Classification of automorphisms of free group of rank 2 by ranks of fixed point subgroups

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Let F_n be a free group of rank n and let $\mathrm{Aut}(F_n)$ be the automorphism group of F_n. For any $\alpha \in \mathrm{Aut}(F_n)$ let $\mathrm{Fix}(\alpha)$ denote the fixed point subgroup of α: $\mathrm{Fix}(\alpha) = \{x \in F_n \,|\, \alpha(x) = x\}$. Bestvina and Handel [BH] proved that $rk(\mathrm{Fix}(\alpha)) \leq n$. Collins and Turner [CT] classified the automorphisms α with $rk(\mathrm{Fix}(\alpha)) = n$. Here we give a classification of automorphisms of the free group of rank 2 by ranks of fixed point subgroups, a classification of fixed point subgroups and a classification of stabilizers of elements. As a corollary we obtain an efficient algorithm solving the conjugacy problem in $\mathrm{Aut}(F_2)$. We give an algorithm for finding a basis of $\mathrm{Fix}(\alpha)$ for $\alpha \in \mathrm{Aut}(F_2)$.

1. Formulations of Theorems

Let F_2 be the free group with the base $\{a, b\}$. Define some automorphisms of F_2:

$$\beta : \begin{cases} a \mapsto a \\ b \mapsto ab \end{cases}, \quad \gamma : \begin{cases} a \mapsto b^{-1}a \\ b \mapsto b \end{cases}, \quad \rho : \begin{cases} a \mapsto ab \\ b \mapsto a^{-1} \end{cases}, \quad \tau : \begin{cases} a \mapsto b \\ b \mapsto a^{-1}b^{-1} \end{cases},$$

$$\sigma : \begin{cases} a \mapsto b^{-1} \\ b \mapsto a \end{cases}, \sigma_1 : \begin{cases} a \mapsto b \\ b \mapsto a \end{cases}, \sigma_2 : \begin{cases} a \mapsto a^{-1} \\ b \mapsto b^{-1} \end{cases}, \sigma_3 : \begin{cases} a \mapsto a^{-1} \\ b \mapsto b \end{cases}.$$

We use the following rule of composition of automorphisms: if $\varphi, \psi \in \mathrm{Aut}(F_2)$ and $x \in F_2$ then $\varphi\psi(x) = \psi(\varphi(x))$. For $x \in F_2$ denote by $\mathrm{St}(x)$ the stabilizer of x in $\mathrm{Aut}(F_2)$. For a subset X of a group denote by $\langle X \rangle$ the subgroup generated by X. Denote $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$. For $g \in F_2$ denote by $\widehat{g}$ the automorphism induced by the conjugation by g: $\widehat{g}(x) = g^{-1}xg$, $x \in F_2$.

Let $- : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(Z)$ be the homomorphism induced by the abelianization of F_2. It is known that the group of inner automorphisms $\mathrm{Inn}(F_2)$ is the kernel of this homomorphism and that

$$\mathrm{GL}_2(Z) \cong D_4 \underset{D_2}{*} D_6 \cong \langle \bar\sigma, \bar\sigma_1 \rangle \underset{\langle \bar\sigma_2, \bar\sigma_1\rangle}{*} \langle \bar\rho, \bar\sigma_1 \rangle \tag{1}$$

where D_n is the dihedral group of order $2n$.

Lemma 1.1. 1) *Any nontrivial periodic automorphism from* $\mathrm{Aut}(F_2)$ *has the order* 2, 3 *or* 4. *There are four conjugacy classes of automorphisms of order* 2, *one of order* 3 *and one of order* 4. *The automorphisms* $\sigma_1, \sigma_2, \sigma_3, \sigma_3\widehat{a}, \tau$ *and* σ *are their representatives.*

2) *Any nontrivial periodic element of* $\mathrm{GL}_2(Z)$ *has the order* 2, 3, 4 *or* 6. *There are three conjugacy classes of elements of order* 2 *and one of order* 3, 4, *and* 6 *respectively. The elements* $\bar\sigma_1, \bar\sigma_2, \bar\sigma_3, \bar\tau, \bar\sigma$ *and* $\bar\rho$ *are their representatives.*

The first assertion is contained in [Me], the second one follows from the decomposition (1).

Note that ρ is the automorphism of order 6 modulo inner automorphisms: $\rho^6 = \widehat{bab^{-1}a^{-1}}$.

The following lemma was proven by Collins and Turner in [CT].

Lemma 1.2. *Let* $\alpha \in \mathrm{Aut}(F_2)$. *Then* $rk(\mathrm{Fix}(\alpha)) = 2$ *if and only if* α *is conjugate to* β^t, $t \in \mathbb{Z}$. Moreover, $\mathrm{Fix}(\beta^t) = \langle a, [a,b]\rangle$ if $t \neq 0$.

Let α be an automorphism such that α^n be an inner automorphism and $n \geq 2$ is minimal. For $u \in F_2$ set $w(\alpha, u) = uu^{\alpha}\dots u^{\alpha^{n-1}}$. For $w \in F_2$, $w \neq 1$ let $\sqrt{w}$ denotes the word u such that $u^k = w$ and k is maximal. Set $\sqrt{1} = 1$.

Theorem 1.3. *Let* $\alpha \in \mathrm{Aut}(F_2) \setminus \mathrm{Inn}(F_2)$. *If* α *fixes a nontrivial word from* F_2 *then after conjugation* $\alpha \in \langle \beta, \gamma\rangle$ *or* α *coincides, modulo inner automorphisms, with one of the following:*

$$\rho, \tau, \sigma, \sigma_1, \sigma_2, \sigma_3, \beta^t, \beta^t\sigma_2.$$

Moreover, the group $\langle\beta,\gamma\rangle$ *coincides with the stabilizer of* $[a,b]$ *and the following holds.*

1. $\mathrm{Fix}(\rho\widehat{u}) = \langle \sqrt{w(\rho, u^{-1})aba^{-1}b^{-1}}\rangle \neq 1.$

2. *If* $\alpha = \tau, \sigma, \sigma_1$ *or* σ_2 *then* $\mathrm{Fix}(\alpha\widehat{u}) = \langle\sqrt{w(\alpha, u^{-1})}\rangle$. *Moreover,* $w(\alpha, u^{-1}) = 1 \Leftrightarrow \alpha\widehat{u}$ *is conjugate to* $\alpha \Leftrightarrow$ *there is* $x \in F_2$ *such that*

$$u = \begin{cases} (x^{-1})^{\alpha}x & \text{for } \alpha = \sigma \text{ and } \sigma_1,\\ (x^{-1})^{\alpha}x \text{ or } (x^{-1})^{\alpha}bx & \text{for } \alpha = \tau,\\ (x^{-1})^{\alpha}x \text{ or } (x^{-1})^{\alpha}ax \text{ or } (x^{-1})^{\alpha}bx & \text{for } \alpha = \sigma_2.\end{cases}$$

3.
$$\mathrm{Fix}(\sigma_3\widehat{u}) = \begin{cases} \langle b^x\rangle & \text{if } u = (x^{-1})^{\sigma_3}x,\\ 1 & \text{if } u = (x^{-1})^{\sigma_3}ax,\\ \langle\sqrt{w(\sigma_3, u^{-1})}\rangle \neq 1 & \text{if } u \neq (x^{-1})^{\sigma_3}a^{\varepsilon}x,\end{cases}$$

where $x \in F_2, \varepsilon = 0, 1$.

In the first case $\sigma_3\widehat{u}$ *is conjugate to* σ_3, *in the second case – to* $\sigma_3\widehat{a}$, *and in the third case* $\sigma_3\widehat{u}$ *is conjugate neither to* σ_3 *nor to* $\sigma_3\widehat{a}$.

4. *Let $\beta^t\widehat{u}$ fixes a nontrivial word from F_2 and $t \neq 0$. Then $\beta^t\widehat{u}$ is conjugate to $\beta^t\widehat{a}^s$ or to $\beta^t\widehat{w_0^s}$ for some $w_0 = a^{k_1}b^{-1}a^{k_2}b\dots b^{-1}a^{k_{2r}}b$ where w_0 is not a proper power, $k_i \neq 0$, $i = 1, \dots, 2r$ and $s \in \mathbb{Z}$. The conjugator is an element from* $\mathrm{Inn}(F_2)$. *Moreover,*

$$\mathrm{Fix}(\beta^t\widehat{a}^s) = \begin{cases} \langle a, bab^{-1}a^{-1}\rangle & \text{if } s = t, \\ \langle a, a^{-1}b^{-1}ab\rangle & \text{if } s = 0, \\ \langle a\rangle & \text{if } s \neq 0, t, \end{cases}$$

$\mathrm{Fix}(\beta^t\widehat{w_0}^s) = \langle w_0\rangle$ *for* $s \neq 0$.

5. *Let $\beta^t\sigma_2\widehat{u}$ fixes a nontrivial word from F_2 and $t \neq 0$. Then $\beta^t\sigma_2\widehat{u}$ is conjugate to $\beta^t\sigma_2\widehat{w_1^{-1}w_0^s}$ for some $w_0 = w_1w_1^{\sigma_2}$ and $s \in \mathbb{Z}$ where $w_1 = a^{k_1}b^{-1}a^{k_2}b\dots a^{k_{2m+1}}b^{-1}, k_i \neq 0, i = 1, \dots 2m+1$ and w_0 is not a proper power. The conjugator is an element from* $\mathrm{Inn}(F_2)$. *Moreover,*

$\mathrm{Fix}(\beta^t\sigma_2\widehat{w_1^{-1}w_0^s}) = \langle w_0\rangle$.

Theorem 1.4. *All possible types, up to isomorphism, of stabilizers of nontrivial elements of F_2 in* $\mathrm{Aut}(F_2)$ *are the following: Z, $Z \times Z$, $\langle x, y \mid x^{-1}yx = y^{-1}\rangle$, $\langle x, y \mid x^2 = [x, y^2] = 1\rangle$ and $\langle x, y \mid xyx = yxy\rangle$.*

Theorem 1.5. *The conjugacy problem in* $\mathrm{Aut}(F_2)$ *is effectively solvable.*

Theorem 1.6. *There is an effective algorithm which for an automorphism $\alpha \in$* $\mathrm{Aut}(F_2)$ *finds a basis of its fixed point subgroup* $\mathrm{Fix}(\alpha)$.

2. Preliminary facts

A. Let Σ be a compact surface with a basepoint x. Denote by $\mathrm{Homeo}(\Sigma)$ the group of all homeomorphisms of Σ and by $\mathrm{Isot}(\Sigma)$ the group of all homeomorphisms of Σ isotopic to the identity. Denote by $\mathrm{Homeo}(\Sigma, x)$ and by $\mathrm{Isot}(\Sigma, x)$ the subgroups of $\mathrm{Homeo}(\Sigma)$ and $\mathrm{Isot}(\Sigma)$, respectively, consisting of all homeomorphisms which fix x. Let $\mathrm{Isot}_x(\Sigma)$ denote the subgroup of $\mathrm{Isot}(\Sigma)$ consisting of those homeomorphisms which are isotopic, relative to x, to the identity.

A closed curve ν in Σ is said to be in *general position* if it is immersed in Σ and intersects itself transversely without triple points. In [HS] J. Hass and P. Scott introduced the following moves not changing the free homotopy class of ν:

1. A monogon bounding a disk in $\Sigma - \nu$ is eliminated.
2. A bigon bounding a disk in $\Sigma - \nu$ is eliminated.
3. One edge of a triangle in $\Sigma - \nu$ is pushed across the opposite vertex (see Figure 1).

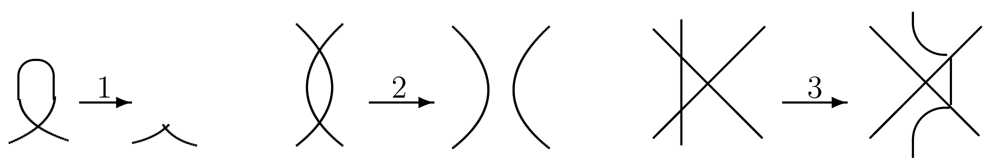


Figure 1.

The first two moves decrease the number of self-intersection points of the curve. The third move is called a *triangle move* and leaves the self-intersection number unchanged.

A closed curve ν in general position in Σ is called a *minimal representative* if it has the minimal possible number of self-intersections among curves in general position freely homotopic to ν.

Theorem 2.1 [HS]. *Let Σ be a compact surface.*

1) *A closed curve ν in general position in Σ can be moved to a minimal representative using finite number of moves* 1,2 *and* 3 *above.*

2) *Let ν and μ be two homotopic minimal representatives. Then there is* a *curve ν' obtained from ν by finite number of triangle moves and an isotopy of Σ sending ν' to μ.*

B. Let T^0 denote a torus with a hole. Recall that F_2 denotes the free group with the basis $\{a, b\}$. Fix a point x in T^0 and identify F_2 with $\pi_1(T^0, x)$ so that the parallel of T^0 corresponds to b and the meridian corresponds to a. Under appropriate orientation the boundary loop of T^0 corresponds to $a^{-1}b^{-1}ab$. Each automorphism $\alpha \in \mathrm{Aut}(\pi_1(T^0, x))$ is induced by some homeomorphism $\widetilde{\alpha} \in \mathrm{Homeo}(T^0, x)$. Moreover, $\mathrm{Homeo}(T^0, x)/\mathrm{Isot}_x(T^0) \cong \mathrm{Aut}(\pi_1(T^0, x)) = \mathrm{Aut}(F_2)$ and $\mathrm{Isot}(T^0, x)/\mathrm{Isot}_x(T^0) \cong \mathrm{Inn}(F_2)$ [Z1].

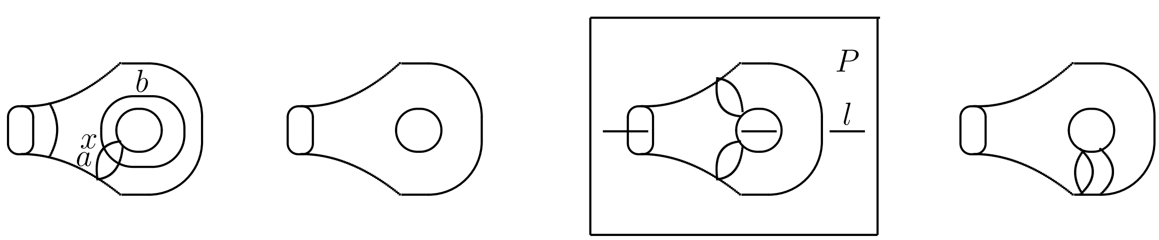
There is a one-to-one correspondence between conjugacy classes of elements of F_2 and free homotopy classes of closed curves in T^0. For $w \in F_2$ denote by $\widetilde{w}$ some closed curve from the homotopy class corresponding to the conjugacy class of w. We may assume that $\widetilde{w}$ is a minimal representative.

Let $\alpha \in \mathrm{Aut}(F_2)$, $w \in F_2$ is not a proper power, $w \neq 1$ and let $\alpha(w) = w$. Then $\widetilde{\alpha}(\widetilde{w})$ is freely homotopic to $\widetilde{w}$. By Theorem 2.1, $\widetilde{\alpha}(\widetilde{w})$ can be moved to $\widetilde{w}$ using finite number of triangle moves and an isotopy. Let $N(\widetilde{w})$ be a subsurface of T^0 consisting of a regular neighborhood of $\widetilde{w}$ together with all disk components of the complement of this neighborhood and annulus component containing ∂T^0 if it exists. Analyzing triangle move we can conclude that there is an isotopy $\widetilde{i}$ of Σ moving $\widetilde{\alpha}(N(\widetilde{w}))$ onto $N(\widetilde{w})$ and such that $\widetilde{i}(x) = x$. Moreover, $(\widetilde{\alpha}\widetilde{i})(\widetilde{w})$ is freely homotopic to $\widetilde{w}$ in $N(\widetilde{w})$.

Lemma 2.2 [LV]. *The following group is finite:*

$$\{\varphi \in \mathrm{Homeo}(N(\widetilde{w})) \,|\, \varphi(\widetilde{w}) \text{ is freely homotopic to } \widetilde{w} \text{ in } N(\widetilde{w})\}/\mathrm{Isot}(N(\widetilde{w})).$$

C. All possible types of the subsurface $N(\widetilde{w})$ up to homeomorphism are pictured on Figure 2 (this can be proven using genus arguments).



Case 1 *Case 2* *Case 3* *Case 4*

Figure 2

This implies a classification of α and w up to conjugacy in $\mathrm{Aut}(F_2)$. Consider these cases.

1) In this case $\widetilde{w}$ corresponds to w which is conjugate in F_2 with $[a,b]^{\pm 1}$. Therefore α is conjugate to an element from $\mathrm{St}([a,b])$. This stabilizer is equal to $\langle \beta, \gamma \rangle$ by [Mal].

2) Since $\mathrm{Homeo}(T^0)/\mathrm{Isot}(T^0) \cong \mathrm{GL}_2(Z)$, αi is an automorphism of finite order modulo $\mathrm{Inn}(F_2)$ by Lemma 2.2. By Lemma 1.1 α is conjugate to $\rho, \tau, \sigma, \sigma_1, \sigma_2$, σ_3 or 1 modulo inner automorphisms.

3) In this case the subgroup of $\mathrm{Homeo}(N(\widetilde{w}))$, which consists of all homeomorphisms leaving the boundary of T^0 invariant, is generated, modulo $\mathrm{Isot}(N(\widetilde{w}))$, by two commuting homeomorphisms. The first one is induced by the reflection in the plane P which contains $\widetilde{b}$ and the second one is induced by the rotation onto 180° around the line $l \in P$ (see Figure 2). These homeomorphisms induce on $\pi_1(T^0, x)$ the automorphisms σ_3 and $\sigma_2\widehat{b}$. Any homeomorphism of $N(\widetilde{w})$ which fixes the boundary of $N(\widetilde{w})$ can be continued to a homeomorphism of T^0 using only a power of the Dehn twist along a simple closed curve which separates $T^0 - N(\widetilde{w})$. Therefore $\alpha = \beta^t \sigma_2^p \sigma_3^q$ modulo inner automorphisms, $p, q \in \{0, 1\}$.

The automorphisms $\beta^t \sigma_2^p \sigma_3 \widehat{u}$ where $u \in F_2$, are conjugate to the more simpler automorphisms:

$$\theta^{-1}(\beta^t \sigma_2^p \sigma_3 \widehat{u})\theta = \begin{cases} \sigma_3 \widehat{u^\theta} & \text{for } t = 2n, p = 0, \ \theta = \beta^n \sigma_3, \\ \sigma_1 \widehat{u^\theta} & \text{for } t = 2n+1, p = 0, \ \theta = \beta^n \gamma^{-1} \sigma_3, \\ \sigma_3 \widehat{b^{-n} u^\theta} & \text{for } t = 2n, p = 1, \ \theta = \beta^n \sigma_1, \\ \sigma_1 \widehat{a^{-1}(ab)^{-n} u^\theta} & \text{for } t = 2n+1, p = 1, \ \theta = \beta^n \sigma_1 \beta. \end{cases}$$

4) This case is dual to Case 3.

3. Proofs of Theorems

Lemma 3.1. *If $\beta^t(w)$ is conjugate to w, $t \neq 0$ then w is conjugate to a^k or with $w_0 = a^{k_1} b^{l_1} \dots a^{k_r} b^{l_r}$ where r is even, $l_i = (-1)^i$, $k_i \neq 0, i = 1, \dots, r$.*

Proof. It is impossible that w is conjugate to b^l for $l \neq 0$. So, assume w is conjugate to $w_0 = a^{k_1} b^{l_1} \dots a^{k_r} b^{l_r}$, $k_i, l_i \neq 0, i = 1, \dots, r$. The word w_0 is conjugate to $\beta^{nt}(w_0) = a^{k_1}(a^{nt}b)^{l_1} \dots a^{k_r}(a^{nt}b)^{l_r}$ for any $n \in \mathbb{Z}$. Take n such that $|k_i \pm nt| > \max\{|k_1|, \dots, |k_r|\}$ for every i. Then there are no cancellations of b-syllables in the cyclic word $\beta^{nt}(w_0)$. Hence $l_i = \pm 1$. Moreover, $l_{i+1} = -l_i$ for every i $(mod\, r)$ since in the opposite case there is a large power of a in the cyclic word $\beta^{nt}(w_0)$. After conjugation we get $l_i = (-1)^i$, $i = 1, \dots, r$. □

Lemma 3.2. *Suppose $\beta^t \widehat{u}(w) = w$ where $w = v^{-1} w_0 v$ and w_0 be as in Lemma 3.1. If w_0 is not a proper power then $\beta^t \widehat{u} = \widehat{v}^{-1}(\beta^t \widehat{w_0}^s)\widehat{v}$ for some $s \in \mathbb{Z}$.*

Proof. Using $w_0 \in \mathrm{Fix}(\beta)$ we have

$$v^{-1} w_0 v = u^{-1} \beta^t(v^{-1} w_0 v) u = u^{-1} \beta^t(v^{-1}) w_0 \beta^t(v) u.$$

Hence $\beta^t(v) u v^{-1} = w_0^s$ for some s and the proof follows. □

Lemma 3.3. 1) *Let w_0 be as in Lemma* 3.1 *and $s \neq 0$. If w_0 is not a proper power then* $\mathrm{Fix}(\beta^t \widehat{w}_0^s) = \langle w_0 \rangle$.

2) $\mathrm{Fix}(\beta^t) = \langle a, a^{-1}b^{-1}ab \rangle$, $\mathrm{Fix}(\beta^t \widehat{a}^t) = \langle a, ba^{-1}b^{-1}a \rangle$, $\mathrm{Fix}(\beta^t \widehat{a}^s) = \langle a \rangle$ *for $t \neq 0$* and *$s \neq 0, t$.*

Proof. 1) Assume $t \neq 0$. In view of Lemma 1.2 it is sufficient to prove that $\beta^t \widehat{w}_0^s$ is not conjugate to a power of β. Suppose the opposite: $\beta^t \widehat{w}_0^s = \alpha^{-1}\beta^l \alpha$. Matrix calculations show that $l = \pm t$. Since $\beta^{-1} = \sigma_3^{-1}\beta\sigma_3$, we may assume that $l = t$. Again, using matrices, we get $\alpha = \beta^r \widehat{x}$ or $\alpha = \beta^r \sigma_2 \widehat{x}$ for some $x \in F_2, r \in \mathbb{Z}$. In the first case $w_0^s = (x^{-1})^{\beta^t} x$, in the second case $w_0^s = (x^{-1}b)^{\beta^t}(b^{-1}x)$.

So, consider the equation $w_0^s = (z^{-1})^{\beta^t} z$. Let z be its solution of minimal length. Then the last letter of z^{-1} is b or b^{-1}: $z^{-1} = a^{p_1}b^{q_1} \dots a^{p_s}b^{q_s}$, $p_2, \dots, p_s \neq 0, q_1, \dots, q_s \neq 0$. Since $w_0 \in \mathrm{Fix}(\beta)$, $w_0^s = (z^{-1})^{\beta^{kt}} z^{\beta^{(k-1)t}}$ for any $k \in \mathbb{Z}$. Multiplying these equations for $k = 1, \dots, n$, we get

$$w_0^{ns} = (z^{-1})^{\beta^{nt}} z = a^{p_1}(a^{nt}b)^{q_1} \dots a^{p_s}(a^{nt}b)^{q_s} \cdot b^{-q_s}a^{-p_s} \dots b^{-q_1}a^{-p_1}.$$

Taking large n, we may assume that there is no cancellation between b-letters in $(z^{-1})^{\beta^{nt}}$. If $q_s \leq -1$ then there is no cancellations on the joint between $(z^{-1})^{\beta^{nt}}$ and z. If $q_s \geq 2$ then there is a cancellation of only one pair of letters between $(z^{-1})^{\beta^{nt}}$ and z. Since the last letter of w_0^{ns} is b, $p_1 = 0$ in both cases. Since the first letter of w_0^{ns} is $a^{\pm 1}$, $q_1 > 0$. But then the first syllable of w_0^{ns} is a^{nt} which is a contradiction for large n. If $q_s = 1$ then $q_{s-1} < 0$, else w_0^{ns} would contain the syllable a^{nt}. Then $z = b^{-1}a^{-p_s}bz_1$, $|z_1| < |z|$ and $w_0^s = (z_1^{-1})^{\beta^t} z_1$ – a contradiction.

2) This assertion follows from an algorithm in [Tu]. □

Lemma 3.4. *Let w be not a proper power, $w \neq 1$ and $t \neq 0$. Then $(\beta^t\sigma_2)(w)$ is conjugate to w if and only if w is conjugate to $w_0 = w_1 w_1^{\sigma_2}$ where $w_1 = a^{k_1}b^{-1}a^{k_2}b \dots a^{k_{2m+1}}b^{-1}$, $k_i \neq 0, i = 1, \dots, 2m+1$, b^{-1} and b alternate.*

Proof. Use $(\beta^t\sigma_2)^2 = \beta^{2t}\widehat{a}^t$ and Lemma 3.1. □

Lemma 3.5. *Suppose $(\beta^t\sigma_2\widehat{u})(w) = w$ where $w = v^{-1}w_0 v$ and w_0 be as in Lemma* 3.4. *If w_0 is not a proper power then $\beta^t\sigma_2\widehat{u} = \widehat{v}^{-1}(\beta^t\sigma_2\widehat{w_1^{-1}w_0^s})\widehat{v}$ for some $s \in \mathbb{Z}$.*

Proof. The proof of this lemma is analogous to the proof of Lemma 3.2 and it uses the fact that $(\beta^t\sigma_2)(w_0) = w_1^{-1}w_0 w_1$. □

Lemma 3.6. *Let w_0 be as in Lemma* 3.4, *$s, t \in \mathbb{Z}$. If w_0 is not a proper power then* $\mathrm{Fix}(\beta^t\sigma_2\widehat{w_1^{-1}w_0^s}) = \langle w_0 \rangle$.

Proof. Denote $x = \beta^t\sigma_2\widehat{w_1^{-1}w_0^s}$. Then $x^2 = \beta^{2t}\widehat{w_0}^{2s-1}$ that follows from the equalities $(\beta^t\sigma_2)^2 = \beta^{2t}\widehat{a}^t$, $(w_1^{-1})^{\beta^t\sigma_2} = a^{-t}(w_1^{-1})^{\sigma_2}$ and $(w_0)^{\beta^t\sigma_2} = w_1^{-1}w_0 w_1$. Hence $\mathrm{Fix}(x) \subseteq \mathrm{Fix}(x^2) = \langle w_0 \rangle$. The converse inclusion is clear. □

Proof of Theorem 1.3.

1) Since $\rho^6 = \widehat{bab^{-1}a^{-1}}$, $w(\rho, \widehat{u^{-1}) \cdot ab}a^{-1}b^{-1} = (\widehat{u}^{-1}\rho^{-1})^6$. This implies $\langle\sqrt{w(\rho, u^{-1}) \cdot aba^{-1}b^{-1}}\rangle \leq \mathrm{Fix}(\rho\widehat{u})$. Moreover, $w(\rho, u^{-1}) \cdot aba^{-1}b^{-1} \neq 1$ since there is no element of order 6 in $\mathrm{Aut}(F_2)$ by Lemma 1.1. Since $\rho\widehat{u}$ and β^t are not conjugate for any t, we get $\langle\sqrt{w(\rho, u^{-1}) \cdot aba^{-1}b^{-1}}\rangle = \mathrm{Fix}(\rho\widehat{u})$ using Lemma 1.2.

2) If $|\alpha| = n$ then $w(\alpha, u^{-1}) = (\widehat{u}^{-1}\alpha^{-1})^n$. Hence $\langle\sqrt{w(\alpha, u^{-1})}\rangle \subseteq \mathrm{Fix}(\alpha\widehat{u})$. If $w(\alpha, u^{-1}) \neq 1$ then $\langle\sqrt{w(\alpha, u^{-1})}\rangle = \mathrm{Fix}(\alpha\widehat{u})$.

Now suppose that $w(\alpha, u^{-1}) = 1$. Then $|\alpha\widehat{u}| = n$.

Case 1. Let $\alpha \in \{\tau, \sigma, \sigma_1, \sigma_2\}$.

By Lemma 1.1, $\alpha\widehat{u} = \varphi^{-1}\alpha\varphi$ for some $\varphi \in \mathrm{Aut}(F_2)$. Hence $\mathrm{Fix}(\alpha\widehat{u}) = (\mathrm{Fix}(\alpha))^{\varphi} = 1$ by [B] and by $\mathrm{Fix}(\sigma) \subseteq \mathrm{Fix}(\sigma^2) = \mathrm{Fix}(\sigma_2) = 1$.

From matrices follows that if $\alpha = \sigma$ then $\varphi = \sigma^i\widehat{x}$ for some $i \in \{0, 1, 2, 3\}$ and $x \in F_2$. This implies $u = (x^{-1})^{\sigma}x$.

If $\alpha = \tau$ then $\varphi = \rho^t\widehat{y} = \tau^{t_1}\rho^{t_2}\widehat{x}$ for some $t \in \{0, \dots, 5\}$, $t_1 \in \{0, 1, 2\}$, $t_2 \in \{0, 1\}$, $y, x \in F_2$. This implies $\widehat{u} = (\widehat{x}^{-1})^{\tau}\tau^{-1}\rho^{-t_2}\tau\rho^{t_2}\widehat{x}$. If $t_2 = 0$ then $u = (x^{-1})^{\tau}x$. If $t_2 = 1$ then $u = (x^{-1})^{\tau}bx$.

If $\alpha = \sigma_1$ or σ_2 then $w(\alpha, u^{-1}) = 1$ implies $u^{\alpha} = u^{-1}$. Using simple analysis of words or matrices it is easy to deduce the desired form of u.

Case 2. Let $\alpha = \sigma_3$.

By Lemma 1.1 there is $\varphi \in \mathrm{Aut}(F_2)$ such that $\sigma_3\widehat{u} = \varphi^{-1}\sigma_3\varphi$ (subcase 1) or $\sigma_3\widehat{u} = \varphi^{-1}(\sigma_3\widehat{a})\varphi$ (subcase 2). From matrices follows that $\varphi = \sigma_3^{\varepsilon_3}\sigma_2^{\varepsilon_2}\widehat{x}$ for some $\varepsilon_2, \varepsilon_3 \in \{0, 1\}$, $x \in F_2$. Since σ_2 and σ_3 commute it follows that $u = (x^{-1})^{\sigma_3}x$ in subcase 1 and $u = (x^{-1})^{\sigma_3}a^{\pm 1}x$ in subcase 2. Note that $(x^{-1})^{\sigma_3}a^{-1}x = (x_1^{-1})^{\sigma_3}ax_1$ where $x_1 = a^{-1}x$.

In the first subcase $\sigma_3\widehat{u} = \widehat{x}^{-1}\sigma_3\widehat{x}$ and $\mathrm{Fix}(\sigma_3\widehat{u}) = (\mathrm{Fix}(\sigma_3))^x = \langle b^x\rangle$.

In the second subcase $\sigma_3\widehat{u} = \widehat{x}^{-1}(\sigma_3\widehat{a})\widehat{x}$ and $\mathrm{Fix}(\sigma_3\widehat{u}) = (\mathrm{Fix}(\sigma_3\widehat{a}))^x = 1$.

If $u \neq (x^{-1})^{\sigma_3}x$ and $u \neq (x^{-1})^{\sigma_3}ax$ where $x \in F_2$ then $|\sigma_3\widehat{u}| \neq 2$. In this case $w(\sigma_3, u^{-1}) \neq 1$ and $\mathrm{Fix}(\sigma_3\widehat{u}) = \langle\sqrt{w(\sigma_3, u^{-1})}\rangle$.

The rest of Theorem 1.3 follows from 2.C and Lemmas 3.1 – 3.6. □

Theorem 1.4. *All possible types, up to isomorphism, of stabilizers of nontrivial elements of* F_2 *in* $\mathrm{Aut}(F_2)$ *are the following:* Z, $Z \times Z$, $\langle x, y \mid x^{-1}yx = y^{-1}\rangle$, $\langle x, y \mid x^2 = y^2\rangle$, $\langle x, y \mid x^2 = [x, y^2] = 1\rangle$ *and* $\langle x, y \mid xyx = yxy\rangle$.

Proof. Let $w \in F_2, w \neq 1$ and w is not a proper power. Denote by $\overline{\mathrm{St}}(w)$ the image of $\mathrm{St}(w)$ in $\mathrm{GL}_2(Z)$. It is clear that $\overline{\mathrm{St}}(w) \cong \mathrm{St}(w)/\langle\widehat{w}\rangle$.

First consider the case where w is conjugate to $[a, b]^{\pm 1}$. We may assume that $w = [a, b]$. It is well-known that $\mathrm{St}(w) = \langle\beta, \gamma\rangle$ (see for example [Mal] or use [Mc]). Prove that

$$\mathrm{St}(w) = \langle\beta, \gamma \mid \gamma\beta\gamma = \beta\gamma\beta\rangle.$$

Note that the relation $\gamma\beta\gamma = \beta\gamma\beta$ implies the relations $[(\beta\gamma\beta)^4, \beta] = 1$ and $[(\beta\gamma\beta)^4, \gamma] = 1$. Since $\mathrm{St}(w) \cap \mathrm{Inn}(F_2) = \langle\widehat{[a, b]}\rangle = \langle(\beta\gamma\beta)^4\rangle$, it is sufficient to prove that $\overline{\mathrm{St}}(w) = \langle\overline{\beta}, \overline{\gamma} \mid \overline{\beta}\overline{\gamma}\overline{\beta} = \overline{\gamma}\overline{\beta}\overline{\gamma},\ (\overline{\beta}\overline{\gamma}\overline{\beta})^4 = 1\rangle$. This follows from the decomposition

$$\mathrm{SL}_2(Z) \cong Z_4 \underset{Z_2}{*} Z_6 \cong \langle\bar{\sigma}\rangle \underset{\langle\bar{\sigma}_2\rangle}{*} \langle\bar{\rho}\rangle$$

and from the expressions $\overline{\beta}\overline{\gamma}\overline{\beta} = \overline{\sigma}$ and $\overline{\gamma}\overline{\beta} = \overline{\rho}^{-1}$.

Now, assume that w is not conjugate to $[a, b]^{\pm 1}$.

Claim 1. $\overline{\mathrm{St}}(w)$ does not contain $Z_2 \times Z_2$.

Proof. Suppose that $\mathrm{St}(w)/\langle\widehat{w}\rangle$ contains $Z_2 \times Z_2$. From (1) follows that $\mathrm{GL}_2(Z)$ contains only two conjugacy classes of subgroups isomorphic to $Z_2 \times Z_2$. The first class contains the subgroup $\langle\bar\sigma_1,\bar\sigma_2\rangle$, the second one contains the subgroup $\langle\bar\sigma_3,\bar\sigma_2\rangle$. After conjugation we need consider two cases.

1) $\mathrm{St}(w)$ contains $\sigma_1\widehat{u}_1$ and $\sigma_2\widehat{u}_2$ for some $u_1,u_2\in F_2$.

Then σ_1 and σ_2 fix the word w up to conjugation. It is easy to show that w is conjugate to cyclically reduced words of kind $v_iv_i^{\sigma_i}$ for $i=1,2$. Therefore there is a nontrivial decomposition $v_1=w_1w_2$ such that $v_2=w_2w_1^{\sigma_1}$ and $v_2^{\sigma_2}=w_2^{\sigma_1}w_1$. This implies $v_2^{\sigma_2\sigma_1}=v_2$. Since $\mathrm{Fix}(\sigma_2\sigma_1)=1$, we get $v_2=w=1$ – a contradiction.

2) $\mathrm{St}(w)$ contains $\sigma_3\widehat{u}_3$ and $\sigma_2\widehat{u}_2$ for some $u_3,u_2\in F_2$.

Since w^{σ_3} is conjugate to w, the word w is conjugate to a power of b or to a word of form $v_3v_3^{\sigma_3}$. On the other hand w is conjugate to a word of form $v_2v_2^{\sigma_2}$. Therefore w is not conjugate to a power of b. Arguing as above, we conclude that $v_2^{\sigma_2\sigma_3}=v_2$. This implies that v_2 is a power of a and $w=1$. We get a contradiction again. □

Claim 2. If $w\neq 1$ is not a proper power and $\mathrm{St}(w)$ contains an involution, then w is a primitive element and $\mathrm{St}(w)$ has the presentation $\langle x,y\,|\,x^2=[x,y^2]=1\rangle$.

Proof. By Lemma 1.1 any involution of $\mathrm{Aut}(F_2)$ is conjugate to $\sigma_1,\sigma_2,\sigma_3$ or to $\sigma_3\widehat{a}$. Since $\sigma_1,\sigma_2,\sigma_3\widehat{a}$ fix only 1 and $\mathrm{Fix}(\sigma_3)=\langle b\rangle$, we have that w is a primitive element. Now the claim follows from [Z2] or from Case 3 below. □

The last part of the proof of Theorem 1.4 deals with two cases where $\overline{\mathrm{St}}(w)$ is finite and infinite.

1) Assume that $\overline{\mathrm{St}}(w)$ is finite. The list of all finite subgroups in $\mathrm{GL}_2(Z)$, up to isomorphism, is the following: $1,Z_2,Z_3,Z_4,Z_6,D_2,D_3,D_4$ and D_6. If $\overline{\mathrm{St}}(w)=1$ then $\mathrm{St}(w)=\langle\widehat{w}\rangle\cong Z$. The groups D_2,D_4 and D_6 are rejected by Claim 1. We consider the other cases using the following assertions.

(i) Any finitely generated virtually free group is the fundamental group of a finite graph of finite groups [St].

(ii) Any element of order 3 in $\mathrm{Aut}(F_2)$ is conjugate to τ and hence fixes only the trivial word.

(iii) Any element of order 4 in $\mathrm{Aut}(F_2)$ is conjugate to σ and hence fixes only the trivial word.

Suppose that $\mathrm{St}(w)/\langle\widehat{w}\rangle$ is isomorphic to D_3,Z_2,Z_3,Z_4 or Z_6. Then the vertex groups of corresponding graph of groups are isomorphic to 1 or Z_2. By Claim 2 the vertex groups can be only trivial. This implies that $\mathrm{St}(w)\cong Z$.

2) Assume that $\overline{\mathrm{St}}(w)$ is infinite. Then there is an element $\alpha\in\mathrm{St}(w)$ such that $|\overline{\alpha}|=\infty$. By Theorem 1.3, α is conjugate to an element from $\langle\beta,\gamma\rangle$ or α is conjugate to $\beta^t\widehat{u}$ or to $\beta^t\sigma_2\widehat{u}$ for some $t\in\mathbb{Z}\setminus\{0\},u\in F_2$. In the first case α fixes some conjugate of $[a,b]$. Recall that α fixes the word w which is not conjugate to any power of $[a,b]$. By Lemma 1.2, $rk(\mathrm{Fix}(\alpha))=2$ and α is conjugate to a power of β. In any case from Theorem 1.3 and the assumption that that w is not a proper power follows that w is automorphic conjugate to $w_0=a^{k_1}b^{-1}a^{k_2}b\dots b^{-1}a^{k_{2r}}b$, $k_1,\cdots,k_{2r}\neq 0$, or to $w_0=a$. So, without loss of generality, assume that $w=w_0$. Note that $\beta\in\mathrm{St}(w_0)$.

The group $\mathrm{GL}_2(Z)$ contains a free nonabelian normal subgroup F of finite index $n\geq 5$. Suppose that $\overline{\mathrm{St}}(w_0)\cap F$ is not cyclic. Then there is $\bar\delta\in\overline{\mathrm{St}}(w_0)\cap F$ such that $\langle\bar\delta,\bar\beta^n\rangle$ is a free group of rank 2. In particular, $|\bar\delta|=\infty$ and $|\bar\beta^n\bar\delta|=\infty$. As it was shown above, any element of $\mathrm{St}(w_0)$ whose image in $\mathrm{GL}_2(Z)$ has infinite order is conjugate to $\beta^t\widehat{u}$ or to $\beta^t\sigma_2\widehat{u}$ for some $t \in \mathbb{Z}, u \in F_2$.

Taking δ^2 instead of δ, we may assume that δ is conjugate to a power of β modulo inner automorphisms. Then $\mathrm{Tr}(\bar{\delta}) = 2$ and $\mathrm{Tr}(\bar{\beta}^n\bar{\delta}) = \pm 2$.

Using easy matrix calculations, we can deduce that $\bar{\delta} = \bar{\beta}^s$. We get a contradiction. Therefore $\overline{\mathrm{St}}(w_0)$ is virtually cyclic and contains $\langle\bar{\beta}\rangle$ as a subgroup of finite index. Hence any $\bar{\alpha} \in \overline{\mathrm{St}}(w_0)$ commutes with $\bar{\beta}^n$ or inverts $\bar{\beta}^n$. Then $\bar{\alpha} = \bar{\beta}^k$ or $\bar{\beta}^k\bar{\sigma}_i$ for some $k \in Z, i = 2, 3, 4$, where $\sigma_4 = \sigma_2\sigma_3$. By Claim 1, $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta\rangle \cong Z \times Z$ or $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_i\widehat{u}\rangle$ for some $u \in F_2$, $i = 2, 3, 4$. Consider these three cases.

Case 1. $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_2\widehat{u}\rangle$.

Then $w_0 = w_1 w_1^{\sigma_2}$ where $w_1 = a^{k_1}b^{-1}a^{k_2}b \dots a^{k_r}b^{-1}$, r is odd and $u = w_1^{-1}w_0^l$. Denote $\sigma_2\widehat{w}_1^{-1}$ by σ_2'. Then $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_2' \mid (\sigma_2')^2 = \widehat{w}_0^{-1}, [\sigma_2', \beta] = [\widehat{w}_0, \beta] = 1\rangle \cong \langle\beta, \sigma_2' \mid [\sigma_2', \beta] = 1\rangle$.

Case 2. $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_3\widehat{u}\rangle$.

Then $w_0 = w_1 w_1^{\sigma_3}$ where $w_1 = a^{k_1}b^{-1}a^{k_2}b \dots a^{k_r}b$, r is even and $u = w_1^{-1}w_0^l$. Denote $\sigma_3\widehat{w}_1^{-1}$ by σ_3'. Then $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_3' \mid (\sigma_3')^2 = \widehat{w}_0^{-1},\ (\sigma_3')^{-1}\beta\sigma_3' = \beta^{-1}, [\widehat{w}_0, \beta] = 1\rangle \cong \langle\beta, \sigma_3' \mid (\sigma_3')^{-1}\beta\sigma_3' = \beta^{-1}\rangle$.

Case 3. $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_4\widehat{u}\rangle$.

There are two subcases here.

(1) $w_0 = a$.

Then $u = a^k$ and $\mathrm{St}(w_0) = \langle\widehat{a}, \beta, \sigma_4 \mid (\beta\sigma_4)^2 = \widehat{a}^{-1}, \sigma_4^2 = 1, [\beta, \widehat{a}] = [\sigma_4, \widehat{a}] = 1\rangle$. Using Tietze transformations, we get $\mathrm{St}(w_0) = \langle\beta_1, \sigma_4 \mid (\sigma_4)^2 = [\sigma_4, \beta_1^2] = 1\rangle$ where $\beta_1 = \beta\sigma_4$.

(2) $w_0 = a^{k_1}b^{-1}a^{k_2}b \dots b^{-1}a^{k_{2r}}b$.

Then $w_0 = w_1 w_1^{\sigma_4}$ where $w_1 = a^{k_1}b^{-1}a^{k_2}b \dots a^{k_r}b^{-1}$, r is odd and $u = w_1^{-1}w_0^l$. Denote $\sigma_4\widehat{w}_1^{-1}$ by σ_4'. Then $\mathrm{St}(w_0) = \langle\widehat{w}_0, \beta, \sigma_4' \mid (\sigma_4')^2 = \widehat{w}_0^{-1},\ (\beta\sigma_4')^2 = \widehat{w}_0^{-1}, [\beta, \widehat{w}_0] = 1\rangle$. Denote $\beta\sigma_4'$ by β'. Then, using Tietze transformations, we get $\mathrm{St}(w_0) = \langle\beta', \sigma_4' \mid (\beta')^2 = (\sigma_4')^2\rangle$.

Note that all groups in Theorem 1.4 are realized. The proof is complete. □

Theorem 1.5. *The conjugacy problem in* $\mathrm{Aut}(F_2)$ *is effectively solvable.*

Proof. Denote by $\mathrm{SAut}(F_2)$ the full preimage of $\mathrm{SL}_2(Z)$ under the homomorphism $- : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(Z)$. It is known that $\mathrm{SAut}(F_2) \cong B_4/Z(B_4)$ where B_4 is the braid group with 4 strands and $Z(B_4)$ is its center [KPS]. Hence, by [Ga] the conjugacy problem in $\mathrm{SAut}(F_2)$ is effectively solvable. It is shown in [B] that there is an algorithm solving the conjugacy problem in $\mathrm{Aut}(F_2)$. This algorithm is effective until the last step where the following question was considered:

Given $\alpha \in \mathrm{Aut}(F_2)$ *and* $z \in F_2$*, is there* $h \in \mathrm{Fix}(\alpha^2)$ *such that* $h^{-1}h^{\alpha} = z$*?*

Now we can give an effective algorithm which gives the answer to this question and finds h if it exists. This can be done using Lemma 3.7. We can make the first part of algorithm in [B] faster if we use the biautomatic structure on B_4 [Th] for finding generators of the centralizer of an element of B_4 [GS] instead of the algorithm of Makanin [Ma].

Lemma 3.7. *Given* $\alpha \in \mathrm{Aut}(F_2)$ *and* $z \in \mathrm{Fix}(\alpha^2)$*, we can decide whether there is* $h \in \mathrm{Fix}(\alpha^2)$ *such that* $h^{-1}h^{\alpha} = z$*. If it is so, we can find one.*

Proof. The necessary condition of the existence of such h is $z^\alpha = z^{-1}$. Assume it is valid and $z \neq 1$. Then $rk(\mathrm{Fix}(\alpha^2)) = 1$ or 2. By Lemma 1.2, $rk(\mathrm{Fix}(\alpha^2)) = 2$ if and only if α^2 is conjugate to β^k for some $k \in \mathbb{Z}$. We can determine all possible k using matrices and then apply the algorithm solving the conjugacy problem in $\mathrm{SAut}(F_2)$. We can also find a conjugator if it exists. If $rk(\mathrm{Fix}(\alpha^2)) = 1$ then α inverts any element of $\mathrm{Fix}(\alpha^2)$ since $z^\alpha = z^{-1}$. Hence the desired h in this case exists if and only if z is a square. Suppose that $rk(\mathrm{Fix}(\alpha^2)) = 2$. Conjugating, we may assume that $\alpha^2 = \beta^k$.

First consider the case where $k \neq 0$. In this case from matrices follows that $\alpha = \beta^t\widehat{v}$ or $\beta^t\sigma_2\widehat{v}$ for some $v \in F_2$ and $k = 2t$. Note that $\mathrm{Fix}(\alpha^2) = \mathrm{Fix}(\beta) = \langle a, b^{-1}ab\rangle$.

Subcase 1. $\alpha = \beta^t\widehat{v}$.

Then $\alpha^2 = \beta^{2t}\widehat{v^{\beta^t}v}$, hence $v^{\beta^t} = v^{-1}$ and $v^{\beta^{2t}} = v$. Since $\mathrm{Fix}(\beta^{2t}) = \mathrm{Fix}(\beta^t)$, $v = 1$. Then α is the identity on $\mathrm{Fix}(\alpha^2)$. Since $z \neq 1$, the desired h does not exist.

Subcase 2. $\alpha = \beta^t\sigma_2\widehat{v}$.

Since $\alpha^2 = \beta^{2t}$, we have $z \in \mathrm{Fix}(\alpha^2) = \mathrm{Fix}(\beta) = \langle a, b^{-1}ab\rangle$. Hence $z^{\sigma_2\widehat{v}} = z^\alpha = z^{-1}$. Let w_0 be the word such that $z = w_0^r$ and r is maximal. Then $w_0^{\sigma_2\widehat{v}} = w_0^{-1}$ and $w_0 \in \langle a, b^{-1}ab\rangle$. It can be shown that w_0 is conjugate by an element from $\langle a, b^{-1}ab\rangle$ to one of the following: $1, a^{\pm1}, b^{-1}a^{\pm1}b$, w_1 where $w_1 = w_2^{-1}b^{-1}w_2^{\sigma_2}b$, $w_2 \in \langle a, b^{-1}ab\rangle$.

The first case is impossible by $z \neq 1$. Consider the second and the third cases. We look for $h \in \mathrm{Fix}(\alpha^2)$ such that $h^{-1}h^\alpha = w_0^r$. If r is even we can take $h = w_0^{-r/2}$. If r is odd then there is no such h since $|h^{-1}h^\alpha| = |h^{-1}h^{\sigma_2\widehat{v}}|$ is even and $|w_0^r|$ is odd.

Now, let $w_0 = u^{-1}w_1u$ where w_1 as above and $u \in \langle a, b^{-1}ab\rangle$. From the condition $w_0^{\sigma_2\widehat{v}} = w_0^{-1}$ follows that $\sigma_2\widehat{v} = \widehat{u}^{-1}(\sigma_2\widehat{bw_1^l})\widehat{u}$ for some $l \in \mathbb{Z}$.

If $r = 2p$, $p \in \mathbb{Z}$, we can take $h = w_0^{-p}$.

Let $r = 2p+1$, $p \in \mathbb{Z}$. If l is even, we can take $h = u^{-1}w_1^{-l/2}w_2w_1^{l/2-p}u$.

If l is odd, we can take $h = u^{-1}w_1^{-(l+1)/2}w_2^{-1}w_1^{(l-1)/2-p}u$.

Finally, consider the case where $\alpha^2 = 1$. By Lemma 1.1, α is conjugate to $\sigma_1, \sigma_2, \sigma_3$ or to $\sigma_3\widehat{a}$. Using canonical homomorphism $- : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(Z)$ one can find $\varphi \in \mathrm{Aut}(F_2), x \in F_2$ and $i \in \{1, 2, 3\}$ such that $\alpha = \varphi\sigma_i\varphi^{-1}\widehat{x} = \varphi\sigma_i\widehat{x^\varphi}\varphi^{-1}$. Conjugating the equation $h^{-1}h^\alpha = z$ by φ, we reduce it to the equation $h_1^{-1}h_1^{\sigma_i\widehat{x^\varphi}} = z_1$ which can be treated as above. Lemma 3.7 and Theorem 1.5 are complete. □

Theorem 1.6. *There is an effective algorithm which for an automorphism* $\alpha \in \mathrm{Aut}(F_2)$ *finds a basis of its fixed point subgroup* $\mathrm{Fix}(\alpha)$.

Proof. Let $\alpha \neq 1$. First consider the case where $\alpha \in \mathrm{SAut}(F_2)$. Let $C(\alpha)$ denotes the centralizer of α in $\mathrm{SAut}(F_2)$ and let $\overline{C}(\alpha)$ denotes the image of $C(\alpha)$ in $\mathrm{SL}_2(Z)$. Identify F_2 with $\mathrm{Inn}(F_2)$. Then $\mathrm{Fix}(\alpha) = C(\alpha) \cap F_2$. As it was mentioned above, we can find generators of $C(\alpha)$. Note that $\overline{C}(\alpha) \leq C(\overline{\alpha})$. Hence the group $\overline{C}(\alpha)$ is virtually cyclic. Using these facts one can find generators of $\mathrm{Fix}(\alpha)$.

Now, let $\alpha \in \mathrm{Aut}(F_2) \setminus \mathrm{SAut}(F_2)$ and α has infinite order. Then $rk(\mathrm{Fix}(\alpha^2)) \leq 1$. Indeed, if $rk(\mathrm{Fix}(\alpha^2)) = 2$ then α^2 is conjugate to β^k for some $k \neq 0$. From matrices follows that k is even and α is conjugate to $\beta^{k/2}\widehat{v}$ or to $\beta^{k/2}\sigma_2\widehat{v}$ for some $v \in F_2$. In particular $\alpha \in \mathrm{SAut}(F_2)$. A contradiction. Since $\mathrm{Fix}(\alpha) \leq \mathrm{Fix}(\alpha^2)$, we can find a generator of $\mathrm{Fix}(\alpha)$.

Finally, let $\alpha \in \mathrm{Aut}(F_2) \setminus \mathrm{SAut}(F_2)$ and α has finite order. By Lemma 1.1, this order is equal to 2 and α is conjugate to σ_1, σ_3 or to $\sigma_3\widehat{a}$. Using canonical homomorphism $- : \mathrm{Aut}(F_2) \to \mathrm{GL}_2(Z)$, one can find $\varphi \in \mathrm{Aut}(F_2), x \in F_2$ and $i \in \{1,3\}$ such that $\alpha = \varphi\sigma_i\varphi^{-1}\widehat{x} = \varphi\sigma_i\widehat{x^\varphi}\varphi^{-1}$. Denote $y = x^\varphi$. Its suffices to find generators of $\mathrm{Fix}(\sigma_i\widehat{y})$. From $(\sigma_i\widehat{y})^2 = 1$ follows $y^{\sigma_i} = y^{-1}$.

If $i = 1$ then $y = 1$. It is clear that $\mathrm{Fix}(\sigma_1) = 1$. If $i = 3$ then we can use Theorem 1.3 to compute a basis of $\mathrm{Fix}(\sigma_3\widehat{y})$. Theorem 1.6 is complete. □

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