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FINITE SUBGROUPS OF HYPERBOLIC GROUPS

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Let G be a hyperbolic group which is δ-thin w.r.t. a finite generating set $\mathcal{X}$. We show that every finite subgroup of G is conjugate to a subgroup each element of which has length at most $2\delta+1$ relative to $\mathcal{X}$.

Let G be a group generated by elements of a finite set $\mathcal{X}$. Denote by $\Gamma_{\mathcal{X}}(G)$ the right Cayley graph of G w.r.t. $\mathcal{X}$. We will consider $\Gamma_{\mathcal{X}}(G)$ as the geodesic-metric space with metric d, in which each edge of the graph has unit length. For any elements a and b in G, $[a,b]$ denotes some geodesic path in $\Gamma_{\mathcal{X}}(G)$ beginning at a and ending at b; $|a|$ is the length of the path $[1,a]$. Put $B(r)=\{g\in G \mid |g|\le r\}$.

A geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ with sides $[a,b]$, $[a,c]$, and $[b,c]$ is called *δ-thin* if, for any points B and C such that $B\in[a,b]$, $C\in[a,c]$, and

$$d(a,B)=d(a,C)\le\frac{1}{2}(d(a,b)+d(a,c)-d(b,c)),$$

the inequality $d(B,C)\le\delta$ holds.

Definition. The subgroup $H\le G$ is said to be *δ-thin* w.r.t. $\mathcal{X}$ if every geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ with vertices in H is δ-thin.

We recall that a group G is called *hyperbolic* if, for some finite generating set $\mathcal{X}$ and number δ, G is δ-thin w.r.t. $\mathcal{X}$.

The main result of this article is the following:

THEOREM. Let G be a group with a finite generating set $\mathcal{X}$. Then any of its finite and δ-thin subgroups H w.r.t. $\mathcal{X}$ is conjugate to some subgroup lying in a ball $B(3\delta+1)$.

If G is a hyperbolic group which is δ-thin w.r.t. $\mathcal{X}$, then any of its finite subgroups H is conjugate to some subgroup lying in the ball $B(2\delta+1)$.

At this point, we cite some results obtained in earlier works on hyperbolic groups G which are δ-thin w.r.t. $\mathcal{X}$.

Every element of finite order in G is conjugate to an element of length not more than $2\delta+1$ (see [1, Prop. 1.3]).

Every finite cyclic subgroup of prime order in G is conjugate to a subgroup lying in $B(4\delta+1)$ (see [2, Chap. 4, Prop. 13]).

Behind the proof of the latter result is the idea that subgroups of prime order have a common fixed point whenever they act on a finite-dimensional contractible Rips complex. For arbitrary finite subgroups, this argument breaks down: in [3, Cor. II.7.4], it was proved that there exist finite groups acting on a

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contractible three-dimensional CW-complex with no fixed point in common (see also [4]). Therefore, here we propose a proof based on a different idea.

COROLLARY. In a hyperbolic group, the number of conjugacy classes of finite subgroups is finite.

Independently, this corollary was proved by Olshanskii (oral communication); see also partial results in [5, Props. 2.2.B and 5.3.C′].

In order to prove our theorem, we use the following:

LEMMA [6]. Let G be a group with a finite generating set $\mathcal{X}$ and let H be its finite subgroup which is δ-thin w.r.t. $\mathcal{X}$. Then, for any four elements x, y, z, and t from H, the following inequality holds: $d(x,y)+d(z,t)\le \max\{d(y,z)+d(t,x), d(y,t)+d(x,z)\}+2\delta$.

Proof of the theorem. Let $l=\max\{|h| \mid h\in H\}$. Then the following statement is valid.

If $[x,y]$ and $[z,t]$ are two geodesics in $\Gamma_{\mathcal{X}}(G)$ of length l with end points in H, then their middles are at distance at most 3δ apart.

In view of the above lemma, at least one of the distances $d(x,z)$, $d(y,z)$, $d(x,t)$, or $d(y,t)$ is not less than $l-\delta$. Without loss of generality, we may assume that $d(y,z)\ge l-\delta$. Let A, B, and C be middles on the geodesics $[x,y]$, $[y,z]$, and $[z,t]$, respectively, and let B_1 and B_2 be points on $[x,y]$ and $[z,t]$ such that $d(y,B_1)=d(z,B_2)=d(y,B)$. Since the geodesic triangle xyz is δ-thin and $d(y,B)\le \frac{1}{2}(d(x,y)+d(y,z)-d(x,z))$, we infer that $d(B,B_1)\le\delta$. Moreover, $d(A,B_1)=\frac{1}{2}(d(x,y)-d(y,z))\le\frac{1}{2}\delta$. Hence $d(A,B)\le\frac{3}{2}\delta$. Similarly, for the geodesic triangle yzt we derive $d(B,C)\le\frac{3}{2}\delta$. Thus, $d(A,C)\le 3\delta$.

Now let b be some element from H of length l and let a be an arbitrary element in H. Consider the geodesics $[1,b]$ and $a[1,b]$. Assume that K and L, respectively, are their middles, and b_1 is an element in G such that $b_1\in[1,b]$ and $d(b_1,K)\le\frac{1}{2}$. Then $|b_1^{-1}ab_1|=d(b_1,ab_1)\le d(b_1,K)+d(K,L)+d(L,ab_1)\le 3\delta+1$. Thus, $H^{b_1}\subseteq B(3\delta+1)$.

Next let G be a hyperbolic group which is δ-thin w.r.t. $\mathcal{X}$ and let H be its finite subgroup. We make use of the following statement which admits a routine proof.

For an arbitrary geodesic triangle with vertices X, Y, and Z and for the middle T on the geodesic $[X,Y]$, the inequality $d(T,Z)\le\max\{d(X,Z),d(Y,Z)\}-\frac{1}{2}d(X,Y)+\delta$ holds.

In view of this, $d(A,z)\le\max\{d(x,z),d(y,z)\}-\frac{l}{2}+\delta$ and $d(A,t)\le\max\{d(x,t),d(y,t)\}-\frac{l}{2}+\delta$.

Let l be an even number. Then A is a vertex and

$$d(C,A)\le\max\{d(A,z),d(A,t)\}-\frac{l}{2}+\delta\le\max\{d(x,z),d(y,z),d(x,t),d(y,t)\}-l+2\delta\le 2\delta.$$

Let l be odd and let M and N be vertices on the geodesics $[x,y]$ and $[z,t]$ such that $d(y,M)=d(z,N)=(l-1)/2$. Then $d(M,z)\le d(A,z)+\frac{1}{2}$, $d(M,t)\le d(A,t)+\frac{1}{2}$, and

$$d(C,M)\le\max\{d(t,M),d(z,M)\}-\frac{l}{2}+\delta\le 2\delta+\frac{1}{2}.$$

Consequently, $d(N,M)\le d(N,C)+d(C,M)\le 2\delta+1$, from which our last claim follows immediately.

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