TREELIKE DECOMPOSABILITY OF AUTOMORPHISM GROUPS OF FREE GROUPS

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A group is said to admit a treelike decomposition if it can be constructed from its proper subgroups by the process of forming free products with amalgamation and HNN extensions. For instance,

$$SL_2(\mathbb{Z}) \simeq \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6 .$$

The groups $SL_n(\mathbb{Z})$ do not admit treelike decomposition for all $n \geq 3$ [2].

In this article, we will prove that for $n \geq 3$ the group of all automorphisms of a free group of rank 3, as well as the group of all outer automorphisms of this group, are treelike indecomposable (Theorems 1 and 2). To this end, we develop a specific theory. The theory is, however, also applicable in other situations. It will be used here to prove that, up to conjugation, the only decomposition of the group of automorphisms of the free group of rank 2, as a nontrivial free product with amalgamation, is the one that arises from the decomposition

$$GL_2(\mathbb{Z}) \simeq D_4 \underset{D_2}{*} D_6 ,$$

(Theorem 3).

1. Preliminaries

It follows from the theory of groups acting on trees [3] that the treelike decomposability of a group is equivalent to the existence of a tree on which the group acts without inversions, in such a way that the stabilizers of all the vertices are proper subgroups. If, on the other hand, every action without inversions of the group on every tree fixes some vertex, then the group does not admit a treelike decomposition.

Let X be a tree. Then every pair of subtrees of X can be joined by a unique shortest path $[A, B]$ of length $\ell(A, B)$. If G is a set of automorphisms of X then X^G will denote the set of vertices and edges of X that are fixed by all the automorphisms in G. The set X^G is obviously connected.

From now on, we will only consider groups of automorphisms of trees that act without inversions. The following Propositions 1.1-1.3 can be found in [3].

Proposition 1.1. Let α be an automorphism of a tree $X, X^{\alpha} \neq \emptyset$, and P any vertex of X. Then the geodesic $[P, P\alpha]$ is of even length, its center lies in X^{α} and the remaining vertices are not in X^{α} (cf. Fig. 1).

An automorphism α having a fixed point will be called a twist. The subtree fixed by α will be denoted by $\widehat{\alpha}$

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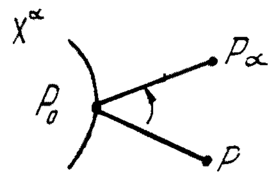

Fig. 1

A straight path is a chain, infinite in both directions, and without backtracking.

Proposition 1.2. Let α be an automorphism of a tree $X, X^\alpha = \emptyset$. We put $m = \min \ell(P, P\alpha)$, over all vertices P, and we let $V = \{P \mid \ell(P, P\alpha) = m\}$. Then:

1) The graph M generated by V is an α-invariant straight path, and α acts on M as a translation of amplitude m:

2) If a vertex $Q \notin M$ and $d = \ell(Q, M)$ then $\ell(Q, Q\alpha) = m + 2d$.

3) The α-invariant straight path α is unique (cf. Fig. 2).

An automorphism α without fixed vertices will be called a transfer. The straight path, on which α acts by translation, will be denoted by $\bar{\alpha}$.

Proposition 1.3. If $\alpha_1, \dots, \alpha_n$ are automorphisms of a tree X with the property that all their pairwise products have fixed vertices, then they have a common fixed vertex.

2. Relative Distribution of Straight Paths and Fixed Sets of Automorphisms

LEMMA 2.1. Suppose that a straight path M is invariant relative to a twist α. Then the restriction of α to M is either the identity, or else it reflects M about a unique fixed vertex in M.

LEMMA 2.2. If α is a twist (resp. a transfer), then every conjugate automorphism $\beta^{-1}\alpha\beta$ is likewise a twist (resp. transfer), and $\widehat{\beta^{-1}\alpha\beta} = \beta(\hat{\alpha})$ (resp. $\overline{\beta^{-1}\alpha\beta} = \beta(\bar{\alpha})$).

From now on, the symbol $a \rightleftarrows b$ will denote the relation $ab = ba$.

LEMMA 2.3. Let α, β, γ be twists and δ, ε transfers. Then:

1) if $\alpha \rightleftarrows \delta$ then $\bar{\delta} \subseteq \hat{\alpha}$;

2) if $\delta \rightleftarrows \beta^{-1}\gamma\beta$ and $\beta(\bar{\delta}) = \bar{\delta}$ then $\bar{\delta} \subseteq \hat{\gamma}$;

3) if $\delta \rightleftarrows \varepsilon$ then $\bar{\delta} = \bar{\varepsilon}$;

4) if $\delta \rightleftarrows \beta^{-1}\delta\beta$ then $\beta(\bar{\delta}) = \bar{\delta}$.

PROOF. 1) In light of 2.2, we have $\hat{\alpha} = \widehat{\delta^{-1}\alpha\delta} = \delta(\hat{\alpha})$; and hence $\bar{\delta} \subseteq \hat{\alpha}$, by virtue of 1.2.

2) It is obvious that $\beta^{-1}\gamma\beta$ is a twist. In light of 2.3.1), we have $\bar{\delta} \subseteq \widehat{\beta^{-1}\gamma\beta} = \beta(\hat{\gamma})$ and $\bar{\delta} = \beta^{-1}(\bar{\delta}) \subseteq \hat{\gamma}$.

3) Obviously $\bar{\delta} = \overline{\varepsilon^{-1}\delta\varepsilon} = \varepsilon(\bar{\delta})$, $\bar{\varepsilon} \subseteq \bar{\delta}$, and, similarly, $\bar{\delta} \subseteq \bar{\varepsilon}$.

4) Since $\beta^{-1}\delta\beta$ is a transfer, it follows from 2.3.3) that $\bar{\delta} = \overline{\beta^{-1}\delta\beta} = \beta(\bar{\delta})$.

LEMMA 2.4. Suppose that φ, ψ are transfers and $\varphi^{k_1}\psi^{\ell_1} \dots \varphi^{k_n}\psi^{\ell_n} = 1$, $k_i\ell_i \neq 0$, $1 \le i \le n$. Then $\bar{\varphi} \cap \bar{\psi}$ contains an edge.

Proof. Let us suppose, to begin, that $\bar{\varphi} \cap \bar{\psi} = \emptyset$. Let $[A, B]$ be the shortest path joining $\bar{\varphi}$ and $\bar{\psi}$, $A \in \bar{\varphi}$, $B \in \bar{\psi}$. Let $w_i = \varphi^{k_1}\psi^{\ell_1} \dots \varphi^{k_i}\psi^{\ell_i}$. We will prove by induction on $i = 1, \dots, n$, that the shortest path $[B, Bw_i]$ contains an edge from $\bar{\psi}$ and does not contain a vertex from $\bar{\varphi}$.

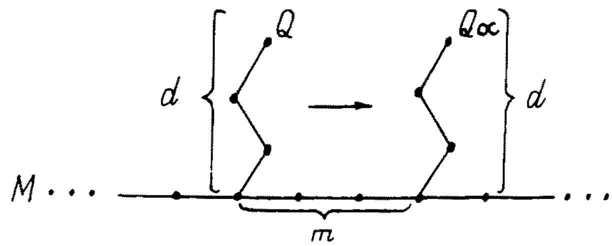

Fig. 2

In Fig. 3, this is shown to be true for $i=1$.

Let us put $C=B\omega_{i-1}$. We may assume that $[B,C]\cap\bar{\psi}=[B,D]$, where $D\neq B$ and $[B,C]\cap\bar{\varphi}=\phi$. Let $B_1=B\varphi^{k_i}$, $A_1=A\varphi^{k_i}$, $C_1=C\varphi^{k_i}$, $D_1=D\varphi^{k_i}$, $B_2=B_1\psi^{\ell_i}$, $A_2=A_1\psi^{\ell_i}$, $C_2=C_1\psi^{\ell_i}$, $D_2=D_1\psi^{\ell_i}$. We see from Fig. 4 that $[B,C_2]\cap\bar{\psi}=[B,B\psi^{\ell_i}]$, $[B,C_2]\cap\bar{\varphi}=\phi$.

By the induction assumption, the shortest path $[B,B\omega_n]$ contains an edge from ψ which contradicts $\omega_n=1$.

If the straight paths $\bar{\varphi}$ and $\bar{\psi}$ have only one common vertex, then a similar argument again leads to a contradiction.

LEMMA 2.5. Let φ be a transfer, ψ a twist and $\varphi^{k_1}\psi^{\ell_1}\dots\varphi^{k_n}\psi^{\ell_n}=1$, $k_i\ell_i\neq 0$, $1\le i\le n$, $n>1$. Then $\widehat{\psi^{\ell_i}}\cap\bar{\varphi}\neq\phi$, $\widehat{\psi^{\ell_j}}\cap\bar{\varphi}\neq\phi$ for some i,j, $i\neq j$.

Proof. First, we will show that $\widehat{\psi^{\ell_i}}\cap\bar{\varphi}\neq\phi$ for some $i<n$. Suppose that this is not so. Then $\widehat{\psi}\cap\bar{\varphi}=\phi$ and there exists a shortest path $[A,B]$ joining $\widehat{\psi}$ and $\bar{\varphi}$. $A\in\widehat{\psi}$, $B\in\bar{\varphi}$. Suppose that A_i is the vertex in $\bar{\varphi}$ closest to $\widehat{\psi^{\ell_i}}$, $i<n$. $A_i\in[A,B]\setminus\bar{\psi}$. Let $\omega_i=\varphi^{k_1}\psi^{\ell_1}\dots\varphi^{k_i}\psi^{\ell_i}$. We will prove by induction on $i=1,\dots,n-1$ that the shortest path $[B,A\omega_i]$ does not contain any edges from $\bar{\varphi}$, and that $[B,A\omega_i]\cap[B,A]=[B,A_i]$. In Fig. 5, this is shown to be true for $i=1$.

We may assume that the shortest path $[B,A\omega_{i-1}]$ does not contain any edges from $\bar{\varphi}$, and that $[B,A\omega_{i-1}]\cap[B,A]=[B,A_{i-1}]$. We see from Fig. 6 that the shortest path $[B,A\omega_i]$ contains no edges from $\bar{\varphi}$, and $[B,A\omega_i]\cap[B,A]=[B,A_i]$.

Since $\omega_{n-1}=\psi^{-\ell_n}\varphi^{-k_n}$, it follows from the induction hypothesis that $[B,A\psi^{-\ell_n}\varphi^{-k_n}]$ contains no edges from $\bar{\varphi}$, which contradicts Proposition 1.2.

So, $\widehat{\psi^{\ell_i}}\cap\bar{\varphi}\neq\phi$ for some i, $i<n$. The proof is completed by applying a similar argument to a cyclic permutation of the work $\varphi^{k_1}\psi^{\ell_1}\dots\varphi^{k_n}\psi^{\ell_n}$.

LEMMA 2.6. Let φ,ψ be twists and $\omega=\psi^{k_1}\varphi^{\ell_1}\dots\psi^{k_n}\varphi^{\ell_n}=1$, $k_i\ell_i\neq 0$, $1\le i\le n$, $n\ge 2$. Then the cyclic word ω contains a pair of adjacent words with a common fixed vertex. It contains, as well, another such pair that does not meet the first pair.

Proof. We will first show that the word $\psi^{k_1}\varphi^{\ell_1}\dots\psi^{k_n}$ contains a pair of adjacent words with a common fixed vertex. Suppose that this is not so. Then $\widehat{\varphi}\cap\widehat{\psi}=\phi$, and there is a shortest path $[A,B]$ joining $\widehat{\varphi}$ and $\widehat{\psi}$, $A\in\widehat{\varphi}$, $B\in\widehat{\psi}$. Let A_i be a vertex of B closest to $\widehat{\varphi^{\ell_i}}$, and B_i a vertex of A closest to $\widehat{\psi^{k_i}}$. We have $A_i\in[A,B]$, $A_i\neq B$ for $1\le i\le n-1$; $B_i\in[A,B]$, $B_i\neq A$, for $1\le i\le n$. We put $a_i=\ell(A_i,A)$, $b_i=\ell(B_i,A)$, $\omega_i=\psi^{k_1}\varphi^{\ell_1}\dots\psi^{k_i}\varphi^{\ell_i}$. Then (cf. Fig. 7)

$$a_i<b_i,\ 1\le i\le n-1;\ a_i<b_{i+1},\ 1\le i\le n-1;\ b_i>0,\ 1\le i\le n. \tag{1}$$

We will prove the following formulas by induction on $i=1,\dots,n-1$:

$$\ell(A,A\omega_i)>0, \tag{2}$$

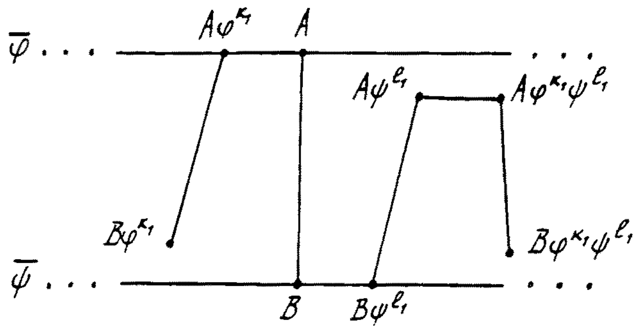

Fig. 3

$$[A,B] \cap [A, A\omega_i] = [A, A_i]. \tag{3}$$

Figure 8 shows that these formulas are true for $i=1$.

Suppose that formulas (2) and (3) are true for some i, $1 \le i \le n-2$. In view of Fig. 9 and the formulas (1)-(3),

$$\ell(A, A\omega_{i+1}) = \ell(A, A\omega_i \psi^{k_{i+1}}) = \ell(A, A\omega_i) + 2(b_{i+1} - a_i) > 0,$$

$[A,B] \cap [A, A\omega_{i+1}] = [A, A_{i+1}]$. By the induction assumption, we get $\ell(A, A\omega_{n-1}) > 0$ and

$$[A,B] \cap [A, A\omega_{n-1}] = [A, A_{n-1}].$$

On the other hand $\omega_{n-1} = \varphi^{-\ell_n}\psi^{-k_n}$. $[A,B] \cap [A, A\omega_{n-1}] = [A,B] \cap [A, A\varphi^{-\ell_n}\psi^{-k_n}] = [A,B] \cap [A, A\psi^{-k_n}] = [A, B_n]$.

From this follows $B_n = A_{n-1}$; a contradiction.

So, the word $\psi^{k_1}\varphi^{\ell_1}\dots\psi^{k_n}$ contains a pair of words with a common fixed vertex. The proof is completed by applying a similar argument to cyclic permutations of the word ω and their inverses.

LEMMA 2.7. Let φ, ψ, ξ_i be twists, $\hat{\varphi} \cap \hat{\xi}_i \ne \emptyset$, $\hat{\psi} \cap \hat{\xi}_i \ne \emptyset$, $1 \le i \le 2n$. Suppose that $\psi^{k_1}\xi_1\varphi^{\ell_1}\xi_2\dots\psi^{k_n}\xi_{2n-1}\varphi^{\ell_n}\xi_{2n} = 1$, $k_i\ell_i \ne 0$, $1 \le i \le n$. Then the statement of Lemma 2.6 remains true for the word $\omega = \psi^{k_1}\varphi^{\ell_1}\dots\psi^{k_n}\varphi^{\ell_n}$, even though in general $\omega \ne 1$.

Proof. The lemma is true if $\hat{\varphi} \cap \hat{\psi} \ne \emptyset$.

Suppose that $\hat{\varphi} \cap \hat{\psi} = \emptyset$ with $[A,B]$ the shortest path joining $\hat{\varphi}$ and $\hat{\psi}$, $A \in \hat{\varphi}$, $B \in \hat{\psi}$. Since $\hat{\xi}_i$ is connected, it follows that $[A,B] \subseteq \hat{\xi}_i$, $1 \le i \le 2n$. The rest of the proof is similar to the arguments used in the proof of Lemma 2.6.

3. Treelike Indecomposibility of the Groups A_n and SA_n for $n \ge 3$

THEOREM 1. The group A_n of automorphisms of the group of rank $n \ge 3$ does not admit a treelike decomposition.

Proof. We will use some of the defining relations of A_n, for $n \ge 3$ (cf. [1, p. 174]):

$$P^2 = \sigma^2 = Q^n = 1, \tag{4}$$

$$(P\sigma)^4 = (QP)^{n-1} = 1, \tag{5}$$

$$(P\sigma PU)^2 = 1, \tag{6}$$

$$\sigma \rightleftarrows Q^{-1}\sigma Q, \tag{7}$$

$$U^{-1}PUP\sigma U\sigma P\sigma = 1, \tag{8}$$

$$U \rightleftarrows Q^{-2}PQ^2, \quad U \rightleftarrows Q^{-2}\sigma Q^2, \tag{9}$$

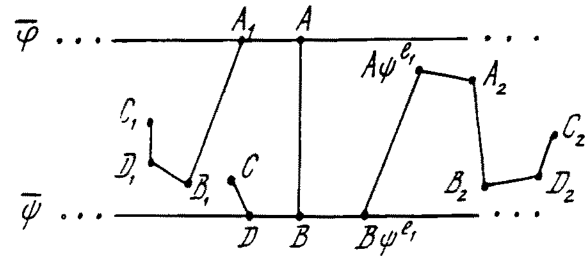

Fig. 4

$$U \doteq \sigma U \sigma, \tag{10}$$

$$U \doteq PQ^{-1}\sigma U \sigma Q P, \tag{11}$$

$$U \doteq Q^{-2} U Q^{2}. \tag{12}$$

Suppose that A_n acts on some tree.

It follows from (4) that P, σ, Q are twists. Suppose that U is a transfer. In light of (12) and Lemma 2.3, $\overline{U}$ is a Q^2 -invariant set. It follows from Lemma 2.3 and (9) that $\overline{U} \subseteq \widehat{P}$, $\overline{U} \subseteq \widehat{\sigma}$. When we now take an arbitrary vertex in the straight path $\overline{U}$ and apply both sides of the relation (6) to it, we obtain a contradiction, which proves that U is a twist.

In light of (7), (5), (10) and Lemma 2.6 we have

$$\widehat{Q} \cap \widehat{\sigma} \neq \phi, \ \widehat{Q} \cap \widehat{P} \neq \phi, \ \widehat{P} \cap \widehat{\sigma} \neq \phi, \ \widehat{U} \cap \widehat{\sigma} \neq \phi.$$

Then, by (8) and Lemma 2.7, $\widehat{U} \cap \widehat{P} \neq \phi$. From this, together with (11) and Lemma 2.7, we get $\widehat{U} \cap \widehat{Q} \neq \phi$.

So, the generators P, σ, Q , U and their pairwise products have fixed vertices; hence, the group A_n admits a fixed vertex for $n \geqslant 3$.

THEOREM 2. The group SA_n of outer automorphisms of the free group of rank $n \geqslant 3$ does not admit a treelike decomposition.

Proof . We will first prove this for odd numbers n . We will use the Reidermeister–Schreier process to choose generators and defining relations for the group SA_n . Obviously $Q, U \in SA_n$, $P, \sigma \notin SA_n$. We choose A_n as a Schreier representative for SA_n relative to σ . The subbroup SA_n is then generated by the set

$$\alpha_1 = \gamma(1, P) = P\overline{P}^{-1} = P\sigma, \ \alpha_2 = \gamma(1, Q) = Q\overline{Q}^{-1} = Q, \ \alpha_3 = \gamma(1, \sigma) = \sigma\overline{\sigma}^{-1} = 1,$$
$$\alpha_4 = \gamma(1, U) = U\overline{U}^{-1} = U, \ \alpha_5 = \gamma(\sigma, P) = \sigma P \overline{\sigma P}^{-1} = \sigma P, \ \alpha_6 = \gamma(\sigma, Q) = \sigma Q \overline{\sigma Q}^{-1} = \sigma Q \sigma,$$
$$\alpha_7 = \gamma(\sigma, \sigma) = \sigma^2 \overline{\sigma^2}^{-1} = 1, \ \alpha_8 = \gamma(\sigma, U) = \sigma U \overline{\sigma U}^{-1} = \sigma U \sigma.$$

Indeed, $SA_n = \mathrm{gr}\,(\alpha_1, \alpha_6, \alpha_8)$ since

$$\alpha_2 = \alpha_6 \alpha_1^{-2}, \ \alpha_4 = \alpha_1^{2} \alpha_8^{-1} \alpha_1^{-2}, \ \alpha_5 = \alpha_1^{-1}.$$

We have

$$\alpha_1^{4} = \alpha_6^{n} = 1, \tag{13}$$

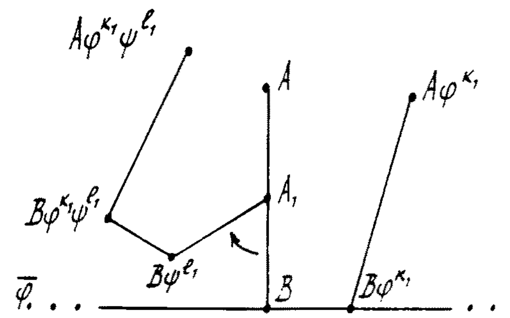

Fig. 5

$$(\alpha_1\alpha_6^{-1})^{n-1} = (\alpha_1\alpha_8^{-1})^3 = 1, \tag{14}$$

$$[\alpha_1^2\alpha_8\alpha_1^{-2}, \alpha_8^{-1}] = 1, \tag{15}$$

$$[\alpha_6\alpha_1^{-1}\alpha_8^{-1}\alpha_1\alpha_6^{-1}, \alpha_8] = 1, \tag{16}$$

$$\alpha_8 \cdot \alpha_1^{-1}\alpha_6^{-1}\alpha_8\alpha_6\alpha_1 = \alpha_6^{-1}\alpha_1^{2}\alpha_8^{-1}\alpha_1^{-2}\alpha_6 \cdot \alpha_8 \cdot \alpha_6^{-1}\alpha_1^{2}\alpha_8\alpha_1^{-2}\alpha_6 . \tag{17}$$

Suppose SA_n acts on a tree. By virtue of (13), α_1 and α_6 are twists. We will show that α_8 is also a twist. Suppose that, on the contrary, α_8 is a transfer. Then $\alpha_1^2\alpha_8\alpha_1^{-2}$ is likewise a transfer and $\overline{\alpha_1^2\alpha_8\alpha_1^{-2}} = \overline{\alpha}_8\alpha_1^{-2}$. By Lemma 2.3.3) and (15), $\overline{\alpha}_8\alpha_1^{2} = \overline{\alpha}_8$. From Lemma 2.1 follows that there are two possible cases:

1) there esists a vertex A such that $\widehat{\alpha_1^2} \cap \overline{\alpha}_8 = \{A\}$ and α_1^2 reflects $\overline{\alpha}_8$ about A ;

2) $\overline{\alpha}_8 \subseteq \alpha_1^2$.

Suppose that the first case applied. It follows from (14) and Lemma 2.6 that $\alpha_1\alpha_8^{-1}$ is a twist, and $\widehat{\alpha_1} \cap \overline{\alpha}_8 \neq \phi$. Let B be a vertex lying in $\widehat{\alpha_1} \cap \overline{\alpha}_8$. Then $B \in \widehat{\alpha_1^2} \cap \overline{\alpha}_8$. It follows from this that $B = A$ and $\widehat{\alpha_1} \cap \overline{\alpha}_8 = \{A\}$. We put $C = A\alpha_8^{-1}$. The $A\alpha_1\alpha_8^{-1} = A\alpha_8^{-1} = C$. Since $\alpha_1\alpha_8^{-1}$ is a twist, there exists a vertex O in the center of the shortest path $[A, C]$ with

$$O\alpha_1 = O\alpha_8 . \tag{18}$$

We put $D = O\alpha_8$ (cf. Fig. 10).

The vertices D and O are symmetrically positioned relative to A , whence $D = O\alpha_1^2$. It follows from (13) and (18) that $O\alpha_1^2 = D = O\alpha_8 = O\alpha_1$. Hence $O\alpha_1 = O$ i.e., $O \in \widehat{\alpha}_1$, $O \in \widehat{\alpha}_1 \cap \overline{\alpha}_8$. This contradicts $\{A\} = \widehat{\alpha}_1 \cap \overline{\alpha}_8$.

Let us now consider the second case: $\overline{\alpha}_8 \subseteq \widehat{\alpha_1^2}$. In light of (14), $\alpha_1\alpha_6^{-1}$ is a twist. From (16) and Lemma 2.5 we deduce that there exists a vertex K such that $K \in \widehat{\alpha_1\alpha_6^{-1}} \cap \overline{\alpha}_8$. Again, there are two possible cases (cf. Lemmas 2.3.3) and 2.1):

1) $\alpha_1\alpha_6^{-1}$ reflects $\overline{\alpha}_8$ about K ;

2) $\overline{\alpha}_8 \subseteq \widehat{\alpha_1\alpha_6^{-1}}$.

Since $\overline{\alpha}_8 \subseteq \widehat{\alpha_1^2}$, these two cases are equivalent to the following two:

1) $\alpha_1^{-1}\alpha_6^{-1}$ reflects $\overline{\alpha}_8$ about K ;

2) $\overline{\alpha}_8 \subseteq \widehat{\alpha_1^{-1}\alpha_6^{-1}}$

We denote the right side of (17) by β . The elements β and α_8^{-1} are conjugate, hence β is a transfer, and its amplitude along the straight path $\overline{\beta}$ is equal to

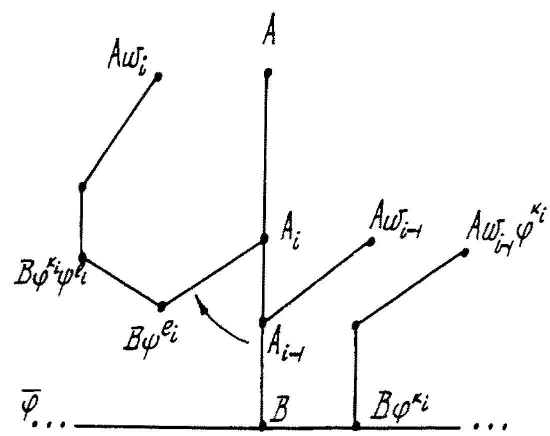

Fig. 6

$c=\ell(K,K\alpha_8)$, the amplitude of the transfer α_8 along $\overline{\alpha}_8$. We put $N=K\alpha_8^{-1}$, $N_1 = K\alpha_8$ (cf. Fig. 11).

In the first case, we have

$$N\beta = N\alpha_8\cdot\alpha_1^{-1}\alpha_6^{-1}\alpha_8\alpha_6\alpha_1 = K\alpha_1^{-1}\alpha_6^{-1}\alpha_8\alpha_6\alpha_1 = K\alpha_8\alpha_6\alpha_1 = N_1\alpha_6\alpha_1 = N,$$

i.e., β has a fixed vertex. This is a contradiction.

In the second case, $\overline{\alpha}_8 \subseteq \widehat{\alpha_1^{-1}\alpha_6^{-1}}$ and hence $\beta=\alpha_8\cdot\alpha_1^{-1}\alpha_6^{-1}\alpha_8\alpha_6\alpha_1$ shifts the straight path $\overline{\alpha}_8$ a distance $2c$. Hence, $\overline{\beta}=\overline{\alpha}_8$ and β shifts the straight path $\overline{\beta}$ a distance $2c$. On the other hand, the element β is a conjugate of α_8^{-1} and hence must shift the straight path $\overline{\beta}$ a distance c. Again a contradiction.

So, α_8 is a twist. It follows from (15) and Lemma 2.6 that $\widehat{\alpha}_1\cap\widehat{\alpha}_8\neq\phi$. From (16) and Lemma 2.7 we deduce that $\widehat{\alpha}_6\cap\widehat{\alpha}_8\neq\phi$.

Thus, for odd $n\geqslant 3$ the group SA_n does not admit a treelike decomposition. We will now prove the treelike indecomposability of SA_n for $n\geqslant 4$.

Let $x_1,\ldots,x_n$ be a basis for the free group F_n and U_{ij} the automorphism of F_n satisfying $U_{ij}(x_i)=x_ix_j$, $U_{ij}(x_\kappa)=x_\kappa$, $\kappa\neq i$. Then $SA_n = \mathrm{gr}\,(U_{ij})$, $1\leqslant i,j\leqslant n$, $i\neq j$. Suppose that SA_n acts on a tree Γ. Let i,j,κ be three distinct integers, $1\leqslant i,j,\kappa\leqslant n$. Put $G_{ij\kappa} = \mathrm{gr}\,(U_{pq})$, where $\{p,q\}\subseteq\{i,j,\kappa\}$, $p\neq q$. Clearly, $G_{ij\kappa}\simeq SA_3$ and $G_{ij\kappa}$ being a subgroup of SA_n, also acts on the tree Γ. It follows from what we proved above that $G_{ij\kappa}$ admits a fixed vertex. The generators U_{ij} are twists, since they lie in the appropriate groups $G_{ij\kappa}$.

We proceed to prove that the generators $U_{\ell m}$ and U_{pq} have a common fixed vertex. If ℓ,m,p,q are distinct integers, then

$$U_{\ell m}\rightleftarrows U_{pq} \text{ and } \widehat{U}_{\ell m}\cap\widehat{U}_{pq}\neq\phi,$$

by virtue of Lemma 2.6.

If some among ℓ,m,p,q are equal, then $U_{\ell m}$ and U_{pq} fall in some suitable group $G_{ij\kappa}$. This completes the proof of the theorem.

4. Uniqueness of the Decomposition of the Group A_2

It is known that

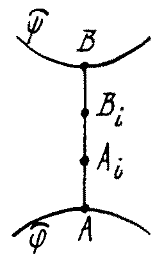

Fig. 7

$$SL_2(\mathbb{Z}) \simeq \mathbb{Z}_4 \underset{\mathbb{Z}_2}{*} \mathbb{Z}_6,$$

where the cyclic groups of order 4, 6, and 2 are generated by the matrices

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

respectively.

The decomposition

$$GL_2(\mathbb{Z}) \simeq D_4 \underset{D_2}{*} D_6$$

arises from the decomposition for $SL_2(\mathbb{Z})$ by adjoining to each factor the dihedral involution

$$D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let F_2 be the free group with basis x_1, x_2, let A_2 be the group of all automorphisms and I_2 the group of inner automorphisms of F_2.

The group A_2 is generated by the automorphisms U, P, σ where

$$U: \begin{cases} x_1 \to x_1 x_2^{-1} \\ x_2 \to x_2 \end{cases}, \quad P: \begin{cases} x_1 \to x_2 \\ x_2 \to x_1 \end{cases}, \quad \sigma: \begin{cases} x_1 \to x_1^{-1} \\ x_2 \to x_2 \end{cases},$$

and are linked by the defining relations $P^2 = \sigma^2 = (\sigma P)^4 = (UP\sigma P)^2 = (\sigma PU)^3 = 1, U^{-1} = \sigma U \sigma$ (cf. [1, p. 179]), bearing in mind that U is the inverse of the element in the reference). The group I_2 is generated by the automorphisms $U\sigma U\sigma, PU\sigma U\sigma P$.

The natural homomorphism, with kernel I_2, of A_2 onto $GL_2(\mathbb{Z})$ maps the elements $P\sigma, \sigma PU^{-1}, P\sigma P\sigma, P$ to the matrices A, B, C, D respectively.

One obtains the decomposition of A_2 as a free product with amalgamation by taking the inverse images of the factors in the decomposition of $GL_2(\mathbb{Z})$. So, $A_2 = G_1 \underset{G_3}{*} G_2$, where

$$G_1 = \mathrm{gr}\,(P\sigma, P, I_2) = \mathrm{gr}\,(\sigma, P, U\sigma U\sigma),$$

$$G_2 = \mathrm{gr}\,(\sigma PU^{-1}, P, I_2) = \mathrm{gr}\,(UP\sigma, P, U\sigma U\sigma), \tag{19}$$

$$G_3 = \mathrm{gr}\,(P\sigma P\sigma, P, I_2) = \mathrm{gr}\,(P\sigma P\sigma, P, U\sigma U\sigma).$$

<u>THEOREM 3.</u> The group A_2 is decomposable as a free product with amalgamation in a way that is unique up to conjugation.

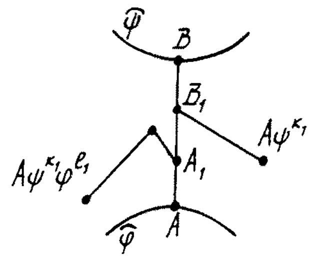

Fig. 8

Proof. Let

$$A_2 = H_1 \underset{H_3}{*} H_2 .$$

We will prove that this decomposition is conjugate to

$$A_2 = G_1 \underset{G_3}{*} G_2 .$$

Let Γ be the tree associated with

$$H_1 \underset{H_3}{*} H_2 ,$$

i.e., the tree on which

$$H_1 \underset{H_3}{*} H_2$$

acts without inversions, such that the fundamental domain is an edge, with H_1 and H_2 being the stabilizers of the ends of the edge, and H_3 is the stabilizer of the edge itself. Since $P^2 = \sigma^2 = (P\sigma)^4 = 1$, it follows that P and σ are twists and $\widehat{P} \cap \widehat{\sigma} \neq \phi$. Suppose that U is a twist. Since $\sigma U \sigma U = U \sigma U \sigma$, it follows in light of Lemma 2.6 that $\widehat{U} \cap \widehat{\sigma} \neq \phi$. From Lemma 2.7 and the relations $(\sigma P U)^3 = 1, \widehat{P} \cap \widehat{\sigma} \neq \phi, \widehat{U} \cap \widehat{\sigma} \neq \phi$ we deduce that $\widehat{U} \cap \widehat{P} \neq \phi$. From Proposition 1.3 we see that the group A_2 has a fixed vertex, which is untrue. So, U is a transfer.

Since $U \rightleftarrows \sigma U \sigma$, it follows that $\overline{U}$ is a σ-invariant straight path, and there are two possible case:

1) there exists a vertex O such that $\widehat{\sigma} \cap \overline{U} = \{O\}$ and σ reflects the straight path $\overline{U}$ about the vertex O ;

2) $\overline{U} \subseteq \widehat{\sigma}$.

We will show that the second case is impossible. Suppose on the contrary that $\overline{U} \subseteq \widehat{\sigma}$. We will show that in this case $\widehat{P} \cap \overline{U} \neq \phi$. Suppose that this is not so: $\widehat{P} \cap \overline{U} = \phi$ and let $[C, D]$ be the shortest path joining $\widehat{P}$ and $\overline{U}$ (Fig. 12). We put $E = CU$.

Since $\widehat{\sigma}$ is a connected set and $\widehat{P} \cap \widehat{\sigma} \neq \phi, \overline{U} \cap \widehat{\sigma} \neq \phi$, it follows that $[C, D] \subseteq \widehat{\sigma}$. We have $C\sigma PU = CPU = CU = E$. Since $(\sigma PU)^3 = 1$, we see that σPU is a twist. Therefore, there exists a vertex N in the middle of $[C, E]$; moreover, $N\sigma PU = N$. We put $M = NU^{-1}$. Since $N \in \widehat{\sigma}$, we obtain $NPU = N, NP = NU^{-1} = M$. Next, P is a twist, and the vertex D is in the middle of $[N, M]$; hence, $D \in \widehat{P}$. This is a contradiction. So, in the case $\overline{U} \subseteq \widehat{\sigma}$ we have $\widehat{P} \cap \overline{U} \neq \phi$. Let A be vertex in $\widehat{P} \cap \overline{U}$, $B = AU$ (Fig. 13).

Let us apply the automorphism of σPU to the vertex $A : A\sigma PU = APU = AU = B$. Since σPU is a twist, it follows that there is a vertex C in the center of the path $[A, B]$;

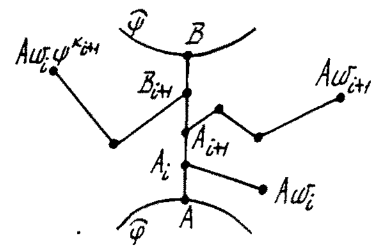

Fig. 9

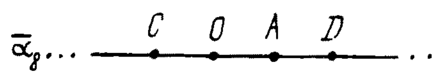

Fig. 10

moreover, $C\sigma PU = C$. It follows from this that $CPU = C$, $CP = CU^{-1} = D$. Since P is a twist, the center of $[C,D]$ must lie in $\widehat{P}$, i.e., $A \in \widehat{P}$, and the other vertices of $[C,D]$ are not in $\widehat{P}$. Hence, $\bar{U} \cap \widehat{P} = \{A\}$. We now apply the automorphism $(UP\sigma P)^2$ to the vertex D:

$$D = DUP\sigma PUP\sigma P = CP\sigma PUP\sigma P = D\sigma PUP\sigma P = DPUP\sigma P = CUP\sigma P = EP\sigma P.$$

On the other hand, $D = CP = C\sigma P = DP\sigma P$, whence $D = E$; a contradiction.

So the assumption $\bar{U} \subseteq \widehat{\sigma}$ must have been false, and it is the first case that applies. We proceed to prove that in this case $\widehat{P} \cap \bar{U} \neq \phi$. Let us suppose that on the contrary that $\widehat{P} \cap \bar{U} = \phi$ and let $[C,D]$ be the shortest path joining $\widehat{P}$ to $\bar{U}$. Since $\widehat{P} \cap \widehat{\sigma} \neq \phi$, $\bar{U} \cap \widehat{\sigma} \neq \phi$ and $\widehat{\sigma}$ is a connected set, $[C,D] \subseteq \widehat{\sigma}$. But $\bar{U} \cap \widehat{\sigma} = \{O\}$; hence $D = O$ (cf. Fig. 14). So we get $E = CU$.

Furthermore, $C\sigma PU = CPU = CU = E$. Since σPU is a twist, there is a vertex N in the middle of $[C,E]$. Moreover, $N\sigma PU = N$. From this follows that $N\sigma P = NU^{-1} = M$. But, $N\sigma = M$ whence $M \in \widehat{P}$; a contradiction.

So, $\widehat{P} \cap \bar{U} \neq \phi$. We will show that the connected set $\widehat{P} \cap \bar{U}$ is a finite segment, symmetric relative to the vertex O. We put $A = OU^{-1}$ (Fig. 15). We have $AUP\sigma P = OP\sigma P = O$; and $AUP\sigma = OP\sigma = O$ and $UP\sigma P$, $UP\sigma$ are twists. Hence, there exists a vertex B in the middle of $[A,O]$. Moreover $B \in \widehat{UP\sigma P}$, $B \in \widehat{UP\sigma}$. Thus, $B \in \widehat{P}$. We put $C = BU$.

Let us apply the automorphism $UP\sigma$ to the vertex B: $B = BUP\sigma = CP\sigma$. But, $C = B\sigma$ hence $C \in \widehat{P}$. Since $\widehat{P}$ is connected, we have $[B,C] \subseteq \widehat{P}$. We will prove that $\widehat{P} \cap \bar{U} = [B,C]$. Suppose that this is not so. Then one of the following two cases must hold.

1) there exists a vertex B_1 such that $B_1 \in \widehat{P} \cap \bar{U}$, $\ell(B,B_1) = 1$ and $B \in [B_1, O]$;

2) there exists a vertex C_1 such that $C_1 \in \widehat{P} \cap \bar{U}$, $\ell(C,C_1) = 1$ and $C \in [O, C_1]$.

Suppose that the first case applies (Fig. 16). Put $B_2 = B_1 U$, $B_3 = B_2\sigma$, $B_4 = B_3 U$. Clearly $B_2, B_3 \in [B,C]$, $B_1 \neq B_3$.

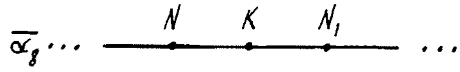

Fig. 11

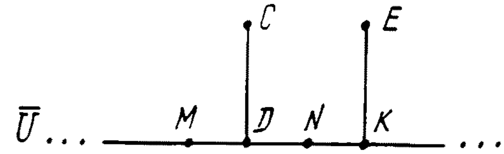

Fig. 12

Fig. 13

Then

$$B_4 = B_4(\sigma PU)^3 = B_1PU(\sigma PU)^2 = B_1U(\sigma PU)^2 = B_2(\sigma PU)^2 = B_3PU\sigma PU =$$

$$= B_3U\sigma PU = B_4\sigma PU = B_2$$

a contradiction.

A similar reasoning, applied to the second case, also leads to a contradiction. So, $\widehat{P}\cap U = [B,C]$ where $C=BU$ and O is the center of $[B,C]$. Put $B_{00}=B$, $A_{00}=C$, $A_{0,i+1}=A_{0i}U$, $B_{0,i+1}=B_{0i}U^{-1}$, $B_{i0}=B_{0i}P$, $A_{i0}=A_{i0}P$ where $i\geq 0$. Since $\widehat{P}\cap\overline{U}=[B_{00},A_{00}]$ it follows that P maps the ray $[B_{00},B_{01})$ to the ray $[B_{00},B_{10})$ and the ray $[A_{00},A_{01})$ to the ray $[A_{00},A_{10})$ (Fig. 17).

By successively applying the left and right sides of the relation $UP\sigma PUP=P\sigma$ to the vertices $OU^{-1}, OU^{-2},\ldots$ and using the relation $P^2=\sigma^2=1$ we deduce that σ relects the centers of the paths $[B_{i0},B_{i+1,0}]$ in the centers of the paths $[A_{i0},A_{i+1,0}]$, $i\geq 0$. It follows from this that $B_{i0}\sigma = A_{i0}$, $i\geq 0$.

So, we have set up a part of the tree Γ, and endowed it with a partial action of σ, P, U. Any segment can be taken as a fundamental domain, seeing that the groups acts transitively on the edges of Γ. If another segment is taken as fundamental domain, then the corresponding decomposition will be conjugate to the first by the element which maps the first segment to the second.

So, $[O,R]$ can be taken as fundamental domain, where $[O,R]\subseteq[O,B_{00}]$. However, we do not know yet that $R=B_{00}$; we therefore suppose that we have a strict inclusion $[O,R]\subset[O,B_{00}]$. We will show that this is impossible. Indeed, the vertex B_{00} must be equivalent to either O or R; i.e., there exists an element $g\in A_2$ such that $B_{00}g$ is equal to O or R. Suppose that $\ell(B_{00},A_{00})=a$. One can prove that $\ell(B_{00},B_{00}g)$ is a

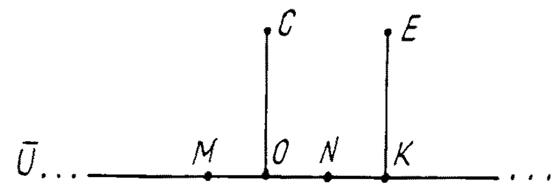

Fig. 14

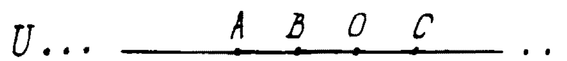

Fig. 15

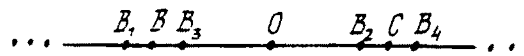

Fig. 16

multiple of a for every g by using induction on the length of g as a word in the letters U, P, σ, and using the reconstructed tree below. But $\ell(B_{00},O)<a$, $\ell(B_{00},R)<a$, hence B_{00} is neither equivalent to O nor to R. A contradiction.

So, $R=B_{00}$ and $[O,B_{00}]$ is a fundamental segment. Suppose that H_1 and H_2 stabilize the vertices O and B_{00} respectively, and H_3 stabilizes the segment $[O,B_{00}]$. It follows from the presented generators for the groups G_1, G_2, G_3 that $G_1\leq H_1$, $G_2\leq H_2$, $G_3 \leq H_3$, $|G_1:G_3|=2$, $G_3\leq G_1\cap H_3$ and σ is a representative for G_1 modulo G_3, $\sigma\notin H_3$; hence

$$G_1 \cap H_3 = G_3, \tag{20}$$

$|G_2:G_3|=3$, $G_3\leq G_2\cap H_3$, and the representatives of G_2 modulo G_3, $(\sigma P U^{-1})^i \notin H_3, i=1,2$. Thus

$$G_2 \cap H_3 = G_3. \tag{21}$$

Similarly,

$$G_2 \cap H_1 = G_3, \quad G_1 \cap H_2 = G_3. \tag{22}$$

Suppose that one of the factors, say H_1 contains G_1 a proper subgroup. We take an element $h\in H_1\setminus G_1$. Let $h=gg_1\dots g_{n-1}g_n$ where $g\in G_3$, $g_i\in G_1\setminus G_3$ or $g_i\in G_2\setminus G_3$ and the successive factors g_i, g_{i+1} lie in different summands, G_1 or G_2. Since $h\in H_1\setminus G_1$ and $H_1\cap G_2=G_3$ we have $n\geq 2$. It follows from formulas (20)-(22) that $g\in H_3$, $g_i\in H_1\setminus H_3$ or $g_i\in H_2\setminus H_3$ and the successive factors g_i, g_{i+1} lie in different summands H_1 and H_2. This contradicts the expression in normal form for elements of a product with amalgamation. So,

$$H_1=G_1,\ H_2=G_2,\ H_3=H_1\cap H_2=G_1\cap G_2=G_3.$$

This completes the proof of the theorem.

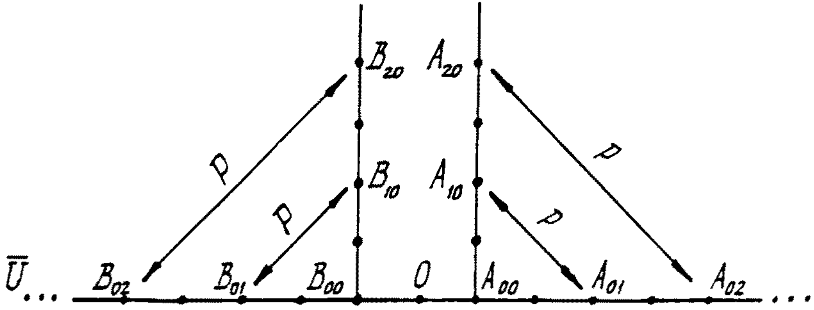

Fig. 17

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