PROBLEM OF THE CONJUGACY OF AUTOMORPHISMS OF FREE GROUPS

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Let F_n be a free group of degree n; let A_n be its automorphism group and let $Fix(\alpha)$ be the subgroup of the fixed points of the automorphism α.

In Sec. 1 of this paper we indicate an effective algorithm solving the conjugacy problem for automorphisms of prime order of the group A_n. In Sec. 2 we indicate an algorithm for the determination of a basis of the subgroup of fixed points of an automorphism for A_2. With the use of this algorithm, in Sec. 3 we solve the conjugacy problem in the group A_2 (Theorem 3). The formulation of Theorem 3 is contained also in [8].

1. Problem of Conjugacy for Automorphisms of Prime Order

Let α be an automorphism of a group F_n of order p, where p is a prime number. In view of [1], the group F_n can be decomposed into a free product of α-invariant subgroups:

$$F_n = Fix(\alpha) * (\underset{i\in M}{*} X_i) * (\underset{j\in K}{*} Y_j), \tag{1}$$

where $M=\{1,\dots,m\}$, $K=\{1,\dots,k\}$ and

1) X_i, $i\in M$, has a basis $x_{i,1},\dots,x_{i,p}$ such that $\alpha(x_{i,r})=x_{i,r+1}$, $1\le r\le p$ (the second indices are taken modulo p);

2) Y_j, $j\in K$, has a basis $y_{j,1},\dots,y_{j,p-1}, z_{j,1},\dots,z_{j,t_j}$ such that

$$\alpha(y_{j,i}) = y_{j,i+1}, \quad i=1,\dots,p-2;$$

$$\alpha(y_{j,p-1}) = (y_{j,1}\dots y_{j,p-1})^{-1};$$

$$\alpha(z_{j,\ell}) = y_{j,1}^{-1} z_{j,\ell}\, y_{j,1}, \quad \ell=1,\dots,t_j.$$

Assume that the degree of the free group $Fix(\alpha)$ is equal to r; $t_1\le\dots\le t_k$.

THEOREM 1. The problem of the conjugacy of automorphisms of prime order of a free group of finite degree, defined by the action on a basis, is algorithmically solvable: The conjugacy class is determined by a collection of numbers r, m, k, $t_1,\dots,t_k$, which can be found effectively.

Proof. We assume that the automorphism α is defined on a fixed basis $b_1,\dots,b_n$ of the group F_n. Assume that to the automorphism α there corresponds a decomposition of the form (1) with some parameters r, m, k, $t_1,\dots,t_k$ and let $d_1,\dots,d_n$ be a basis of the group F_n, being the union of the above indicated bases of the groups X_i, $i\in M$, Y_j, $j\in K$, and of some

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basis of the group $Fix(\alpha)$. For an element x of some group, we denote by $\bar{x}$ the image of x in the quotient with respect to the commutator.

Since $b_1,\dots,b_n$ and $d_1,\dots,d_n$ are bases of the group F_n, we have

$$\{\alpha(w)\cdot w^{-1};\, w\in F_n\}^{F_n}=\{\alpha(b_i)b_i^{-1};\, i=1,\dots,n\}^{F_n}=\{\alpha(d_i)d_i^{-1};\, i=1,\dots,n\}^{F_n}.$$

We introduce the notations:

$$H=\mathrm{gr}\,(b_1,\dots,b_n)/\{\alpha(b_i)b_i^{-1};\; i=1,\dots,n\}^{F_n},$$
$$T=\mathrm{gr}\,(d_1,\dots,d_n)/\{\alpha(d_i)d_i^{-1};\; i=1,\dots,n\}^{F_n}.$$

We have

$$H\simeq T\simeq F_{z+m}*(\mathbb{Z}_p\times F_{t_1})*\dots*(\mathbb{Z}_p\times F_{t_k}). \tag{2}$$

Let $f_1,\dots,f_{z+m},e_1,h_1,\dots,h_{t_1},\dots,e_k,h_{t_1+\dots+t_{k-1}+1},h_{t_1+\dots+t_k}$ be a sequence of elements from T, consisting of the bases of the corresponding subgroups. Let $G=H/H_3(H')^p$, $S=T/T_3(T')^p$, $\Delta=z+m+t_1+\dots+t_k$. We have $T/T'=\mathrm{gr}\,(\bar{e}_1,\dots,\bar{e}_k)\oplus\mathrm{gr}\,(\bar{f}_1,\dots,\bar{f}_{z+m},\bar{h}_1,\dots,\bar{h}_{t_1+\dots+t_k})\simeq \mathbb{Z}_p^k\oplus\mathbb{Z}^\Delta$. Therefore, in H one can find effectively the elements $a_1,\dots,a_k,a_{k+1},\dots,a_{k+\Delta}$ such that

$$H/H'\simeq \mathrm{gr}\,(\bar{a}_1,\dots,\bar{a}_k)\oplus\mathrm{gr}\,(\bar{a}_{k+1},\dots,\bar{a}_{k+\Delta})\simeq\mathbb{Z}_p^k\oplus\mathbb{Z}^\Delta,$$

and, thus, one can find k and Δ. From (1) there follows that $n=z+pm+(p-1)k+(t_1+\dots+t_k)=(p-1)(m+k)+\Delta$. From here $m=\frac{n-\Delta}{p-1}-k$. From (2) there follows that $G'\simeq S'\simeq\mathbb{Z}_p^\ell$ where $\ell=C^2_{z+m+t_1+\dots+t_k+k}-(t_1+\dots+t_k)=\dfrac{(\Delta+k)(\Delta+k-1)}{2}-\Delta+(z+m)$. From here $z=\ell+\Delta-m-\dfrac{(\Delta+k)(\Delta+k-1)}{2}$.

Taking into account that G is a finitely generated, finitely presented nilpotent group of level ≤ 2, one can effectively find a basis of the group G' and the numbers ℓ and z. It remains to determine $t_1,\dots,t_k$.

We have $C_S(\bar{e}_1)/S^pS'\simeq\mathbb{Z}_p^{t_1+1},\dots,C_S(\bar{e}_k)/S^pS'\simeq\mathbb{Z}_p^{t_k+1}$. The notation $C_S(\bar{e}_i)$ makes sense since the centralizers of the elements, differing by an element from S', coincide. We note that

$$C_S(\bar{e}_1^{\delta_1}\dots\bar{e}_k^{\delta_k})/S^pS'\simeq\mathbb{Z}_p,$$

if for some $i,j,1\leq i<j\leq k$, we have $\delta_i\neq 0,\ \delta_j\neq 0$ modulo p. Thus, the basis $\bar{e}_1,\dots,\bar{e}_k$ of the periodic component of the group T/T' is distinguished among all the bases (to within a permutation and the raising to powers of the basis elements) by the property that for it the sum

$$|C_S(\bar{e}_1)/S^pS'|+\dots+|C_S(\bar{e}_k)/S^pS'|$$

is maximal.

Thus, in order to find $t_1,\dots,t_k$ we have:

1) to fix a basis $\bar{a}_{k+1},\dots,\bar{a}_{k+\Delta}$ not the periodic part in some decomposition of the group $H/H'\simeq\mathbb{Z}_p^k\oplus\mathbb{Z}^\Delta$;

2) to enumerate all the bases of the periodic component of the group H/H'. There are at most $|GL_k(\mathbb{Z}_p)|$;

3) for each such basis $\bar{a}_1,\dots,\bar{a}_k$ compute $|C_G(\bar{a}_i)/G^pG'|$, $1\le i\le k$. This can be done by going through all the elements of the form

$$c_{j_1,\dots,j_{k+s}}=\bar{a}_1^{j_1}\dots\bar{a}_{k+s}^{j_{k+s}},\quad 0\le j_1,\dots,j_{k+s}<p,$$

and verifying the validity of the equalities $[c_{j_1,\dots,j_{k+s}},\bar{a}_i]=1$ in the group G. The latter is possible in view of the fact that the equality problem in a finitely generated, finitely presented nilpotent group of level ≤ 2 is solvable;

4) to find a basis $\bar{a}_1,\dots,\bar{a}_k$ for which the sum

$$\sum_{i=1}^{k}|C_G(\bar{a}_i)/G^pG'|$$

is maximal. Performing a permutation, we can assume that $|C_G(\bar{a}_1)/G^pG'|\le\dots\le|C_G(\bar{a}_k)/G^pG'|$. Then $t_i=\log_p|C_G(\bar{a}_i)/G^pG'|-1$.

2. Basis of $Fix(\alpha)$

Let F_n be a free group with basis $x_1,\dots,x_n$; let $|x|$ be the length of the element x in this basis, and let $A_n=Aut\,F_n$. For $\alpha\in A_n$ we set $\|\alpha\|=\max|x_i\alpha|$, $i=1,\dots,n$. By deg F we denote the free group F.

LEMMA 1. Let F_n be a free group with basis $x_1,\dots,x_n$; $u_1,\dots,u_m\in F_n$; $\alpha_1,\dots,\alpha_k\in A_n$, $A=$ gr $(\alpha_1,\dots,\alpha_k)$ and assume that it is known that the degree of the subgroup $H=\mathrm{gr}(u_1,\dots,u_m)^A$ does not exceed z. Then there exists an algorithm which allows us to find in a finite number of steps a basis of the subgroup H.

Proof. Assume that $|u_1|\le\dots\le|u_m|$; $\|\alpha_1\|\le\dots\le\|\alpha_k\|$; $M=|u_m|$, $K=\|\alpha_k\|$; $v_1,\dots,v_t$ is an N-reduced basis of the group H (see [2]), $t\le z$, $|v_1|\le\dots\le|v_t|$. We have $|v_1|\le|u_1|\le M$. We prove that

$$|v_t|\le\left(2(M+K)\right)^z. \tag{3}$$

We assume the opposite. Then there exists ℓ, $1\le\ell<t$, such that

$$|v_{\ell+1}|>2(M+K)|v_\ell|. \tag{4}$$

We prove that gr $(v_1,\dots,v_\ell)$ is an invariant group with respect to $\alpha_1,\dots,\alpha_K$. We assume the opposite. Then for some v_p and α_j, $1\le p\le\ell$, $1\le j\le K$, there exists at least one basis element v_s, $s>\ell$, which occurs in the irreducible notation of the element $v_p\alpha_j$ in the form of the product of the basis elements $v_1,\dots,v_t$ of the group H. Let $v_p\alpha_j=v_{i_1}^{\pm1}\dots v_{i_q}^{\pm1}v_s^{\pm1}\dots v^{\pm1}$ be this notation; $i_1,\dots,i_q\le\ell$, $s>\ell$.

By virtue of the N-reducibility of the basis $v_1,\dots,v_t$ and formula (4), we have

$$|v_p\alpha_j|\ge\frac{|v_s|}{2}-\frac{|v_{i_q}|}{2}\ge\frac{|v_{\ell+1}|-|v_\ell|}{2}>\left(M+K-\frac{1}{2}\right)|v_\ell|>K|v_p|.$$

We obtain a contradiction with the fact that $|v_\rho \alpha_j| \leq \|\alpha_j\| \cdot |v_\rho| \leq K|v_\rho|$.

Thus, the group gr $(v_1,\dots,v_\ell)$ is A-admissible. In a similar way one proves that $u_1,\dots,u_m \in$ gr $(v_1,\dots,v_\ell)$. From here it follows that $H \leq$ gr $(v_1,\dots,v_\ell) <$ gr $(v_1,\dots,v_t) = H$ is a contradiction. Therefore, formula (3) holds.

One can effectively enumerate all the collections of t elements, $t \leq \tau$, ordered according to increasing length, for which formula (3) is valid. Let $H_1,\dots,H_d$ be their corresponding subgroups. We select from them A-admissible subgroups $H_{i_1},\dots,H_{i_c}$ containing the elements $u_1,\dots,u_m$. This is possible by virtue of the fact that the problem of the occurrence of an element of a free group in a subgroup of finite degree is effectively solvable. We have

$$H = \bigcap_{j=\overline{1,c}} H_{i_j},$$

and the basis H is effectively sought in terms of the bases of the groups $H_{i_1},\dots,H_{i_c}$ (see [3]).

LEMMA 2. Let F_n be a free group with basis $X=\{x_1,\dots,x_n\}$, $\alpha \in A_n$. Then deg. $Fix(\alpha) \leq 1 + (2n-1)n\|\alpha\|$.

The proof follows from [6] with slight modifications in the notations. Let G be a graph, let G^0 be the set of its vertices, and let G' be the set of its edges. For a vertex v, by $St(v)$ we denote the set of the edges incident to v, together with the vertex v.

In [6] one constructs a graph Γ, whose vertices are the elements of the group F_n. The vertices u and v are joined by an edge with label (u,x,v) if $x \in X$, $v = \alpha(x)^{-1}ux$. Let Γ_1 be the connected component of the graph Γ, containing 1. It is easy to see that $Fix(\alpha) \simeq \pi(\Gamma_1)$. Then a finite set V of vertices of the graph Γ_1 is selected and the graph $\Gamma_2 = \Gamma_1 \setminus (\bigcup_{v \in V} St(v))$ is oriented in such a manner that at most one edge starts from each of its vertices. The set V consists of all initial segments of the words $\alpha(x)$, $x \in X$, lying in Γ_1^0. By E we denote the set of the edges in the set

$$\bigcup_{v \in V} St(v).$$

Obviously, $|V| \leq n\|\alpha\|$, $|E| \leq 2n|V|$. The graph Γ_2 is oriented in the following manner. Let $e \in \Gamma_2'$ be an edge with label (u,x,v) and $u \neq v$. Then in the product $\alpha(x)^{-1} \cdot u$ at least the letter of the word u is not cancelled. If this letter is cancelled in the product ux, then we direct the edge e from u to v; otherwise, from v to u. In the case $u=v$ we say that u is the origin and the end of the edge e.

Let $G_1,\dots,G_m$ be the connected components of the graph Γ_2. In [6] one proves, with the use of the indicated orientation property, that deg. $\pi(G_i) \leq 1$, $1 \leq i \leq m$. Let $K_1,\dots,K_m$ be finite subgraphs of the graphs $G_1,\dots,G_m$ such that $\pi(K_i) \simeq \pi(G_i)$ and the graph

$$K = \left(\bigcup_{i=1}^{m} K_i\right) \cup \left(\bigcup_{v \in V} St(v)\right)$$

is connected. Then $\pi(\Gamma_1) \simeq \pi(K)$ and

$$\deg.\, \pi(K) = |K^1| - |K^0| + 1 = \sum_{i=1}^{m} \left(|K_i^1| - |K_i^0|\right) + |E| - |V| + 1 =$$
$$= \sum_{i=1}^{m} \left(\deg.\, \pi(G_i) - 1\right) + |E| - |V| + 1 \le |E| - |V| + 1 \le 1 + (2n-1)n|\alpha|.$$

Let F_2 be a free group with basis x_1, x_2. There exists a unique homomorphism from A_2 onto $GL_2(\mathbb{Z})$ with kernel F_2. We denote by A_2^+ the subgroup of those automorphisms which are mapped in $SL_2(\mathbb{Z})$. For $\alpha \in A_2$, $G \le A_2$, we denote by $\tilde{\alpha}$ and $\tilde{G}$ the images of α and G in $GL_2(\mathbb{Z})$.

THEOREM 2. There exists an algorithm which allows us to find in a finite number of steps a basis of the subgroup of the fixed points of an automorphism for A_2. For an automorphism from A_2^+, the number of steps in this algorithm can be estimated from above.

Proof. It is known [4, pp. 341, 351] that $A_2^+ \simeq B_4 / Z(B_4)$ where $B_4 \simeq (\sigma_1, \sigma_2, \sigma_3 \,\|\, \sigma_1\sigma_3 = \sigma_3\sigma_1,$ $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2,\ \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3)$ is the braid group, $Z(B_4) =$ gr $(\sigma_1\sigma_2\sigma_3)^4$ is its center, isomorphic to the infinite cyclic group. The generators σ_1, σ_2, σ_3 can be selected in the following manner:

$$\sigma_1: \begin{cases} x_1 \to x_1 x_2^{-1} \\ x_2 \to x_2 \end{cases}, \quad \sigma_2: \begin{cases} x_1 \to x_1 \\ x_2 \to x_2 x_1 \end{cases}, \quad \sigma_3: \begin{cases} x_1 \to x_2^{-1} x_1 \\ x_2 \to x_2 \end{cases}.$$

Assume first that $\alpha \in A_2^+$; $C(\alpha)$ is the centralizer of α in the group A_2^+. We have $\mathrm{Fix}(\alpha) = F_2 \cap C(\alpha)$. We show that

$$C(\alpha) = N(\alpha) / Z(B_4), \tag{5}$$

where $N(\alpha)$ is the centralizer of some preimage of the element α in the group B_4. Let $c \in C(\alpha)$, c and α being written in the generators σ_1, σ_2, σ_3. Then in the group B_4 we have $c^{-1}\alpha c = \alpha z$ for some $z \in Z(B_4)$. Mapping the generators σ_1, σ_2, σ_3 of the group B_4 onto the generator a of the infinite cyclic group gr (a) and extending this mapping to a homomorphism from B_4 onto gr (a), we can see that $z = 1$ and formula (5) holds.

Makanin's algorithm [5] allows us to find effectively a finite set of elements, generating $N(\alpha)$ and, therefore, also $C(\alpha)$. Let $C(\alpha) = \mathrm{gr}\,(c_1, \ldots, c_k)$.

Since $SL_2(\mathbb{Z}) \simeq \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6$ is an almost free group, one can effectively obtain the representation of each of its finitely generated subgroups, including also that of the group $\widetilde{C(\alpha)}$, with a given generating set. Indeed, assume that the cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are generated by the elements a and b, respectively. There exists a homomorphism from the group $\mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6$ into the group $\mathbb{Z}_{12}$, under which the factors $\mathbb{Z}_4$ and $\mathbb{Z}_6$ are imbedded in the group $\mathbb{Z}_{12}$. In view of the description of the subgroups of a free product with the union, the kernel H of this homomorphism is a free group. It is easy to see that the elements $g_1 = [a, b]$ and $g_2 = [a, b^2]$ form a basis of the group H. The group $\widetilde{C(\alpha)}$ is a finite cyclic extension of the free group $\widetilde{C(\alpha)} \cap H$ and in order to find its representation with the generating set $\{\tilde{c}_1, \ldots, \tilde{c}_k\}$ it is necessary to find a basis of the group $\widetilde{C(\alpha)} \cap H$ in the form of words of $\tilde{c}_1, \ldots, \tilde{c}_k$. Making use of Schreier's method, first we find the generating elements of the group $\widetilde{C(\alpha)} \cap H$

in the form of words of $\tilde{c}_1,\dots,\tilde{c}_k$ Then we express these words in terms of g_1, g_2 and we apply Nielsen transformations in order to find a basis of the group $\widetilde{C(\alpha)}\cap H$ in the form of words of g_1, g_2. Knowing the used Nielson transformations, one can find this basis in the form of words of $\tilde{c}_1,\dots,\tilde{c}_k$.

Let $\widetilde{C(\alpha)}=(\tilde{c}_1,\dots,\tilde{c}_k \mid w_1(\tilde{c}_1,\dots,\tilde{c}_k)=1,\dots,w_m(\tilde{c}_1,\dots,\tilde{c}_k)=1)$, where $w_1,\dots,w_m$ are some words, in which the letters $\tilde{c}_1,\dots,\tilde{c}_k$ have been substituted; N is the normal closure in $C(\alpha)$ of the set $\{w_1(c_1,\dots,c_k),\dots,w_m(c_1,\dots,c_k)\}\subseteq F_2$. Then $Fix(\alpha)=C(\alpha)\cap F_2=N$. In view of Lemmas 1 and 2, we can find effectively a basis of the group $Fix(\alpha)$.

Assume now that $\alpha\in A_2$. Then $\alpha^2\in A_2^+$ and one can find a basis of $Fix(\alpha^2)$. If α acts as the identity on $Fix(\alpha^2)$ then $Fix(\alpha)=Fix(\alpha^2)$. Otherwise, α is an automorphism of the second order on the group $Fix(\alpha^2)$ and there exists for it a suitable basis of the group $Fix(\alpha^2)$ whose initial piece is a basis of the group $Fix(\alpha)$ (see Sec. 1). Knowing one basis of the group $Fix(\alpha^2)$ one can enumerate all the bases. At some step we encounter a suitable basis for α and we find a basis of the group $Fix(\alpha)$.

3. Conjugacy Problem in A_2

THEOREM 3. The conjugacy problem in the group A_2 is solvable.

Proof. First we prove that the conjugacy problem (CP) in the group A_2 is solvable for the elements from A_2^+. It is sufficient to prove the solvability of the CP for the elements from A_2^+ in the group A_2^+ itself. Let β, $\gamma\in A_2^+$. We fix the notation of β and γ in the generators $\sigma_1,\sigma_2,\sigma_3$ (see Sec. 2). The elements β and γ are conjugate in $A_2^+\simeq B_4/Z(B_4)$ if and only if there exists $\delta\in B_4$ such that $\delta^{-1}\beta\delta=\gamma z$ in the group B_4 for some $z\in Z(B_4)$. The element z is defined uniquely by β and γ from the condition of the equality of the sums of the exponents of the elements β and γz. Therefore, the CP for the elements β and γ in A_2^+ reduces to the CP for the elements β and γz in B_4. The CP in B_4 is solvable [7].

Let $\beta,\gamma\in A_2$. A necessary condition for the conjugacy of the elements γ and β in the group A_2 is the conjugacy of the elements $\tilde{\gamma}$ and $\tilde{\beta}$ in $GL_2(\mathbb{Z})$. Let $\tilde{\delta}^{-1}\tilde{\gamma}\tilde{\delta}=\tilde{\beta}$ for some $\delta\in A_2$. We find $y\in F_2$ such that $\delta^{-1}\gamma\delta=\beta y$.

From here it follows that the CP for γ and β is equivlent to the CP for βy and β. We assume that βy and β are conjugate: $c^{-1}\beta c=\beta y$ for some $c\in A_2$. Then $\tilde{c}\in C(\tilde{\beta})$. We describe the centralizer $C(\tilde{\beta})$, assuming that $\tilde{\beta}\neq\pm E$ (the case when $\beta\in A_2^+$ has been considered above). One can find an element $\tilde{\beta}_1\in GL_2(\mathbb{Z})$ and a number k such that $\tilde{\beta}=\tilde{\beta}_1^k$ or $\tilde{\beta}=-\tilde{\beta}_1^k$ and k is maximal. Making use of the decomposition $GL_2(\mathbb{Z})\simeq D_4 \underset{D_2}{*} D_6$ if $\tilde{\beta}$ is an element of infinite order and of matrix computations if $\tilde{\beta}$ is an element of finite order, we can prove that $C(\tilde{\beta})=\mathrm{gr}(\pm\tilde{\beta}_1)$. Let $\tilde{c}=\pm\tilde{\beta}_1^j$, $j=qk+\tau$, $0\le\tau<k$. Replacing c by $\beta^{-q}c$, we can assume that $\tilde{c}=\pm\tilde{\beta}_1^\tau$. Let x_1,x_2 be a basis of F_2, $\sigma\in A_2$, $\sigma(x_1)=x_1^{-1}$, $\sigma(x_2)=x_2^{-1}$. Then $c=\sigma^\varepsilon\beta_1^\tau f$ for some $f\in F_2$, $0\le\tau<k$, $\varepsilon\in\{0,1\}$. Thus, β and βy are conjugate if and only if, for some τ and ε, $0\le\tau<k$, $\varepsilon\in\{0,1\}$, the elements $\beta_1^{-\tau}\sigma^{-\varepsilon}\beta\sigma^\varepsilon\beta_1^\tau$ and βy are conjugate to an element of F_2. By virtue of the fact that $(\beta_1^{-\tau}\sigma^{-\varepsilon}\beta\sigma^\varepsilon\beta_1^\tau)(\beta y)^{-1}\in F_2$ it is sufficient to answer the follow-

ing question. For given $\alpha \in A_2$, $x \in F_2$ is there an $f \in F_2$ such that $f^{-1}\alpha f = \alpha x$? A necessary condition for the existence of f is the existence of $f \in F_2$ with the property

$$f^{-1}\alpha^2 f = (\alpha x)^2. \tag{6}$$

The elements α^2 and $(\alpha x)^2$ belong to A_2^+ and for them the CP is solvable in the group A_2^+. We find an element $g_0 \in A_2^+$ (if it exists) such that $g_0^{-1}\alpha^2 g_0 = (\alpha x)^2$. The element g_0 need not belong to F_2. The set of all the solutions of the equation $g^{-1}\alpha^2 g = (\alpha x)^2, g \in A_2^+$ coincides with the right coset $C(\alpha^2)g_0$ where $C(\alpha^2)$ is the centralizer of α^2 in the group A_2^+. Therefore, the solution of Eq. (6) in the group F_2 exists if and only if $\tilde{g}_0^{-1} \in \widetilde{C(\alpha^2)}$. The group $\widetilde{C(\alpha^2)}$ is a finitely generated subgroup (see Sec. 2) of the almost free group $SL_2(\mathbb{Z}) \simeq \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6$. Therefore, the problem of the occurrence in the group $\widetilde{C(\alpha^2)}$ is solvable.

We assume that $\tilde{g}_0^{-1} \in \widetilde{C(\alpha^2)}$ i.e., $\tilde{g}_0^{-1} = \tilde{c}$ for some $c \in C(\alpha^2)$. Setting $f_0 = cg_0$ we find that f_0 is a solution of Eq. (6), lying in F_2.

Thus, assume that there exists a solution f_0 of Eq. (6), lying in F_2. The remaining solutions of Eq. (6), lying in F_2, have the form hf_0, where $h \in Fix(\alpha^2)$. Therefore, all the solutions (if they exist) of the equation $f^{-1}\alpha f = \alpha x$, $f \in F_2$ have the form hf_0 for some $h \in Fix(\alpha^2)$.

Thus, the CP reduces to the question of the existence of an element $h \in Fix(\alpha^2)$ such that $f_0^{-1}h^{-1}\alpha h f_0 = \alpha x$. We set $z = f_0 x \alpha^{-1} f_0^{-1} \alpha$. Then one has to answer the following question: Is there an element $h \in Fix(\alpha^2)$ such that $h^\alpha = hz$?

If $z \notin Fix(\alpha^2)$ then solutions do not exist. Let $z \in Fix(\alpha^2)$. We find a basis of the group $Fix(\alpha^2)$ (see Sec. 2). If α is the identity automorphism on $Fix(\alpha^2)$ then a solution exists only in the case $z=1$ (then one can take, for example, $h=1$). If α is an automorphism of the second order on $Fix(\alpha^2)$ then we find a basis of the group $Fix(\alpha^2)$ suitable for α. For each element x from $Fix(\alpha^2)$ by $|x|$ we denote its length in this basis. Let $L = \{h \mid h^\alpha = hz,\ h \in Fix(\alpha^2)\}$.

We prove that if $L \neq \emptyset$ then there exists $h \in L$ such that $|h| \leq |z|$. We assume the opposite: $L \neq \emptyset$, h is a solution of minimal length, and $|h| \geq |z| + 1$. Then the irreducible notation of the element h cannot start with a basis element of the group $Fix(\alpha)$. Since in the products hz at least the first letter of the word h does not cancel, it follows that h cannot start with other basis elements. Contradiction.

Thus, it is sufficient to verify only those h whose lengths do not exceed the length of the word z in a suitable basis.

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