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INFINITE COMMENSURABLE HYPERBOLIC GROUPS ARE BI-LIPSCHITZ EQUIVALENT

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It is proved that commensurable hyperbolic groups are bi-Lipschitz equivalent. Therefore, subgroups of finite index in an arbitrary hyperbolic group also share this property. In addition, it is shown that any two separated nets Γ_1 and Γ_2 in the hyperbolic space H^n of dimension $n \geq 2$ are bi-Lipschitz-equivalent. These results answer the questions posed in [1].

Gromov in [1, p. 23] posed the following questions:

(1) Under which conditions do we say that two subgroups of finite index in a finitely generated group are bi-Lipschitz equivalent? — In particular, is it true that so defined are the free groups F_2 and F_3?

(2) Is it true that any two separated nets in the hyperbolic space H^n, $n \geq 2$, are bi-Lipschitz equivalent?

In [2], it has been affirmed that free groups of finite rank $n \geq 2$ are bi-Lipschitz equivalent. In the present article, we prove the following more general result.

THEOREM 1. Any two infinite commensurable hyperbolic groups are bi-Lipschitz equivalent.

This is equivalent to

THEOREM 1′. Any two subgroups of finite index in an infinite hyperbolic group are bi-Lipschitz equivalent.

This theorem answers the first question for the case of hyperbolic groups. The positive answer to the second question is given by the following:

THEOREM 2. Any two separated nets Γ_1 and Γ_2 in the hyperbolic space H^n, $n \geq 2$, are bi-Lipschitz equivalent. Moreover, there exist a constant $c > 0$ and a bijection $\varphi : \Gamma_1 \longrightarrow \Gamma_2$ such that $d(x, \varphi(x)) \leq c$ $\forall x \in \Gamma_1$ (here d is the standard metric on H^n).

The basic definitions are given in Secs. 1 and 3; Theorem 1′ is proved in Secs. 1 and 2, and Theorem 2 — in Sec. 3.

In attempting to answer the second question, one of the major difficulties that we have to get through is that a group may have the so-called dead elements with respect to some finite generating system.

Definition 1. An element g of a group G is called *dead* in G with respect to the finite generating set X if $|gx| \leq |g|$ $\forall x \in X^{\pm 1}$, where $|g|$ denotes the length of g in the word metric with respect to X.

Remark. If the language L of all geodesic words of G in the alphabet X ($X = X^{-1}$) is regular and M is an automaton recognizing L, then the dead words in G with respect to X are precisely those that can be read from along the paths leading to the dead states of M (see [3]).

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Example 1. Let $G = SL_2(Z)$. It is known that $G = G_1 *_{G_3} G_2$, where $G_1 = \langle x \rangle \cong Z_4$, $G_2 = \langle y \rangle \cong Z_6$, and $G_3 \cong Z_2$. Then y^2 is a dead element of G with respect to $X = \{x, y\}$, since $y^2 x = y^{-1}x^{-1}$, $y^2 x^{-1} = y^{-1}x$, and $y^2 y = x^2$. That y^{-2} and x^2 are dead elements can be verified similarly. There is no other dead element in G with respect to X. Indeed, assume $g = g_1 g_2 \cdot \ldots \cdot g_n$, where g_i and g_{i+1} lie in different factors but do not lie in G_3, $1 \leq i \leq n-1$, and the sum $\sum_{i=1}^{n} |g_i|$ is minimal among all such decompositions. It is easy to see that if $n \geq 2$ then $g_i \in \{x^{\pm 1}, y^{\pm 1}\}$. Hence $|g| = n$ and $|gg_{n+1}| = n+1$, where $g_{n+1} \in \{x^{\pm 1}, y^{\pm 1}\} \setminus \{g_n^{\pm 1}\}$. So, if g is a dead element with respect to X then $n = 1$ and $g \in \{y^2, y^{-2}, x^2\}$.

Example 2. Let $G_k = \langle x, y \mid x^3 = y^3 = (xy)^k = 1 \rangle$, $k \geq 3$. It is known that the group G_k is automatic but not hyperbolic for $k = 3$ and hyperbolic for $k \geq 4$. The Cayley graph of G_3 with respect to $X = \{x, y\}$ is depicted in Fig. 1. Define w as $(xy)^{k/2}$, if k is even, and as $(xy)^{(k-1)/2}x$ if k is odd. Then elements w^n, $n \in Z \setminus \{0\}$, are precisely the dead elements of G_k with respect to X. However, G_k has no dead elements with respect to the generating set $X = \{x, y, z\}$, where $z = xy$.

This setting gives rise to the following:

Questions. (1) Is it true that for every infinite finitely generated (hyperbolic) group there exists a finite generating set X such that G has no dead words with respect to X?

(2) Let G be a group and X an arbitrary finite generating system. Is it true that the portion of dead elements in all elements of the ball $B(r) = \{g \in G \mid |g| \leq r\}$ tends to 0 at $r \to \infty$?

(3) Let G be an infinite finitely presented group and X an arbitrary finite system of generators. Does there exist a constant l (depending on X) such that for any $g \in G$ there exists a $w \in G$ with $|w| \leq l$ and $|gw| = |g| + 1$? — In view of Lemma 3, this question has an affirmative answer for the case of nonelementary hyperbolic groups.

1. PRELIMINARY LEMMAS

Definition 2. Let G_1 and G_2 be finitely generated groups and let d_1 and d_2 be word metrics on G_1 and G_2 with respect to some finite generating sets. We say that G_1 and G_2 are *bi-Lipschitz equivalent* if there exist a bijection $\varphi : G_1 \longrightarrow G_2$ and a constant β such that $\frac{1}{\beta} d_1(g_1, g_2) \leq d_2(\varphi(g_1), \varphi(g_2)) \leq \beta d_1(g_1, g_2)$ $\forall g_1, g_2 \in G_1$.

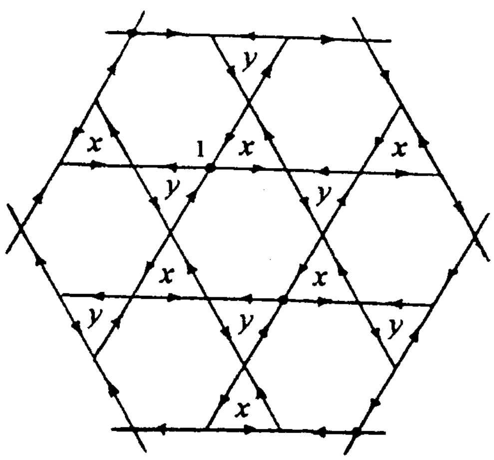

Fig. 1

Note that this property does not depend on the choice of finite generating systems.

Definition 3. Groups G_1 and G_2 are *commensurable* if there exists a group H which is embeddable in G_1 and G_2 as a subgroup of finite index.

Let G be a group generated by a finite generating set $\mathcal{X}$. Denote by $\Gamma_{\mathcal{X}}(G)$ the right Cayley graph of G with respect to $\mathcal{X}$. We consider $\Gamma_{\mathcal{X}}(G)$ as a geodesic metric space whose metric d is the path metric of $\Gamma_{\mathcal{X}}(G)$, where every edge has length 1. For any elements a and b in G, denote by $[a,b]$ some geodesic path in $\Gamma_{\mathcal{X}}(G)$, with initial point a and terminal point b. Denote by $|a|$ the length of the path $[1,a]$. Let $B_x(r) = \{g \in G \mid d(x,g) \le r\}$.

A geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ with sides $[a,b]$, $[a,c]$, and $[b,c]$ is called *δ-thin* if, for any points B and C satisfying $B \in [a,b]$, $C \in [a,c]$, and $d(a,B) = d(a,C) \le \frac{1}{2}(d(a,b) + d(a,c) - d(b,c))$, the inequality $d(B,C) \le \delta$ holds.

Definition 4. A group G is called *δ-hyperbolic with respect to $\mathcal{X}$* if every geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ is δ-thin.

Let Γ be an infinite elementary hyperbolic group, i.e., Γ has a subgroup of finite index isomorphic to Z. We identify that subgroup with Z, letting $T = \{t_1, ..., t_m\}$ be a system of right coset representatives of Γ in Z. It is easy to verify that the map $\varphi : \Gamma \longrightarrow Z$, given by the rule $zt_i \longrightarrow mz + i$ where $t_i \in T$ and $z \in Z$, is a bi-Lipschitz equivalence.

Further, let Γ be a nonelementary δ-hyperbolic group with respect to the generating set $X = \{\gamma_1, ..., \gamma_n\}$ and let Γ_1 be its subgroup of index m. To prove Theorem 1′, it is sufficient to show that there exist a constant $c > 0$ and a bijection $\varphi : \Gamma \longrightarrow \Gamma_1$ such that $d(x, \varphi(x)) \le c \ \forall x \in \Gamma$.

We will use the fact that a nonelementary hyperbolic group Γ contains a free non-Abelian group of rank 2 (see [4]), and hence there exist constants $a > 1$ and $r_0 > 0$ such that $\forall r > r_0$,

$$a^r \le |B(r)| \le (2n)^{r+1}, \tag{1}$$

where $B(r) = B_1(r)$.

LEMMA 1. For $x, y \in \Gamma$, $x_1 \in [1,x]$, $y_1 \in [1,y]$, and $|x_1| = |y_1|$, the following inequality holds: $d(x_1, y_1) \le d(x,y) + 2\delta$.

Proof. If $|x_1| \le \frac{|x|+|y|-d(x,y)}{2}$, then $d(x_1,y_1) \le \delta$. Suppose that $|x_1| > \frac{|x|+|y|-d(x,y)}{2}$. Then $d(x_1,x_2) \le \delta$ and $d(y_1,y_2) \le \delta$, where x_2 and y_2 are points on the geodesic $[x,y]$ such that $d(x,x_2) = d(x,x_1)$ and $d(y,y_2) = d(y,y_1)$. It follows that $d(x_1,y_1) \le d(x_1,x_2) + d(x_2,y_2) + d(y_2,y_1) \le d(x,y) + 2\delta$.

LEMMA 2. For arbitrary $s \ge \delta$ and $x \in \Gamma$, the number of elements $y \in B_x(s)$ for which $|y| \ge |x| + d(x,y) - (4\delta + 2)$ is not less than $|B(s)| - |B(s-\delta)|$.

Proof. Let $y \in B_x(s)$ and suppose that the opposite inequality holds, i.e., $|y| < |x| + d(x,y) - (4\delta + 2)$. Let x_1 and z be points on the geodesics $[1,x]$ and $[x,y]$ such that $d(x,x_1) = d(x,z) = 2\delta + 1$. These points do exist, and $d(x_1,z) \le \delta$ since $\frac{1}{2}(|x| + d(x,y) - |y|) > 2\delta + 1$. Hence $d(x_1,y) \le d(x_1,z) + d(x,y) - d(x,z) \le d(x,y) - \delta - 1 \le s - \delta - 1$.

Let x_2 be a vertex on the geodesic $[1,x]$ which is closest to the point x_1 and is such that $x_1 \in [1,x_2]$. Then $y \in B_{x_2}(s-\delta)$ and the number of those y is not more than $|B(s-\delta)|$.

Definition 5. We call a positive number s *beautiful* if for any $x \in \Gamma$ the number of elements $y \in B_x(s)$ such that $|y| \ge |x| + d(x,y) - (4\delta + 2)$ is not less than $\left|B(\frac{s}{2})\right|$. An element y is referred to as a *near-continuation* of the element x.

LEMMA 3. There exists a number k_0 such that, for all $k \ge k_0$, at least one beautiful number exists in the interval $[k \log_2 k, k^2]$.

Proof. If $\delta = 0$, then Γ is a free group (see [4]), whence the lemma. Assume $\delta > 0$. If the lemma was false, then, for all k_0 for which $k_0 \log_2 k_0 \geq \delta$, there would exist a $k \geq k_0$ such that the interval $[k \log_2 k, k^2]$ did not contain a beautiful number. Let s be an arbitrary number in that interval. Then $|B(s)| - |B(s-\delta)| < |B(\frac{s}{2})|$. If $s - \delta \in [k \log_2 k, k^2]$, then $|B(s-\delta)| - |B(s-2\delta)| < |B(\frac{s-\delta}{2})| \leq |B(\frac{s}{2})|$. We can write a series of similar inequalities of which the last one is $|B(s-t\delta)| - |B(s-(t+1)\delta)| < |B(\frac{s}{2})|$, where for $t \in N$, $s - t\delta \geq k \log_2 k > s - (t+1)\delta$. Adding these inequalities and taking into account that $|B(s-(t+1)\delta)| \leq |B(k \log_2 k)|$ and $t \leq \frac{k^2}{\delta} - 1$, we deduce that

$$|B(s)| < |B(k \log_2 k)| + \frac{k^2}{\delta} |B(\frac{s}{2})|.$$

Putting $s = k^2, \frac{k^2}{2}, \ldots, \frac{k^2}{2^p}$, where for $p \in N$, $\frac{k^2}{2^p} \geq k \log_2 k > \frac{k^2}{2^{p+1}}$, we obtain a series of the inequalities

$$|B(k^2)| < |B(k \log_2 k)| + \frac{k^2}{\delta} \left| B\left(\frac{k^2}{2}\right) \right|,$$

$$\left| B\left(\frac{k^2}{2}\right) \right| < |B(k \log_2 k)| + \frac{k^2}{\delta} \left| B\left(\frac{k^2}{4}\right) \right|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\left| B\left(\frac{k^2}{2^p}\right) \right| < |B(k \log_2 k)| + \frac{k^2}{\delta} \left| B\left(\frac{k^2}{2^{p+1}}\right) \right|.$$

Combining them gives

$$|B(k^2)| < |B(k \log_2 k)| \left(1 + \frac{k^2}{\delta} + \left(\frac{k^2}{\delta}\right)^2 + \cdots + \left(\frac{k^2}{\delta}\right)^p\right) + \left(\frac{k^2}{\delta}\right)^{p+1} \left| B\left(\frac{k^2}{2^{p+1}}\right) \right|.$$

If $\frac{k_0^2}{\delta} \geq 2$, we apply $|B(\frac{k^2}{2^{p+1}})| \leq |B(k \log_2 k)|$ to arrive at

$$|B(k^2)| \leq |B(k \log_2 k)| \cdot \left(\frac{k^2}{\delta}\right)^{\log_2(\frac{k}{\log_2 k})+2}.$$

The last inequality contradicts (1), with k_0 sufficiently large.

If we apply Lemma 3 twice we see that there exist beautiful numbers $k \geq k_0$ and $s \in [k \log_2 k, k^2]$ and, moreover, it is possible to choose a sufficiently large k such that $k > 12\delta + 8$,

$$\frac{k^2 + m + 1}{k/2 - 3 - 4\delta} \leq 3k \tag{2}$$

and

$$\frac{|B(\frac{k \log_2 k}{2})| - |B(4\delta + 3 + m)|}{|B(m)|} \geq C_1^2 + C_1, \tag{3}$$

where $C_1 = (k^2 + m + 2)|B((3\delta + 1)k + 20\delta + 2m + 6)|$.

Let $r = s + m + 1$ and foliate Γ with spherical strata $T_j = \{x \in \Gamma \mid r \cdot (j-1) \leq |x| < r \cdot j\}$, $j = 1, 2, \ldots$, of width $r - 1$. Our nearest goal is to construct a 1-1 mapping $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$ such that $d(x, \varphi_j(x))$ is bounded above by a constant which does not depend on j and $x \in T_j$. This will be done while carrying on the constructions in the next section.

2. CONSTRUCTIONS

Briefly, our plan is as follows. First, to an element $x \in T_j$ we ascribe some element y, such that $|y| = [r \cdot j]$, and ascribe the element $y' \in [1, y]$ for which $|x| = |y'|$, and

$$d(x, y') \leq (3\delta + 1)k + 6\delta + 2, \tag{4}$$

$$d(x, y) \leq (3\delta + 1)k + 6\delta + 2 + r. \tag{5}$$

We call an element y a *continuation* of the element x. The *construction* defined on the *source* x is the set

$$S_x = O_m(\{z \in B_y(s) \mid |z| \geq |y| + d(z, y) - (4\delta + 2), \ z \notin B_y(4\delta + 3 + m)\}) \bigcap \Gamma_1. \tag{6}$$

Here O_m denotes the m-neighborhood of the set in brackets. It is clear that $S_x \subseteq T_{j+1}$.

Next, we prove the following statements:

(1) the number of sources producing the same construction is bounded above by some constant $C = C(k, \delta, n, m)$;

(2) in every construction, we can choose C elements so that the elements chosen in different constructions are pairwise distinct.

This will permit us to construct a 1-1 correspondence $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$ which assigns to each source $x \in T_j$ an element from the construction S_x. In addition,

$$d(x, \varphi_j(x)) \leq d(x, y) + d(y, \varphi_j(x)) \leq (3\delta + 1)k + 6\delta + 2s + m + 3. \tag{7}$$

Thus, let $x \in T_j$. Since k is a beautiful number (cf. Definition 5), there exists a sequence of elements $x = x_0, x_1, \dots$ in Γ such that

$$\frac{k}{2} - 1 \leq d(x_i, x_{i+1}) \leq k\,; \ |x_{i+1}| \geq |x_i| + d(x_i, x_{i+1}) - (4\delta + 2), \tag{8}$$

$i = 1, 2, \dots$ (see Fig. 2 below).

Let t be a number such that $x_t \in T_j$ and $x_{t+1} \in T_{j+1}$. Then

$$r \geq |x_t| - |x_0| = \sum_{i=0}^{t-1}(|x_{i+1}| - |x_i|) \geq t \cdot (\frac{k}{2} - 3 - 4\delta),$$

and hence $t \leq 3k$ in view of (2).

Suppose x_i' is a point on the geodesic $[1, x_i]$ with $|x_i'| = |x|$, $1 \leq i \leq t$. Such a point exists since $|x_i| \geq |x_{i-1}| + \frac{k}{2} - 3 - 4\delta \geq |x_{i-1}| \geq \cdots \geq |x|$. Let y be a first vertex on the geodesic $[x_t, x_{t+1}]$, for which $|y| = [r \cdot j]$, and let y' be a vertex on the geodesic $[1, y]$ such that $|y'| = |x|$. Below we give estimates for the distances $d(x, x_1')$, $d(x_i', x_{i+1}')$, and $d(x_t', y')$, $1 \leq i \leq t - 1$.

Let e and e_1 be points on the geodesics $[1, x]$ and $[1, x_1]$ such that $|e| = |e_1| = \frac{1}{2}(|x| + |x_1| - d(x, x_1))$. Then $d(e, e_1) \leq \delta$, and in view of (8),

$$d(x, x_1') = d(x, e) + d(e, e_1) + d(e_1, x_1') \leq 2(|x| - |e|) + \delta \leq 5\delta + 2.$$

Further, if $i \geq 1$ then

$$|x_i| - |x_i'| = \sum_{l=0}^{i-1}(|x_{l+1}| - |x_l|) \geq i \cdot (\frac{k}{2} - 3 - 4\delta) \geq 2\delta + 1.$$

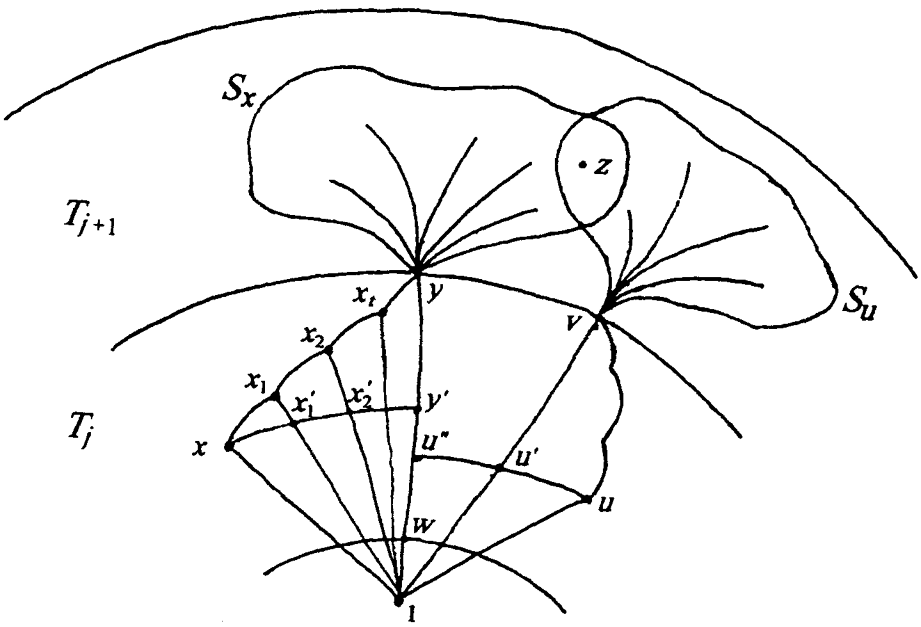

Fig. 2

In view of (8), $\frac{1}{2}(|x_i| + |x_{i+1}| - d(x_i, x_{i+1})) \geq |x_i| - 2\delta - 1 \geq |x_i'| = |x_{i+1}'|$. Therefore, $d(x_i', x_{i+1}') \leq \delta$. By Lemma 1 and inequalities (8), we have $d(x_t', y') \leq d(x_t, y) + 2\delta \leq d(x_t, x_{t+1}) + 2\delta \leq k + 2\delta$, and hence $d(x, y') \leq d(x, x_1') + \sum_{i=1}^{t-1} d(x_i', x_{i+1}') + d(x_t', y') \leq 5\delta + 2 + (t-1)\delta + k + 2\delta \leq (3\delta + 1)k + 6\delta + 2$. Since $d(y', y) \leq r$, we obtain $d(x, y) \leq (3\delta + 1)k + 6\delta + 2 + r$.

Now we define the construction S_x via formula (6). Suppose that two constructions have a common element: $h \in S_x \bigcap S_u$. Let y and v be continuations of the sources x and u. Assume that z_1 and z_2 are elements in the sets

$$\{z \in B_y(s) \mid |z| \geq |y| + d(z, y) - (4\delta + 2);\ z \notin B_y(4\delta + 3 + m)\}$$

and

$$\{z \in B_v(s) \mid |z| \geq |v| + d(z, v) - (4\delta + 2);\ z \notin B_v(4\delta + 3 + m)\},$$

such that $d(h, z_1) \leq m$ and $d(h, z_2) \leq m$. Let z_1' and z_2' be points on the geodesics $[1, z_1]$ and $[1, z_2]$, respectively, for which $|z_1'| = |z_2'| = [r \cdot j](= |y| = |v|)$. The inequalities $d(y, z_1') \leq 5\delta + 2$ and $d(v, z_2') \leq 5\delta + 2$ can be derived in the same way as was done for $d(x, x_1') \leq 5\delta + 2$. Since $d(z_1, z_2) \leq 2m$, we obtain $d(z_1', z_2') \leq 2m + 2\delta$ by Lemma 1. It follows that $d(y, v) \leq 12\delta + 2m + 4$.

Let u'' and u' be vertices on the geodesics $[1, y]$ and $[1, v]$ such that $|u''| = |u'| = |u|$. By Lemma 1, $d(u'', u') \leq d(y, v) + 2\delta \leq 14\delta + 2m + 4$. In view of (4), $d(u, u') \leq (3\delta + 1)k + 6\delta + 2$, and so $d(u, u'') \leq (3\delta + 1)k + 20\delta + 2m + 6$.

Let w be a point on the geodesic $[1, y]$ such that $|w| = r \cdot (j - 1)$. Then $u'' \in [w, y]$, and hence u lies in the $((3\delta + 1)k + 20\delta + 2m + 6)$-neighborhood of the geodesic $[w, y]$. Therefore, the number of sources giving rise to a construction intersecting a given construction S_x does not exceed $C = (r + 1) \cdot |B((3\delta + 1)k + 20\delta + 2m + 6)|$. In particular, no construction can be produced by more than C sources, and the number of constructions intersecting a given one does not exceed C either.

We observe that the distance between the sources x and u may be too great: $d(x,u) \le d(x,y') + d(y',u'') + d(u'',u) \le ((3\delta+1)k + 6\delta + 2) + r + ((3\delta+1)k + 20\delta + 2m + 6)$. This means that it cannot be used in further calculations and so we confine ourselves to the estimation for $d(u,[w,y])$.

Since s is a beautiful number, the number of elements in the set $W = \{z \in B_y(s) \mid |z| \ge |y| + d(z,y) - (4\delta+2);\ z \notin B_y(4\delta+3+m)\}$ is not less than $|B(\frac{s}{2})| - |B(4\delta+3+m)|$. Since in the m-neighborhood of any element of Γ we can find some element from Γ_1, that set W is contained in the m-neighborhood of the set S_x. Therefore,

$$|S_x| \ge \frac{|B(\frac{s}{2})| - |B(4\delta+3+m)|}{|B(m)|} \ge \frac{|B(\frac{k\log_2 k}{2})| - |B(4\delta+3+m)|}{|B(m)|} \ge C^2 + C \tag{9}$$

by the choice of k; see (3).

We prove that, in every construction, it is possible to choose C elements so that the following condition is satisfied:

$$\text{Samplings do not intersect if constructions are different.} \tag{10}$$

Let us enumerate constructions lying in the $(j+1)$th stratum. In the first construction, C elements can be chosen in view of (9). Assume that, in each of the constructions $S_1, \dots, S_i$, C elements have been chosen so as to satisfy (10). The construction S_{i+1} intersects not more than C other constructions, and hence it contains at most C^2 elements, chosen before. In view of (9), in S_{i+1} we can choose C elements for which condition (10) is satisfied.

For every j, we are now in a position to construct a 1-1 correspondence $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$ satisfying (7). Define the mappings

$$\varphi_{j,1} = \varphi_j \mid (T_j \setminus \Gamma_1) \bigcup Im\varphi_{j-1,1} \quad \text{and} \quad \varphi_{j,2} = \mathrm{id} \mid T_j \bigcap (\Gamma_1 \setminus Im\varphi_{j-1,1}).$$

Finally, define the mapping $\varphi : \Gamma \longrightarrow \Gamma_1$ by setting $\varphi(x) = \varphi_{j,i}(x)$ if $x \in \operatorname{dom}\varphi_{j,i}$. It is easy to verify that φ is a bijection, and in view of (7), we have $d(x,\varphi(x)) \le (3\delta+1)k + 6\delta + 2s + m + 3$. Theorem 1$'$ is proved.

3. SEPARATED NETS IN THE HYPERBOLIC SPACE H^n

Definition 6. Let (X,d) be a metric space. The subset $Y \subseteq X$ is called a *separated net* in X if there exist constants $\varepsilon > 0$ and $\mu > 0$ such that the following conditions are satisfied:

(1) $\forall x \in X\ \exists y \in Y : d(x,y) \le \varepsilon$;

(2) $\forall y_1, y_2 \in Y$, if $y_1 \ne y_2$, then $d(y_1,y_2) \ge \mu$.

Subgroups of finite index in G can be taken to be separated nets in G. This is the reason why the proof of Theorem 1' can be reduced to that of Theorem 2.

Proof of Theorem 2. Let H^n be a hyperbolic space of dimension $n \ge 2$, with d and δ its metric and hyperbolicity constant, respectively. Write $V(r)$ to denote the volume of a ball of radius r in H^n. It is known from [5] that $V(r)$ grows exponentially.

If G is an arbitrary μ-separated ε-net in H^n, and $r > \varepsilon$, then it follows easily from the definition that

$$\frac{V(r-\varepsilon)}{V(\varepsilon)} \le |B_x(r) \bigcap G| \le \frac{V(r+\frac{\mu}{2})}{V(\frac{\mu}{2})}.$$

Therefore, there exist positive constants a, b, and c such that any $r \ge 0$ satisfies the inequality

$$ae^{(n-1)r} - b \le |B_x(r) \bigcap G| \le ce^{(n-1)r}. \tag{11}$$

Let 1 be an arbitrary point in H^n and $|x| = d(1,x)$ for an arbitrary point $x \in H^n$. Similarly to Lemma 2 we can prove the following:

LEMMA 2′. For arbitrary $\alpha \geq 0, s \geq 0$, and $x \in H^n$, the number of elements $y \in B_x(s) \bigcap G$ such that $|y| \geq |x| + d(x,y) - \alpha\delta$ is not less than

$$|B_x(s) \bigcap G| - ce^{(n-1)(s - \frac{\alpha-2}{2}\delta)}.$$

Choose α so that $\alpha > \frac{2\ln(\frac{c}{a})}{(n-1)\delta} + 2$.

Definition 5′. We call a positive real number s *beautiful for the net* G if, for every $\forall x \in H^n$, the number of elements $y \in B_x(s) \bigcap G$ such that $|y| \geq |x| + d(x,y) - \alpha\delta$ is at least $|B_x(\frac{s}{2}) \bigcap G|$.

LEMMA 3′. There exists a number s_0 such that every $s \geq s_0$ is a beautiful number for the net G.

Proof. In view of Lemma 2′, it is sufficient to prove that the inequality $|B_x(s) \bigcap G| - ce^{(n-1)(s-\frac{\alpha-2}{2}\delta)} \geq |B_x(\frac{s}{2}) \bigcap G|$ holds for large s. This in turn follows by the choice of α, using inequalities (11).

Now let Γ_1 and Γ_2 be two arbitrary separated nets in H^n with constants $\mu_i, \varepsilon_i, a_i, b_i$, and c_i $(i = 1,2)$. Choose a beautiful number s for the net Γ_1, so that the following inequality holds:

$$a_1 e^{(n-1)\frac{s}{2}} - b_1 - c_1 e^{(n-1)\alpha\delta} \geq C^2 + C, \tag{12}$$

where

$$C = \left(\frac{s+1}{4(\alpha+2)\delta} + 2\right)(c_1 + c_2)e^{4(\alpha+2)\delta(n-1)}.$$

For the case of H^n, the definition of constructions will be simpler, since every geodesic in H^n can be continued infinitely in both directions. Let $r = s + 1$,

$$T_j = \{x \in H^n \mid r\cdot(j-1) \leq |x| < r\cdot j\}, \quad j = 1,2,\ldots.$$

For the sources $x \in (\Gamma_1 \bigcup \Gamma_2) \bigcap T_j$ and only for them, the construction $S_x \subseteq T_{j+1}$ is defined thus:

$$S_x = \{z \in B_y(s) \bigcap \Gamma_1 \mid |z| \geq |y| + d(z,y) - \alpha\delta\,;\ z \notin B_y(\alpha\delta)\},$$

where y is a point such that $x \in [1,y]$ and $|y| = r \cdot j$.

Suppose that two constructions intersect: $z \in S_x \bigcap S_u$. Assume that v is a point for which $u \in [1,v]$ and $|v| = r\cdot j$, and that z' is a point on the geodesic $[1,z]$ such that $|z'| = r \cdot j$.

By analogy with the proof of $d(x,x_1') \leq 5\delta + 2$ in Sec. 2, we can derive the inequalities $d(y,z') \leq (\alpha+1)\delta$ and $d(z',v) \leq (\alpha+1)\delta$. It follows that $d(y,v) \leq 2(\alpha+1)\delta$. Let w be a point on the geodesic $[1,y]$, with $|w| = r\cdot(j-1)$, and let u' be a point on the geodesic $[w,y]$ with $|u'| = |u|$. Lemma 1 applied to the triangle $[1,y,v]$ then yields $d(u,u') \leq d(y,v) + 2\delta \leq 2(\alpha+2)\delta$. Write $A = 2(\alpha+2)\delta$.

Now $u \in (\Gamma_1 \bigcup \Gamma_2) \bigcap T_j$ lies in the A-neighborhood of the geodesic $[w,y]$, and hence it lies in the union of at most $\frac{r}{2A} + 2$ balls of radius $2A$, covering this geodesic of length r. Therefore, the number of those sources u is at most

$$\left(\frac{r}{2A} + 2\right)|B(2A) \bigcap (\Gamma_1 \bigcup \Gamma_2)| \leq \left(\frac{s+1}{2A} + 2\right)(c_1 + c_2)e^{2A(n-1)} = C.$$

From this, it follows that

(1) the number of constructions intersecting a given one is at most C;

(2) every construction is produced by not more than C sources;

(3) $|S_x| \geq |B_y(\frac{s}{2}) \bigcap \Gamma_1| - |B_y(\alpha\delta) \bigcap \Gamma_1| \geq a_1 e^{(n-1)\frac{s}{2}} - b_1 - c_1 e^{(n-1)\alpha\delta} \geq C^2 + C$ since s is a beautiful number and C satisfies (12).

Arguing in the same way as at the end of Sec. 2, we can construct a bijection $\varphi_1 : \Gamma_1 \bigcup \Gamma_2 \longrightarrow \Gamma_1$ such that $d(x, \varphi_1(x)) \leq 2r$ $\forall x \in \Gamma_1 \bigcup \Gamma_2$. Likewise we can find a constant t and a bijection $\varphi_2 : \Gamma_1 \bigcup \Gamma_2 \longrightarrow \Gamma_2$, so that $d(x, \varphi_2(x)) \leq 2t$ $\forall x \in \Gamma_1 \bigcup \Gamma_2$. Then $\varphi = \varphi_1^{-1} \circ \varphi_2 : \Gamma_1 \longrightarrow \Gamma_2$ is a bijection for which $d(x, \varphi(x)) \leq 2(r+t)$.

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REFERENCES

1. M. Gromov, *Geometric Group Theory, Asymptotic Invariants of Infinite Groups, Lect. Note Ser.*, 182 (1993).

2. P. Papasoglu, "Homogeneous trees are bilipschitz equivalent," *Geom. Ded.*, 54, 301-306 (1995).

3. D. Epstein, J. Cannon, D. Holt, et al., *Word Processing in Groups*, Jones & Barlett (1992).

4. E. Ghys and P. de la Harpe, *Espaces Metriques Hyperbolicues sur les Groupes Hyperbolicues dàprés Mikhael Gromov*, Birkhaüser (1991).

5. L. V. Ahlfors, *Mōbius Transformations in Several Dimensions*, University of Minnesota (1981).