pi_1(\Sigma, x),\ j = 1, \ldots, k\};$$

$$M = \{\gamma \in \mathrm{Aut}(\pi_1(\Sigma, x)) \mid \gamma(f_j) \text{ is conjugate to } f_j \text{ in } \pi_1(\Sigma, x),\ j = 1, \dots, k\}.$$

By Theorem 1.3, we can determine whether the set K is non-empty and find at least one element of K if $K \neq \varnothing$. If $K = \varnothing$, then the answer to question (A_1) is negative. Suppose that some element $\gamma_1 \in K$ is found. Then $K = \gamma_1 M$.

We describe the set $\gamma_1 M \cap \mathrm{St}(H_1^*)$. Using Lemmas 5.1 and 5.3 (cf. below), we can find a finite set V of generators for the group $M \cap \mathrm{St}(H_1^*)$ and a finite system U of representatives of left cosets of M in $M \cap \mathrm{St}(H_1^*)$.

The set $\gamma_1 M \cap \mathrm{St}(H_1^*)$ is non-empty if and only if $\gamma_1 u \in \mathrm{St}(H_1^*)$ for some $u \in U$. Assume that such u exists. Then $\gamma_1 M \cap \mathrm{St}(H_1^*) = \gamma_1 u(M \cap \mathrm{St}(H_1^*))$. We are thus faced up to

$$\{\gamma \in \mathrm{St}(H_1^*) \mid \gamma(g_j) \text{ is conjugate to } f_j \text{ in } \pi_1(\Sigma, x),\ j = 1, \dots, k\}.$$

Now we describe the set $\mathcal{O}$ using the graph Γ specified thus. Let $\{h_1, \dots, h_m\}$ be a complete system of representatives of left cosets of $\pi_1(\Sigma, x)$ in H_1^*. Set

$$\Gamma^0 = \{(c_1^{-1} f_1 c_1, \dots, c_k^{-1} f_k c_k) \mid c_1, \dots, c_k \in \{h_1, \dots, h_m\}\} \approx,$$

where $\approx$ is an equivalence relation defined thus:

$$(y_1, \dots, y_k) \approx (z_1, \dots, z_k)$$

if and only if $y_j = b_j^{-1} z_j b_j$ for some $b_j \in H_1^*$, $j = 1, \dots, k$. Join the vertices represented by sets $(y_1, \dots, y_k)$ and $(z_1, \dots, z_k)$ with an oriented edge labelled $v \in V$, if

$$(v(y_1), \dots, v(y_k)) \approx (z_1, \dots, z_k).$$

The graph Γ can be constructed algorithmically using $\pi_1(\Sigma, x)$, which is a torsion-free hyperbolic group satisfying the following:

(a) the conjugacy problem for $\pi_1(\Sigma, x)$ is decidable;

(b) the centralizer of a non-trivial element in $\pi_1(\Sigma, x)$ is isomorphic to an infinite cyclic group, and there exists an algorithm for finding its generator;

(c) given a coset of a cyclic subgroup in $\pi_1(\Sigma, x)$, we can decide whether that coset contains an element of H_1^*.

There exist elements $c_1, \dots, c_k \in \{h_1, \dots, h_m\}$ such that

$$(u(\gamma_1(g_1)), \dots, u(\gamma_1(g_k))) \approx (c_1^{-1} f_1 c_1, \dots, c_k^{-1} f_k c_k).$$

Let σ and τ be two vertices in Γ with representatives $(c_1^{-1} f_1 c_1, \dots, c_k^{-1} f_k c_k)$ and $(f_1, \dots, f_k)$. The set $\mathcal{O}$ is non-empty if and only if the vertices σ and τ lie in a same component of Γ. If so, let ω be the label of some path going from σ to τ. Then $\mathcal{O} = \beta \mathcal{O}_1$, where $\beta = \gamma_1 u w$ and $\mathcal{O}_1$ is a group consisting of labels of the loops in Γ with basepoint τ. It is clear that $\mathcal{O}_1$ is finitely generated.

Similarly we can describe the set

$$\mathcal{O}^* = \bigcup_{\varepsilon,\, \pi} \{\gamma \in \mathrm{St}(H_1^*) \mid \gamma(g_j) \text{ is conjugate to } f_{\pi(j)}^{\varepsilon(j)} \text{ in } H_1^*,\ j = 1, \dots, k\},$$

where ε runs over all functions from $\{1, \dots, k\}$ into $\{-1, 1\}$ and π runs over all permutations on the set $\{1, \dots, k\}$.

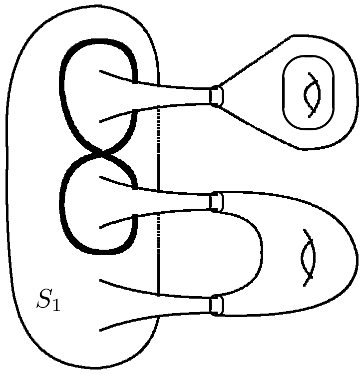

Fig. 4

The answer to (G_1) is positive if and only if the set $\mathcal{O}^*$ is non-empty. The negative reply implies that (G) too is answered in the negative. Therefore we assume that $\mathcal{O}^*$ is non-empty. Let $\mathcal{O}_1^* = \beta\langle\beta_1, \ldots, \beta_m\rangle$. If the subsurfaces $S_2' = \overline{\Sigma_1 \setminus S_2}$ and S_2 are not homeomorphic then questions (G) and (G_1) are equivalent. Therefore we assume that $S_2' = \overline{\Sigma_1 \setminus S_2}$ and S_2 are homeomorphic. In this case each of the surfaces $\widetilde{\beta}(S_1), \widetilde{\beta_1}(\widetilde{\beta}(S_1)), \ldots, \widetilde{\beta_m}(\widetilde{\beta}(S_1))$ is isotopic to S_2 or to S_2'. The answer to (G) is positive if and only if one of these surfaces is isotopic to S_2. This can be verified by using Claims 1 and 2 below and the following definition.

Definition. A closed curve γ in Σ_1 is called an *N-curve* if γ is in the general position, is primitive, and none of the Hass–Scott moves can be applied to it.

Let G_0 be an N-curve in $\mathrm{int}(S_1)$ which is not freely homotopic to a boundary component of S_1. Such a curve exists because S_1 is not a cylinder by our assumption (see Fig. 4). Let g_0 be an element of $\pi_1(S_1, x_1)$ whose conjugacy class in the group $\pi_1(\Sigma_1, x_1)$ corresponds to the curve G_0.

Claim 1. For $\alpha \in \mathcal{O}^*$, the following are equivalent:

(1) the element $\alpha(g_0)$ is conjugate in $\pi_1(\Sigma_1, x_1)$ to an element of $\pi_1(S_2, x_1)$;

(2) the curve $\widetilde{\alpha}(G_0)$ is freely homotopic to a curve in S_2;

(3) the curve $\widetilde{\alpha}(G_0)$ is isotopic to a curve in S_2.

(4) the subsurface $\widetilde{\alpha}(S_1)$ is isotopic to the subsurface S_2.

Proof. The equivalence (1) $\Leftrightarrow$ (2) and the implication (4) $\Rightarrow$ (2) are obvious.

(2) $\Rightarrow$ (3). Suppose that $\widetilde{\alpha}(G_0)$ is freely homotopic to a curve F_0 in S_2. We may assume that F_0 is a minimal representative. Since $\widetilde{\alpha}(G_0)$ is an N-curve, $\widetilde{\alpha}(G_0)$ and F_0 are isotopic by Theorem 2.1.

(3) $\Rightarrow$ (4). Since $\alpha \in \mathcal{O}^*$, the components of $\widetilde{\alpha}(\partial S_1)$ are isotopic to those of ∂S_2. Assume that the curve $\widetilde{\alpha}(G_0)$ is isotopic to a curve F_0 which lies in S_2. Claim 2 in Proposition 4.5 remains true if we replace the condition that γ_t and γ_t' are simple curves by the condition that γ_t and γ_t' are N-curves. Therefore there is a common isotopy i of Σ_1 such that $\widetilde{\alpha}(\partial S_1) = i(\partial S_2)$ and $\widetilde{\alpha}(G_0) = i(F_0)$. This implies that $\widetilde{\alpha}(S_1) = i(S_2)$.

Claim 2. For every element g of $\pi_1(\Sigma_1, x_1)$, we can verify whether g is conjugate in $\pi_1(\Sigma_1, x_1)$ to an element of $\pi_1(S_2, x_1)$.

Proof. We may assume that $x_1 \in S_1 \cap S_2'$. Since the surface Σ_1 is obtained from surfaces S_2 and S_2' by identifying their boundary components, the group $\pi_1(\Sigma_1, x_1)$ can be represented thus:

$$\langle \pi_1(S_2, x_1),\, \pi_1(S_2', x_1),\, t_1, \ldots, t_r \mid s = s',\, t_1^{-1} s_1 t_1 = s_1',\, \ldots,\, t_r^{-1} s_r t_r = s_r' \rangle,$$

where $s, s_1, \ldots, s_r \in \pi_1(S_2, x_1)$ and $s', s_1', \ldots, s_r' \in \pi_1(S_2', x_1)$. We can verify whether g is conjugate to an element of $\pi_1(S_2, x_1)$ using normal forms in amalgamated free products and in HNN-extensions (cf. [13]).

Set $a_1 = p_1(F_1), \ldots, a_k = p_1(F_k)$. Note that the homotopy class of a curve a_i corresponds to a conjugacy class of the element f_i in $\pi_1(\Sigma, x)$, $i = 1, \ldots, k$. Let

$$L = \{f \in \mathrm{Homeo}(\Sigma) \mid f(a_i) \text{ is freely homotopic to } a_i, i = 1, \ldots, k\}.$$

LEMMA 5.3. (1) $M/\mathrm{Inn}(\pi_1(\Sigma, x)) \cong L/\mathrm{Isot}(\Sigma)$.

(2) The group M is finitely generated. There exists an algorithm for finding a finite set of generators for M.

Proof. The first assertion follows from Theorem 1.1. To prove that the group M is finitely generated, it is sufficient to show that $L/\mathrm{Isot}(\Sigma)$ is finitely generated. We follow the line of argument of Lemma 2.1 in [5]. Let $f \in L$. There is no loss of generality in assuming that curves in the set $a = \{a_1, \ldots, a_k\}$ are primitive and pairwise non freely homotopic. Moreover, we may suppose that a is a minimal representative. Then the set $\{f(a_1), \ldots, f(a_k)\}$ of curves can be obtained from $\{a_1, \ldots, a_k\}$ by applying triangle moves and an isotopy.

Let $a^1, \ldots, a^s$ all be components of the graph $\bigcup_{j=1}^{k} a_j$. Let N^t be a subsurface of Σ consisting of a regular neighborhood of a^t, together with all disk components of the complement to that neighborhood, $t = 1, \ldots, s$. Choose N^t so that N^{t_1} and N^{t_2} are disjoint for $t_1 \neq t_2$. Set $N = \bigcup_{t=1}^{s} N^t$. Let $\Sigma^1, \ldots, \Sigma^l$ all be components of $\overline{\Sigma \setminus N}$.

Analysis of triangle moves shows that there is an isotopy i of Σ mapping $f(N^t)$ onto N^t for every t. Write f instead of $f \cdot i$. Then f has the following property: $f(N^t) = N^t$ and $f(a_j)$ is freely homotopic to a_j in N^t if $a_j \in N^t$, $j = 1, \ldots, k$, $t = 1, \ldots, s$.

Let D be a subgroup of $\mathrm{Homeo}(N)$ consisting of all homeomorphisms with this property. Let D' be a subgroup of D consisting of those homeomorphisms which can be extended to an homeomorphism of Σ. Every isotopy of N is extendable to one of Σ. An homeomorphism α of N is extendable to an homeomorphism of Σ if and only if α induces a suitable permutation of the boundary components of N. In particular, D' is a subgroup of finite index in D.

Using Lemma 5.4 (cf. below), we work to algorithmically find a complete system of representatives of right cosets of D' in $\mathrm{Isot}(N)$. For each representative, choose its certain extension to an homeomorphism of Σ. Multiplying f by such one extension and by a suitable isotopy of Σ, we may assume that f fixes N pointwise. It remains to observe that the group $\mathrm{Homeo}(\Sigma, N)/\mathrm{Isot}(\Sigma, N)$ is finitely generated, which follows from the following facts. First,

$$\mathrm{Homeo}(\Sigma, N)/\mathrm{Isot}(\Sigma, N) \cong \mathrm{Homeo}(\Sigma^1, \partial\Sigma^1)/\mathrm{Isot}(\Sigma^1, \partial\Sigma^1) \times \cdots \times \mathrm{Homeo}(\Sigma^l, \partial\Sigma^l)/\mathrm{Isot}(\Sigma^l, \partial\Sigma^l).$$

Second, every group $\mathrm{Homeo}(\Sigma^j, \partial\Sigma^j)/\mathrm{Isot}(\Sigma^j, \partial\Sigma^j)$ is a subgroup of finite index in $\mathrm{Homeot}(\Sigma^j)$. Hence it is finitely generated by Theorem 1.1 and by McCool's theorem in [7].

LEMMA 5.4 [5]. The group $D/\mathrm{Isot}(N)$ is finite. There is an algorithm which determines a complete system of representatives of right cosets of D in $\mathrm{Isot}(N)$.

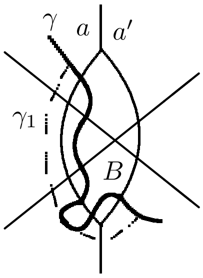

Fig. 5

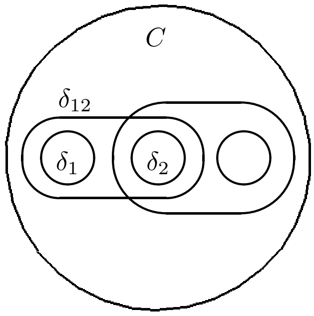

Fig. 6

Proof. Here, we need to extend the argument of [5, Lemma 2.1]. Let $a = \{a_1, \ldots, a_k\}$ be the same minimal representative as was used to construct N. Without loss of generality, we may assume that $\bigcup_{i=1}^{k} a_i$ is connected. Then N too is connected. Let γ be a closed curve in N and $[\gamma]$ its free homotopy class. Denote by $|\gamma \cap a|$ the number of intersections of γ and $\bigcup a_i$. Write $|[\gamma] \cap a|$ for the minimal possible number of intersections of γ' and $\bigcup a_i$ among all curves $\gamma' \in [\gamma]$ which are in the general position and intersect $\bigcup a_i$ transversely, without triple points.

Claim 1. If a' is obtained from a via a triangle move then $|[\gamma] \cap a| = |[\gamma] \cap a'|$.

Proof. Suppose that γ is a closed curve which is in the general position in N and intersects $\bigcup a_i$ and $\bigcup a'_i$ transversely, without triple points. Denote by B a bigon such that a and a' coincide outside ∂B, and one side of B is a subarc of a curve in a, and the other — in a'. Then γ can be pushed out of B in a way that the number of intersections of the new curve γ_1 with $\bigcup a_i$ and $\bigcup a'_i$ will not increase (see Fig. 5). If γ does not intersect B then $|\gamma \cap a| = |\gamma \cap a'|$, whence Claim 1. f **Claim 2.** If a' is obtained from a via an isotopy i of N, then $|[\gamma] \cap a| = |[\gamma] \cap a'|$.

The **proof** follows from the equalities $|\gamma \cap a'| = |\gamma \cap i(a)| = |i^{-1}(\gamma) \cap a|$.

Claim 3. Let F_n be a free group with basis $x_1, \ldots, x_n$ and α be an automorphism of F_n such that $\alpha(x_i) = v_i^{-1} x_i v_i$, $\alpha(x_j x_{j+1}) = u_j^{-1}(x_j x_{j+1}) u_j$, and $\alpha(x_1 \cdots x_n) = w^{-1}(x_1 \cdots x_n) w$, where $w, v_i, u_j \in F_n$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n-1$. Then α is an inner automorphism.

Proof. Allowing conjugation, we may assume that $v_1 = 1$ and that the first letter of v_i is distinct from x_i and x_i^{-1}, $i = 2, \ldots, n$. Since the word $\alpha(x_1 x_2) = x_1 v_2^{-1} x_2 v_2$ is conjugate to $x_1 x_2$, $v_2 = x_1^{t_1}$ for some $t_1 \in \mathbb{Z}$. By induction, there are integers $t_1, \ldots, t_{n-1}$ such that $v_i = x_{i-1}^{t_{i-1}} \cdots x_1^{t_1}$, $i = 2, \ldots, n$. Finally, the condition that $\alpha(x_1 \cdots x_n) = w^{-1}(x_1 \cdots x_n) w$ implies that $t_2 = \cdots = t_{n-1} = 0$. Therefore α is an inner automorphism.

Claim 4. Let Σ be a compact surface which is not a disk or sphere. There exists a finite set Γ of simple closed curves in Σ such that if f is an homeomorphism of Σ with $f(\gamma)$ freely homotopic to γ for every γ in Γ then f is isotopic to the identity homeomorphism.

Proof. Let $\beta_1, \ldots, \beta_n$ all be boundary components of Σ (n may be equal to 0) and $\alpha_1, \ldots, \alpha_l$ be a maximal non-separating system of simple closed curves in $\mathrm{int}(\Sigma)$. Denote by C a disk with n holes, obtained by cutting Σ along $\alpha_1, \ldots, \alpha_l$. Assume that $p : C \to \Sigma$ is the canonical projection and that $\delta_0, \delta_1, \ldots, \delta_n$

are boundary loops in C such that $p(\delta_0) = \bigcup \alpha_i$, $p(\delta_1) = \beta_1, \dots, p(\delta_n) = \beta_n$. For each $i = 1, \dots, n-1$, choose a simple closed curve $\delta_{i,i+1}$ in $\text{int}(C)$ with the following properties:

— one of the components of $C \setminus \delta_{i,i+1}$ contains δ_i and δ_{i+1} but does not contain any other δ_j;

— the number of intersections of $\delta_{i,i+1}$ and $\delta_{j,j+1}$ is equal to 2, if $|i-j| = 1$, and to 0 if $|i-j| \neq 1$ (see Fig. 6). We can then set $\Gamma = \{\alpha_1, \dots, \alpha_l, \beta_1, \dots, \beta_n, p(\delta_{12}), \dots, p(\delta_{n-1,n})\}$.

Indeed, let f be an homeomorphism of Σ such that the curve $f(\gamma)$ is freely homotopic to γ for every γ in Γ. By Theorem 2.1, there is an isotopy i such that $f \cdot i$ fixes all curves in Γ. Then $f \cdot i$ induces an homeomorphism g of C such that $g(\delta_s) = \delta_s$ and $g(\delta_{t,t+1}) = \delta_{t,t+1}$ for all s and t. Claim 3 implies that g is isotopic to the identity homeomorphism of C. Hence f is isotopic to the identity homeomorphism of Σ.

Claim 5. For any integer m, there are (up to isotopy) not more than finitely many simple closed curves γ in N such that $|\gamma \cap a| = m$.

Proof. Let P be a set of double points in $\bigcup a_i$. Every component of $\bigcup a_i \setminus P$ is called a P-*segment*. Choose m points in such each P-segment. We call them X-*points*. Let γ be a simple closed curve in N such that $|\gamma \cap a| = m$. Applying a suitable isotopy of N, we may assume that γ intersects each P-segment at X-points only.

Let K be the closure of a component of $N \setminus \bigcup a_i$. Then K is a disk or annulus. One boundary component of K is a union of the closures of some P-segments. The other (for the case of annulus) coincides with some boundary component of N.

Suppose that γ is not null-homotopic and nor isotopic to a boundary component of N. Then the components of $K \cap \gamma$ are arcs in K ending in X-points. Therefore there are (up to isotopy) not more than finitely many possibilities for the arcs of γ to be aligned in K. The claim is proved.

We continue to prove Lemma 5.4. Let Γ be the set of simple closed curves in N specified as in Claim 4. There is no loss of generality in assuming that the set $\Gamma \cup a$ of curves is in the general position. Set $M = \max\{|\gamma \cap a| \mid \gamma \in \Gamma\}$. By Claim 5, we can construct a finite set Γ_1 of representatives of isotopy classes of simple closed curves γ with the property that $|\gamma \cap a| \leqslant M$. We may also assume that $\Gamma \subseteq \Gamma_1$.

Let $\gamma \in \Gamma_1$ and $f \in D$. Then $f(a_j)$ is freely homotopic to a_j for each $a_j \in a$, and $f(a)$ is a minimal representative as is a. By Claims 1 and 2 and Theorem 2.1, we have $|[f(\gamma)] \cap a| = |[f(\gamma)] \cap f(a)| = |[\gamma] \cap a| \leqslant M$. Hence $f(\gamma)$ is freely homotopic and so isotopic to some curve in Γ_1. Therefore f induces a permutation on a finite set $A = \{[\gamma] \mid \gamma \in \Gamma_1\}$. There is then a subgroup D_1 of finite index in D such that D_1 induces an identity permutation on A. By Claim 4, $D_1 \leq \text{Isot}(N)$. Hence the group $D/\text{Isot}(N)$ is finite. In order to construct a complete system of representatives of cosets of D in $\text{Isot}(N)$, we need to answer the following question: Which permutations of the set A are induced by homeomorphisms of D? Which can well be answered by appealing to Theorem 1.3. Lemma 5.4 is proved, thus completing the proof of Theorem 5.2.

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