Homotopical categories of logics

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Abstract. Categories of logics and translations usually come with a natural notion of when a translation is an equivalence. The datum of a category with a distinguished class of weak equivalences places one into the realm of abstract homotopy theory where notions like homotopy (co)limits and derived functors become available. We analyze some of these notions for categories of logics. We show that, while logics and flexible translations form a badly behaved category with only few (co)limits, they form a well behaved homotopical category which has all homotopy (co)limits. We then outline several natural questions and directions for further research suggested by a homotopy theoretical viewpoint on categories of logics.

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1. Introduction

In his opening lecture at Unilog 2010 Jean-Yves Béziau named the following as the main questions of Universal Logic:

1. What is a logic?
2. What is a translation between logics?
3. When are two logics equivalent?
4. How to combine logics?

Lots of different answers to these questions have been proposed over time, with a recent increase of activity spurred by the contests of the Unilog conference series.

The consideration of categories of logics is a way of evaluating and comparing such answers. First observe that answering questions 1 and 2 usually results in a category whose objects are logics and whose morphisms are translations. One then gets tentative answers to questions 3 – two logics might be called equivalent if they are isomorphic in that category – and 4 – a combination of logics may be seen as the formation of a colimit in this category, following [47]. However, these answers to questions 3. and 4. are rarely satisfying.

To see this, let us place ourselves in the setting of Hilbert systems, i.e. formal languages generated by some primitive connectives and variables and endowed with a consequence relation. A *strict translation* is a map of the formal languages sending generating connectives to generating connectives and preserving consequence, while a *flexible translation* may send generating connectives to more complex formulas (both are required to map n-ary connectives to n-ary connectives).

Now for question 3 consider the two presentations of classical propositional logic $CPL_1 := \langle \wedge, \neg \mid \text{rules} \ldots \rangle$ and $CPL_2 := \langle \wedge, \neg, \vee, \to \mid \text{rules} \ldots \rangle$. Clearly one would say that both are presentations of "the same" logic, since the connectives $\vee$ and $\to$ appearing additionally in the second logic, are expressible, up to logical equivalence, by compositions of $\wedge$ and $\neg$, and do not need to be present as primitive symbols. Indeed, the inclusion of formal languages $CPL_1 \to CPL_2$ has the property that it is conservative, i.e. inferences hold in the target logic if and only if they hold in the domain logic, and that any formula of the target logic is logically equivalent to one in the image. We will call a translation with these properties a *weak equivalence*.

But there can be no isomorphism between these logics. There is no strict translation at all from CPL_2 to CPL_1, since it would have to map the binary connectives $\vee$ and $\to$ to the only binary connective $\wedge$ of CPL_1, but such a map cannot preserve consequence, since the connectives satisfy different rules. We do have a flexible translation from CPL_2 to CPL_1 which maps, for example, $- \vee -$ to the derived formula $\neg(\neg(-) \wedge \neg(-))$. But this cannot be part of an isomorphism since going back via the inclusion results in the map $CPL_2 \to CPL_2$ which sends $- \vee -$ to $\neg(\neg(-) \wedge \neg(-))$ – the formulas are logically equivalent, but not equal, and an the composition of an isomorphism with its inverse has to give the identity.

Thus we found that a sensible notion of equivalence is an extra notion, and does not emerge from the categorical structure.

For question 4 about combining logics, the answer that the combination of logics should be a colimit is often a good one where it applies, but this is only the case for a restricted class of diagrams of logics. Essentially, only colimits of diagrams of strict morphisms exist and behave well[1]. This includes a lot of cases from practice, but it would be even nicer to be able to combine logics along flexible morphisms.

Consider for example a modal extension of classical propositional logic, presented by $L := \langle \wedge, \neg, \vee, \to, \Box, \Diamond \mid \text{rules} \ldots \rangle$. It receives an inclusion of classical propositional logic $CPL := \langle \wedge, \neg, \vee, \to \mid \text{rules} \ldots \rangle$. Now we might be interested in what happens if we make the underlying propositional logic of L intuitionistic by removing the law of excluded middle from its rules. For example we might ask whether properties like algebraizability or the validity of a metatheorem of deduction will still hold and how to construct a semantics for the new logic. Such questions have been amply adressed in the theory of fibring of logics, so we could try to express our "intuitionistified" logic L^{int} as a

[1] see Example 2.8 for a colimit of flexible morphisms which does exist, but does not behave right.

fibring, i.e. a pushout, of logics:

$$\begin{array}{ccc} CPL & \hookrightarrow & L \\ {\scriptstyle \neg\neg}\downarrow & & \downarrow \\ IPL & \longrightarrow & L^{int} \end{array}$$

The idea is that we embed the classical propositional sublogic of L into intuitionistic logic IPL along the double negation translation (the left vertical arrow), and glue the extra, not doubly negated, layer that intuitionistic logic has in comparison to CPL, to the modal logic L while maintaining the place that was formerly occupied by CPL.

The problem is that this colimit does not exist in the category of Hilbert systems and flexible translations, and indeed very few colimits exist there.

The conclusion that we draw from these observations is that it is better to regard questions 1, 2 and 3 as fundamental questions: We should first ask for notions of logic, translation and weak equivalence. Then we have a category with an additional structure, a distinguished class of morphisms given by the weak equivalences. As harmless as it looks, this has vast implications: Such a pair consisting of a category and a class of morphisms, also called a relative category, is all one needs, to do an abstract form of homotopy theory.

The usual categorical notions and constructions can now be accompanied with their "derived" versions. For example there is the notion of homotopy colimit: Usual colimits do not need to preserve weak equivalences, i.e. given two weakly equivalent diagrams, their colimits need not be weakly equivalent. A homotopy colimit can roughly be thought of as the best approximation of a colimit construction which preserves weak equivalences. One could argue that, if one devises a logically meaningful construction of a new logic from some given other logics, then one would like equivalent inputs to lead to equivalent outputs and that thus the derived notions are the better ones. Maybe more importantly, homotopy (co)limits can exist where (co)limits do not exist. Indeed, for Hilbert systems all homotopy colimits do exist.

Another benefit from working with relative categories is this: Relative categories are commonly regarded not as the important objects in themselves, but rather as *presentations* of a so-called $(\infty, 1)$-category. As an analogy, in group theory one can have different presentations by generators and relations of a group, and these can be useful for answering different questions about the group, while the actual object of interest is still (the isomorphism class of) the group itself. Analogously there can be different relative categories which are presentations of the same $(\infty, 1)$-category, and when asking questions whose answers are invariant under equivalence, there is no harm in switching to a better suited presentation. There is a theory of $(\infty, 1)$-categories, very much parallel to usual category theory, where one studies such invariant properties and the $(\infty, 1)$-categories of logics that we consider here have much better properties in this realm, than in the usual category theoretical world where they arose.

Overview of the article. In this article we explore a bit of the homotopy theoretical perspective on logics that we have hinted at. In Chapter 2 we review usual categories of logics and pin down, what is the problem with categories of flexible morphisms. In Chapter 3 we give a quick tour through some concepts of abstract homotopy theory to explain the setting in which we wish to study categories of logics. Here we can only serve some rough ideas of a huge area, but we need very little of the full scope of abstract homotopy theory and the next chapter – particularly Section 4.2 – can be read with very few prerequisites. The main ingredients that will be used from here are simplicial categories and how they arise from 2-categories as well as the the notion of equivalence of simplicial categories.

Chapter 4 is the technical heart of the article, where we investigate categories of Tarski style logics. We first introduce, in Section 4.1, the two different notions of homotopy equivalence and weak equivalence and see how they relate differently in the strict and flexible settings. We next, in Section 4.2, give a short preview on work to appear that adresses the so-called the hammock localization of the category of Hilbert systems. It is in this setting that we can particularly well handle the combination of logics along flexible translations, which we hinted at above. In particular we get preservation results for homotopy colimits parallel to those for fibring, which was what motivated our discussion of question 4. Our treatment of this, however, uses a particular kind of presentation of an $(\infty, 1)$-category, which we felt was too much to expose here in detail.

Section 4.3 exploits the fact that the set of translations between two logics carries a natural equivalence relation: Two translations $f, g\colon L \to L'$ can be called equivalent if for every formula φ of L the images $f(\varphi)$ and $g(\varphi)$ are mutually derivable from each other in L'. Equivalence relations are a special kind of groupoid and thus categories of logics can be seen as 2-categories and come with a natural notion of equivalence, which under mild hypotheses coincides with the ones of section 4.1. The resulting $(\infty, 1)$-categories, which we call the 2-categorical localizations, are homotopy theoretically very simple; their mapping spaces are homotopy discrete. This means that they are equivalent to the quotient categories with respect to the above equivalence relations. These quotient categories have been studied by Mariano and Mendes in [41] and [42], where they show, among other things, that the quotient category of congruential Hilbert systems is complete and cocomplete. In section 4.3.2 we show how these results of Mariano and Mendes, can be cast into the language of $(\infty, 1)$-categories, here embodied by categories enriched in simplicial sets. On the one hand this is because, in our view, the simplicial categories are the natural objects one would want to study, and that this boils down to the study of their homotopy categories could be seen as merely a technical convenience. On the other hand this is to offer the reader an easy entry point to get acquainted with the language – it is the language that will be needed for the more refined categories of logics of Section 5.3.

It is in Section 4.3 that the main technical results of the article appear. These are: Theorem 4.26 (crucially relying on work of Mariano and Mendes), which asserts that in the world of $(\infty, 1)$-categories the category of logics and flexible morphisms is a reflective subcategory of that of strict morphisms, Theorem 4.39, which asserts that the category of logics and flexible morphisms has all homotopy limits (contrary to the 1-categorical case) and the discussion of Section 4.3.3, which asserts the existence of all homotopy colimits. We chose to construct homotopy limits in a pedestrian way to give a feeling of how one can handle single logics homotopically, and to sketch a proof of the existence of homotopy colimits by abstract results, to give a different sample of homotopy theoretical methods.

In the remaining Chapter 5 we gather questions and prospects for further developments suggested by the homotopy theoretical viewpoint on logic.

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Happy birthday, Jean-Yves!!

2. Categories of logics

2.1. Signatures

Definition 2.1. A *signature* S is a sequence of sets $(S_n, n \in \mathbb{N})$.

We think of the elements of S_n as the generating n-ary connectives of a formal language. We fix once and for all a set $\mathrm{Var} := \{x_n \mid n \in \mathbb{N}\}$ of variables and denote as usual by $\mathrm{Fm}(S)$ the absolutely free algebra with signature S generated by Var. We have a decomposition $\mathrm{Fm}(S) = \coprod_{n\in\mathbb{N}} \mathrm{Fm}(S)[n]$, where $\mathrm{Fm}(S)[n]$ denotes the set of formulas with n free variables. We also denote by $\mathrm{Fm}(S)[x_1, \ldots, x_n]$ the set of formulas containing exactly the variables $x_1, \ldots, x_n$.

Definition 2.2. A *strict morphism* $f : S \to S'$ of signatures is a sequence of maps $(f_n : S_n \to S'_n, n \in \mathbb{N})$.

A *flexible morphism*, or simply a *morphism*,$f : S \to S'$ of signatures is a sequence of maps $(f_n : S_n \to \mathrm{Fm}(S')[x_1, \ldots, x_n], n \in \mathbb{N})$.

Thus a strict morphism is an arity-preserving map sending generating connectives to generating connectives while a flexible morphism can be seen as a map sending generating connectives to derived connectives. A strict morphism can be seen as a flexible morphism which happens to send generating connectives to generating connectives (where a generating connective $c \in S_n$ is seen as the formula $c(x_1, \ldots, x_n) \in \mathrm{Fm}(S')[x_1, \ldots, x_n]$), see Definition 2.5.1.

A morphism $f : S \to S'$, either strict or flexible, induces a map $f : \mathrm{Fm}(S) \to \mathrm{Fm}(S')$ which is inductively defined as usual.

Example 2.3. The usual double negation translation from the standard signature of classical propositional logic to the standard signature of intuitionistic logic is a flexible morphism which is not strict as it sends, for example, the binary connective $\wedge$ (or $(x_1 \wedge x_2)$) to the derived connective $(\neg\neg(x_1) \wedge \neg\neg(x_2))$. One can make the double negation translation into a strict morphism, if one chooses to present intuitionistic logic with extra connectives and axioms: To resolve, for example, the above obstacle to a strict translation, one could add a binary connective $\wedge^{class}$ to the presentation of intuitionistic logic and add the axiom $(x_1 \wedge^{class} x_2) \dashv\vdash (\neg\neg(x_1) \wedge \neg\neg(x_2))$. Then a translation from classical logic could be defined by sending $\wedge$ to $\wedge^{class}$.

Definition 2.4. The category $\mathcal{S}ig^{strict}$ is the category whose objects are signatures and whose morphisms are strict morphisms. The category $\mathcal{S}ig$ is the category whose objects are signatures and whose morphisms are flexible morphisms.

We note that the category $\mathcal{S}ig^{strict}$ is equivalent to the category $\mathbf{Set}^{\mathbb{N}}$ of sequences of sets and morphisms, in particular it is complete and cocomplete.

Central to our main results in Section 4.3 is the following adjunction established by Mariano and Mendes in [42].

Definition 2.5 ([42, Prop. 1.5, Mariano/Mendes])**.**

1. The functor $i\colon \mathcal{S}ig^{strict} \to \mathcal{S}ig$ is defined on objects by the identity and on morphisms by associating to $f = (f_n)_{n\in\mathbb{N}}\colon S \to S'$ the flexible morphism $i(f)$ with $i(f)_n\colon S_n \to \mathrm{Fm}(S')[x_1,\ldots,x_n]$, $c \mapsto (f_n(c))(x_1,\ldots,x_n)$.
2. The functor $Q\colon \mathcal{S}ig \to \mathcal{S}ig^{strict}$ is defined on objects by $S \mapsto Q(S)$ where $Q(S)_n := \mathrm{Fm}(S)[x_1,\ldots,x_n]$ and by sending a flexible morphism $f\colon S \to S'$, given by $(f_n\colon S_n \to \mathrm{Fm}(S')[x_1,\ldots,x_n])$, to the sequence of induced maps $\mathrm{Fm}(S)[x_1,\ldots,x_n] \to \mathrm{Fm}(S')[x_1,\ldots,x_n]$.

Theorem 2.6 ([42, Thm. 1.6, Mariano/Mendes])**.** *The functor i is left adjoint to Q.*

Proof. The natural isomorphisms

$$\mathrm{Hom}_{\mathcal{S}ig}(i(S), S') \cong \{(f_n\colon S_n \to \mathrm{Fm}(S')[x_1,\ldots,x_n])_{n\in\mathbb{N}}\} \cong \mathrm{Hom}_{\mathcal{S}ig^{strict}}(S, Q(S'))$$

follow straight from the definitions of the morphisms of $\mathcal{S}ig$, resp. $\mathcal{S}ig^{strict}$, and the functors i and Q. □

The unit $S \to i(Q(S))$ of the adjunction is given by the inclusions $S_n \to \mathrm{Fm}(S)[x_1,\ldots,x_n]$, $c \mapsto c(x_1,\ldots,x_n)$. The counit $Q(i(S)) \to S$ is the flexible morphism given by the identity maps $\mathrm{Fm}(S)[x_1,\ldots,x_n] \to \mathrm{Fm}(S)[x_1,\ldots,x_n]$.

In fact, by [42, Thm. 1.12, Mariano/Mendes] the category $\mathcal{S}ig$ is the Kleisli category of the above adjunction. Thus it is a category of free algebras and has much worse categorical properties than $\mathcal{S}ig^{strict}$. It is not complete nor cocomplete. This is to be expected, as (co)limits of free algebras, formed within their (co)complete ambient category of all algebras, are not usually free again.

Example 2.7. The category $\mathcal{S}ig$ has no terminal object. Indeed, a terminal signature would have to have a generating connective of arity ≥ 2, since if there were only generating connectives of arities 0 and 1 the sets of n-ary formulas $\mathrm{Fm}(S)[x_1,\ldots,x_n]$ would be empty and there could be no morphism from a signature with n-ary connectives. But if c is an n-ary connective, then $\mathrm{Fm}(S)[x_1,\ldots,x_{2n-1}]$ contains the two different formulas

$c(c(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1})$ and $c(x_1, \dots, x_{n-1}, c(x_n, \dots, x_{2n-1}))$ and hence admits two different morphisms from the signature with just one generating $(2n-1)$-ary connective.

Some (co)limits do exist. Among these the colimits imported via the functor i (which is left adjoint, hence colimit preserving) from the cocomplete category $\mathcal{S}ig^{strict}$ are well behaved, but others are degenerate and do not express what we would like to achieve with them in logic.

Example 2.8. Consider the signature S generated by a single unary connective $\Box$. We have the two flexible morphisms $f, g\colon S \to S$ defined by $f(\Box) := \Box\Box x_1$, $g(\Box) := \Box\Box\Box x_1$, respectively. Any flexible morphism $h\colon S \to S'$ which satisfies $h \circ f = h \circ g$ (i.e. which "coequalizes" f and g) can only map $\Box$ to the variable $x_1 \in \mathrm{Fm}(S')[x_1]$: If $\Box$ is mapped to any other formula $\varphi(x_1)$ then the formulas $\Box\Box x_1$ and $\Box\Box\Box x_1$ will have the images $\varphi(\varphi(x_1))$ and $\varphi(\varphi(\varphi(x_1)))$, and these images will be different because the target is an absolutely free algebra. The coequalizer can then easily be seen to be empty signature $\emptyset$ with no generating connectives: By the usual definition of formulas, the set of formulas is the smallest set containing all variables and closed under application of connectives, so there we have $x_1 \in \mathrm{Fm}(\emptyset)[x_1]$ and the uniqueness property of a colimit is satisfied since this signature is initial. However, this is not what one would like in practice. The coequalizer should remember that there was a connective $\Box$ and that "$\Box\Box = \Box\Box\Box$", not just forget it completely.

Remark 2.9. One could adopt a yet more flexible notion of morphism by considering the set $\mathrm{Fm}(S')\langle x_1, \dots, x_n\rangle$ of formulas which contain no other variables than $x_1, \dots x_n$ (but are allowed to contain less than these), i.e. $\mathrm{Fm}(S')\langle x_1, \dots, x_n\rangle := \coprod_{i=0}^{n} \mathrm{Fm}(S)[x_1, \dots, x_i]$, and then defining the set of morphisms as $\mathrm{Hom}(S, S') := \{(f_n\colon S_n \to \mathrm{Fm}(S')\langle x_1, \dots, x_n\rangle)_{n\in\mathbb{N}}\}$. Such morphisms would no longer preserve the arity of formulas, and for example one could "delete" n-ary connectives by mapping them to the single variable x_1 (substitution into which would correspond to the identity operation). Much of what will be said in this article would carry over to this setting, as well as to many other variants, but the notions of morphism we chose to consider seem to be the ones of biggest interest in practice.

Definition 2.10. A *substitution* is a map $\sigma\colon \mathrm{Var} \to \mathrm{Fm}(S)$.

Again a substitution induces an inductively defined map $\sigma\colon \mathrm{Fm}(S) \to \mathrm{Fm}(S)$.

2.2. Logics

Definition 2.11. 1. Let S be a signature. A *consequence relation* over S is a relation $\vdash \subseteq \mathscr{P}(\mathrm{Fm}(S)) \times \mathrm{Fm}(S)$ between subsets of $\mathrm{Fm}(S)$ and elements of $\mathrm{Fm}(S)$. As usual we write it in infix notation $\Gamma \vdash \varphi$.

2. A *logic* is a pair $L = (S, \vdash)$, where S is a signature and $\vdash$ a consequence relation on $\mathrm{Fm}(S)$.

Given a logic L, we will sometimes denote its underlying signature by S_L and its consequence relation by $\vdash_L$.

Definition 2.12. A consequence relation is *Tarskian* if the associated operation $\mathrm{Cn}\colon \mathscr{P}(\mathrm{Fm}(S)) \to \mathscr{P}(\mathrm{Fm}(S))$, $\Gamma \mapsto \{\varphi \mid \Gamma \vdash \varphi\}$ satisfies

1. (increasingness) $\Gamma \subseteq \mathrm{Cn}(\Gamma)$ for all $\Gamma \subseteq \mathrm{Fm}(S)$
2. (idempotence) $\mathrm{Cn}(\mathrm{Cn}(\Gamma)) \subseteq \mathrm{Cn}(\Gamma)$ for all $\Gamma \subseteq \mathrm{Fm}(S)$
3. (monotonicity) $\Gamma \subseteq \Gamma' \Rightarrow \mathrm{Cn}(\Gamma) \subseteq \mathrm{Cn}(\Gamma')$ for all $\Gamma, \Gamma' \subseteq \mathrm{Fm}(S)$

These conditions say exactly that Cn is a closure operator on $\mathscr{P}(\mathrm{Fm}(S))$.

Two common further additional conditions that one likes to impose on consequence relations are:

(finitarity) If $\Gamma \vdash \varphi$ then there exists a finite subset $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \varphi$

(substitution invariance) If $\Gamma \vdash \varphi$, then for any substitution σ we have $\sigma(\Gamma) \vdash \sigma(\varphi)$

Substitution invariance is also called *structurality*.

Finally, the role of the following notion for the study of categories of logics has been brought to light by Mariano and Mendes in [41], [42].

Definition 2.13. A logic $(S, \vdash)$ is *congruential* if for every two sequences of formulas $\varphi_1, \ldots \varphi_n$ and $\psi_1, \ldots, \psi_n$ with $\varphi_i \dashv\vdash \psi_i$ and every pair of formulas $\beta(x_{i_1}, \ldots, x_{i_n})$, $\gamma(x_{i_1}, \ldots, x_{i_n})$ with $\beta \dashv\vdash \gamma$ we have $\beta(\varphi_1, \ldots \varphi_n) \dashv\vdash \gamma(\psi_1, \ldots, \psi_n)$.

Congruentiality is a notion taking its place in the Leibniz hierarchy of degrees of algebraizability.

By the results of Łoś and Suszko in [38], the consequence relations that are finitary, substitution invariant and Tarskian are exactly the provability relations coming from a Hilbert style system, where consequence is given by finite derivations using axioms, rules and substitutions. We therefore call a logic $(S, \vdash)$ a *Hilbert system*, if the consequence relation has these properties. On the semantical side there is Wójcicki's result from [52], saying that a finitary and substitution invariant Tarskian logic is sound and complete for an appropriate finitary matrix semantics (see also [53, Thm. 3.1.6]).

The consequence relations over a fixed signature S can be ordered by setwise inclusion: $\mathrm{Cn}_1 \leq \mathrm{Cn}_2 :\Leftrightarrow \mathrm{Cn}_1(\Gamma) \subseteq \mathrm{Cn}_2(\Gamma)\ \forall \Gamma \subseteq \mathrm{Fm}(S)$. Obviously consequence relations (without further conditions) form a complete lattice with respect to the above order. By [53, Thm. 1.5.4] also the subset of Tarskian consequence relations on $\mathrm{Fm}(S)$ forms a complete lattice with respect to this order, by [53, Theorems 1.5.5-1.5.6] the same is true for the subsets of finitary, resp. structural Tarskian consequence relations and finally by [53, Thm. 1.5.7] the same is true for Hilbert systems.

About congruential Hilbert systems there is the following result by Mariano and Mendes:

Proposition 2.14 ([42, Prop. 2.18, Mariano/Mendes])**.** *The category of congruential Hilbert systems is a reflective subcategory of the category of all Hilbert systems.*

By considering colimits of diagrams of congruential logics whose underlying signature morphisms are the identity, one can conclude that congruential Hilbert systems form a complete lattice of consequence relations as well. It is also easy to see that intersections of congruential consequence relations are congruential again

In particular for a signature S we have on $\mathrm{Fm}(S)$ a maximal and a minimal consequence relation of each of the types just listed.

The completeness of the considered lattices of consequence relations gives us the possibility of defining direct and inverse image logics, as done for Hilbert systems in [2, Def 2.9]:

Definition 2.15. Given a logic $(S, \vdash)$ and a signature morphism $f\colon S \to S'$ we can view $\vdash$ as a subset of $\mathscr{P}(\mathrm{Fm}(S)) \times \mathrm{Fm}(S)$, take the set-theoretic image $f(\vdash) \subseteq \mathscr{P}(\mathrm{Fm}_{S'}) \times \mathrm{Fm}_{S'}$ and define the *direct image* $f_*(\vdash)$ as the infimum of all consequence relations of the given type (e.g. Tarskian, resp. finitary and/or structural Tarskian consequence relations) containing $f(\vdash)$.

Likewise, given a logic $(S', \vdash')$ and a signature morphism $f\colon S \to S'$, the *inverse image* $f^*(\vdash')$ can be defined as the infimum of all consequence relations of the given type (i.e. Tarskian, resp. finitary and/or substitution invariant Tarskian consequence relations) on $\mathrm{Fm}(S)$ containing $f^{-1}(\vdash')$, the set-theoretic pre-image of $\vdash'$.

Definition 2.16. A *translation* (resp. *strict translation*) $L = (S, \vdash) \to (S', \vdash') = L'$ of logics is a signature morphism (resp. *strict signature morphism*) $f\colon S \to S'$ such that $\Gamma \vdash \varphi \Rightarrow f(\Gamma) \vdash' f(\varphi)$.

Remark 2.17. As noted in [2, Fact 5 (p.12)], given two logics $L = (S, \vdash), L' = (S', \vdash')$, a signature morphism $f\colon S \to S'$ is a translation iff $\vdash \leq f^*(\vdash')$ iff $f_*(\vdash) \leq \vdash'$.

Definition 2.18. We denote by $\mathcal{LOG}$ the category of logics whose objects are logics and whose morphisms are translations. We denote by $\mathcal{L}og^{(Tarsk)}$ (resp. $\mathcal{L}og^{(fin,Tarsk)}$, resp. $\mathcal{L}og^{(subst,Tarsk)}$, resp. $\mathcal{L}og^{(subst,Tarsk,con)}$ etc.) the full subcategory of Tarskian (resp. of finitary Tarskian, resp. substitution invariant Tarskian, resp. substitution invariant congruential Tarskian, etc.) logics. Finally we denote by $\mathcal{H}ilb$ the full subcategory of Hilbert systems and by $\mathcal{H}ilb^{(con)}$ the full subcategory of congruential Hilbert systems.

Convention 2.19. For the remainder of the article we denote by $\mathcal{L}og$ any of the following full subcategories of $\mathcal{LOG}$: $\mathcal{LOG}$, $\mathcal{L}og^{(Tarsk)}$, $\mathcal{L}og^{(con)}$, $\mathcal{L}og^{(Tarsk,con)}$, $\mathcal{L}og^{(fin,Tarsk)}$, $\mathcal{L}og^{(subst,Tarsk)}$, $\mathcal{L}og^{(subst,Tarsk,con)}$, $\mathcal{H}ilb$, $\mathcal{H}ilb^{(con)}$. By the terms "logic" and "consequence relation" we will mean a logic, resp. a consequence relation, taken from this chosen category $\mathcal{L}og$. If we need to distinguish consequence relations from $\mathcal{LOG}$ and consequence relations defining objects of $\mathcal{L}og$, we will call the latter "admissible consequence relations". In parts of Chapter 4 we will need to assume additional properties of our logics and will then say so.

We invite the reader to read the article with a specific category of logics, such as $\mathcal{L}og^{(Tarsk)}$ or $\mathcal{H}ilb$ in mind.

Remark 2.20. For much of what follows we could be rather flexible about what properties exactly we demand from our consequence relations. Much of the article can be read by fixing a set of properties that one wishes our consequence relations to have and that satisfy the following assumptions:

1. The lattice of consequence relations satisfying the properties is complete
2. The direct image maps satisfy $g_*(f_*(\vdash)) = (g \circ f)_*(\vdash)$

These assumptions are exactly what is needed for Proposition 2.24 below to hold, i.e. that one can construct (co)limits in the corresponding category of logics by constructing them in $\mathcal{S}ig$ (resp. $\mathcal{S}ig^{strict}$) and then endow the resulting signature with an appropriate consequence relation. Much of Chapter 4, however, needs the property of idempotence.

Remark 2.21. The completeness of the lattice of consequence relations of a chosen kind also makes it possible to define a consequence relation by giving generating rules. For example, given a logic $(S, \vdash)$ and $\varphi, \psi \in \mathrm{Fm}(S)$, the Tarskian consequence relation generated by $\vdash$ and the rules $\{\varphi\} \vdash \psi$, $\{\psi\} \vdash \varphi$ is defined to be the infimum in the lattice of Tarskian consequence relations of all those consequence relations containing $\vdash$ and which satisfy $\{\varphi\} \vdash \psi$ and $\{\psi\} \vdash \varphi$. We will freely make use of such constructions and if we talk of the consequence relation generated by some given rules, we will always mean the consequence relation in our chosen category $\mathcal{L}og$.

Definition 2.22. Let $U \colon \mathcal{L}og \to \mathcal{S}ig$, $(S, \vdash) \mapsto S$ denote the obvious faithful forgetful functor forgetting the consequence relation. We will denote its restriction to the subcategories of strict morphisms $\mathcal{L}og^{strict} \to \mathcal{S}ig^{strict}$ by U as well.

Lemma 2.23. *The functor U has a left adjoint $Min \colon \mathcal{S}ig \to \mathcal{L}og$, $S \mapsto (S, \vdash_{min})$ which endows a signature S with the minimal consequence relation on $Fm(S)$, as well as a right adjoint $Max \colon \mathcal{S}ig \to \mathcal{L}og$, $S \mapsto (S, \vdash_{max})$ placing the maximal consequence relation on $Fm(S)$. These adjunctions restrict to the subcategories of strict morphisms.*

Proof. The direct/inverse image characterization of translations of Remark 2.17 implies that, for a morphism of signatures $f \colon S \to S'$, the pair (f_*, f^*) is a pair of adjoint functors between the preorders of consequence relations, seen as categories. Here f_* is the left adjoint, hence it preserves colimits, i.e. suprema of consequence relations. In particular it preserves the supremum of the empty family, i.e. the minimal consequence relation, i.e. $f_*(\vdash^S_{min}) = \vdash^{S'}_{min}$. Now we have natural bijections

$$\mathrm{Hom}_{\mathcal{L}og}(Min(S), L) = \{f \in \mathrm{Hom}_{\mathcal{S}ig}(S, S_L) \mid f_*(\vdash_{min}) \leq \vdash_L\} = \mathrm{Hom}_{\mathcal{S}ig}(S, S_L)$$

where the left equality is th direct image characterization of translations of Remark 2.17 and the right equality can be seen from the fact that $f_*(\vdash^S_{min}) = \vdash^{S_L}_{min}$ and thus the condition in the middle set is empty.

This shows that Min is left adjoint to U. The right adjointness of Max works by dualizing the proof, the restriction statement is clear. □

Note that we have $U \circ \mathrm{Min} = U \circ \mathrm{Max} = id_{\mathcal{S}ig}$.

Proposition 2.24. *The category $\mathcal{L}og$ has (co)limits of a given diagram shape, if and only if the (co)limits of this shape exist in $\mathcal{S}ig$.*

Proof. Since $U \colon \mathcal{L}og \to \mathcal{S}ig$ has a left and a right adjoint, it preserves colimits and limits. Thus if (co)limits of shape D exist in $\mathcal{L}og$, then, given a diagram of shape D in $\mathcal{S}ig$, we can lift it to $\mathcal{L}og$ e.g. via the functor Min, take the colimit in $\mathcal{L}og$ and apply U. This will yield the (co)limit of the diagram we started with.

Conversely, if colimits of shape D exist in $\mathcal{S}ig$, then given a diagram $F \colon D \to \mathcal{L}og$, we can take the colimit of the underlying diagram $U \circ F$ in $\mathcal{S}ig$ and then endow the resulting signature with the smallest consequence relation such that all the incoming signature morphisms from the original diagram become translations (this exists because of the completeness of the lattice of consequence relations). This construction of colimits is done in all details in [2, Prop. 2.11].

Likewise, limits in $\mathcal{L}og$ can be constructed by taking them in $\mathcal{S}ig$ and endowing the resulting signature with the maximal consequence relation such that all outgoing signature morphisms into the diagram become translations.

An inspection of the proofs in [2] (which are written for Hilbert systems) shows that the assumptions of Remark 2.20 are all that is needed. □

On the one hand this shows that the category of $\mathcal{L}og^{strict}$ of logics and strict morphisms is complete and cocomplete, since the underlying category $\mathcal{S}ig^{strict} \simeq \mathbf{Set}^{\mathbb{N}}$ of signatures is complete and cocomplete. On the other hand the category $\mathcal{S}ig$ of signatures and flexible translations is not (co)complete as seen in the last section and hence the same is true for $\mathcal{L}og$.

Clearly we would like a category with the more flexible morphisms, in which we can perform constructions such as (co)limits and which has good categorical properties such local presentability. Proposition 2.24 seems to rule this out. It is one of the main points of this article to argue that in fact we do not just have a category $\mathcal{L}og$, but instead a category endowed with an extra structure: A distinguished class of morphisms which we want to see as "equivalences". This extra structure tells us that $\mathcal{L}og$ is most naturally seen not as a category but as a so-called $(\infty, 1)$-category. There is a theory of $(\infty, 1)$-categories, largely parallel to usual category theory, whose basic ideas we will sketch in the following section. It will then turn out in Section 4 that, seen as an $(\infty, 1)$-category, $\mathcal{L}og$ does not have the defects that it has as a 1-category.

3. Abstract homotopy theory

In this section we review how the basic datum of a category with a distinguished class of morphisms gives rise to structures and notions of a homotopy theoretical flavor.

A *relative category* is a pair (C, W) where C is a category and $W \subseteq \mathrm{Mor}C$ is a class of morphisms containing the identity morphisms. We will call the morphisms in W the *weak equivalences* and will try to find constructions and notions that are invariant under weak equivalences.

3.1. Simplicial sets and nerves

The archetypical example of a category with a distinguished class of weak equivalences is the category **Top** of topological spaces and continuous maps, where a weak equivalence $X \to Y$ is a map inducing isomorphisms $\pi_n(X) \to \pi_n(Y)$ between all homotopy groups π_n, $n \geq 1$ (with respect to all possible base points) and between the sets of connected components $\pi_0(X)$, $\pi_0(Y)$. One important feature of abstract homotopy theory is that one can often replace a relative category by another one which is better behaved but contains the same information regarding "weak equivalence types". For the category of topological spaces the most common choice is the relative category of simplicial sets:

Definition 3.1. A *simplicial set* is a functor $\Delta^{op} \to \mathbf{Set}$, where Δ denotes the category of finite linearly ordered sets and order preserving maps. A morphism of simplicial sets is a natural transformation. We denote the category of simplicial sets by **sSet**.

The category Δ (or a skeleton thereof) can be described by generators and relations: We can take as objects the standard linearly ordered sets $[n] := \{0 \leq \ldots \leq n\}$ and observe that any order preserving map arises as a composition of the basic maps $[n] \to [n-1]$ which identify two neighbouring numbers and the maps $[n] \to [n+1]$ which leave a gap between two neighbouring numbers. To give a functor from Δ (or Δ^{op}) to some category, it is then enough to say what it does on objects and on these basic maps and to ensure that the choice of values on the maps satisfies some relations. Thus a simplicial set can alternatively be described as a diagram of sets of the following shape

$$X_0 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_1 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_2 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; X_3 \;\substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow}\; \cdots$$

in which the arrows satisfy certain relations (see e.g. [22, p. 4]).

One may think about a simplicial set as follows: The set X_n is the set of n-simplices of some abstract simplicial complex and the maps $X_n \to X_{n-1}$ identify the faces of these n-simplices with certain $(n-1)$-simplices from X_{n-1}. Thus two n-simplices can share a common face and the whole diagram can be seen as giving combinatorial data for pasting together the several collections of simplices (let us not care about the index increasing maps).

We record the following standard result from category theory (see e.g. [33, 2.7.1]):

Lemma 3.2. *Let* $\mathbf{C}$ *be a small category,* $\mathcal{D}$ *a cocomplete category and* $F\colon \mathbf{C} \to \mathcal{D}$ *a functor. Then there is an adjunction* $\overline{F} : \mathbf{Set}^{\mathbf{C}^{op}} \rightleftarrows \mathcal{D} : \mathrm{Hom}_{\mathcal{D}}(F(\cdot), -)$.

Here the left adjoint $\overline{F}$ is given by left Kan extension of F along the Yoneda embedding, i.e. by expressing a presheaf as a colimit of representables, mapping the representables to $\mathcal{D}$ as prescribed by F and then taking the colimit there.

There is a cosimplicial object in **Top**, i.e. a functor $\Delta \to \mathbf{Top}$, which associates to the object $[n]$ the standard n-simplex $\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \Sigma x_i = 1,\ x_i \geq 0\ \forall\, i\}$, which send increasing maps to inclusions of faces of the standard simplex and decreasing maps to continuous maps collapsing a simplex to one of its faces, see [22, Example 1.1]. By Lemma 3.2 this induces an adjunction $|\cdot|\colon \mathbf{sSet} \rightleftarrows \mathbf{Top} : \mathit{Sing}$. Here the value $|X|$ of the left adjoint at a simplicial set X, called the *geometric realization* of X, is given by taking a standard n-simplex in **Top** for each element of the set X_n and glueing these simplices together as suggested by the face maps. The right adjoint is defined on objects by $\mathit{Sing}(X)_n := \mathrm{Hom}_{\mathbf{Top}}(\Delta^n, X)$ and on maps by precomposition of face inclusions/retractions. Geometric realization can be seen as a formalization of the intuition about simplicial sets offered in the previous paragraph.

One says that a map of simplicial sets is a *weak equivalence* if its geometric realization is a weak equivalence of topological spaces. A first indication (but not the complete story) that for the purpose of studying spaces up to weak equivalence one can replace **Top** by **sSet** is the fact that the unit and counit of the above adjunction are weak equivalences at each object, i.e. one has weak equivalences $|Sing(X)| \to X$ and $Y \to Sing(|Y|)$ so that going back and forth between the two categories results in weakly equivalent objects. A more complete statement is that **Top** and **sSet** form equivalent $(\infty, 1)$-categories, see below. We therefore sometimes refer to simplicial sets as "spaces", for example when we will talk about mapping "spaces" below.

Similarly we can define the *nerve functor* $N\colon$ **Cat** $\to$ **sSet** by applying Lemma 3.2: There is a cosimplicial object in the category **Cat** of (small) categories simply given by considering the linearly ordered sets of Δ as categories in the usual way, i.e. taking the numbers $0, \dots, n$ as objects and declaring that there is a unique morphism from i to j if $i \leq j$. Mapping out of this cosimplicial object into a fixed category C produces a simplicial set $N(C) := \mathrm{Hom}_{\mathbf{Cat}}(\Delta, C)$ as above, and the nerve functor is the right adjoint of 3.2 in this special case.

More concretely the nerve of a category C is the simplicial set with $N(C)_0 = \mathrm{Ob}\, C$, $N(C)_1 = \mathrm{Mor}\, C$, $N(C)_n = \{$chains of n composable morphisms of $C\}$ and whose structure maps are given by composing arrows, resp. inserting identity morphisms into a chain.

3.2. Localization of categories and homotopy (co)limits

One thing we can do with a relative category (C, W) is to force the morphisms from W to become isomorphisms: That means we can construct a category $C[W^{-1}]$ together with a functor $L\colon C \to C[W^{-1}]$ mapping morphisms from W to isomorphisms and with the property that any other functor $C \to \mathcal{D}$ mapping morphisms from W to isomorphisms factorizes through L uniquely up to unique natural isomorphism.

3.2.1. Localizing categories. Here is a concrete construction of $C[W^{-1}]$: As objects we take the objects of C. To define the morphisms from A to B, first say that a zig-zag from A to B is a sequence of morphisms of C with arrows pointing in either direction

$$A \leftarrow X_1 \to X_2 \leftarrow \dots \leftarrow X_n \to B$$

and in which the arrows pointing from right to left are from W. Two zig-zags can be related if one arises from the other by (a) composing two consecutive arrows which point into the same directions (b) deleting identity arrows or (c) deleting two arrows which are equal in C and point into opposite directions in the zig-zag. Now consider the equivalence relation $\sim$ generated by the relations (a),(b) and (c) and define $\mathrm{Hom}_{C[W^{-1}]}(A, B) := \{$zig-zags from A to $B\}/\sim$. Composition in $C[W^{-1}]$ is induced by concatenation of zig-zags. We have an obvious functor $L\colon C \to C[W^{-1}]$ mapping the morphisms of C to equivalence classes of zig-zags of length one. The arrows pointing from right to left can be seen as newly added inverses to the arrows of W and with this in mind it is not hard to prove the desired universal property of the functor L. Note that since the zig-zags can range over all the objects of C, the class $\mathrm{Hom}_{C[W^{-1}]}(A, B)$ is not a set in general (or lives in a higher Grothendieck universe), but in many cases in practice it turns out to be a set again. One also writes $Ho(C, W) := C[W^{-1}]$ and calls this category the *homotopy category* of C with respect to W. If the class W is clear, one often suppresses it from the notation and simply writes $Ho(C)$.

3.2.2. Homotopy (co)limits. Suppose that (C, W) is a relative category and C has colimits of some diagram shape $\mathbf{D}$. The diagram category $C^{\mathbf{D}}$ is itself a relative category with weak equivalences given by objectwise weak equivalences of C, hence we also have a homotopy category $Ho(C^{\mathbf{D}})$. By the universal property of the localization, if the colimit functor preserves weak equivalences we can find a functor (represented by the dotted arrow below) making the following diagram commute:

$$
\begin{array}{ccc}
C^{\mathbf{D}} & \xrightarrow{colim} & C \\
\downarrow & & \downarrow \\
Ho(C^{\mathbf{D}}) & \dashrightarrow & Ho(C)
\end{array}
$$

However, the colimit functor has no reason to preserve weak equivalences. A standard example in the category of topological spaces is to consider the pushouts $S^1 = \mathrm{colim}([0,1] \leftarrow \{0,1\} \rightarrow [0,1])$ (the circle, obtained by glueing two unit intervals along their end points) and $* = \mathrm{colim}(* \leftarrow \{0,1\} \rightarrow *)$ (where $*$ denotes the 1-point space): The unit interval $[0,1]$ is contractible, so the obvious transformation from the first to the second diagram is an equivalence. Yet the respective colimits of the two diagrams are not equivalent (since S^1 has nontrivial fundamental group).

The next best thing that we can do then is to form the right Kan extension of $C^{\mathbf{D}} \rightarrow C \rightarrow Ho(C)$ along $C^{\mathbf{D}} \rightarrow Ho(C^{\mathbf{D}})$ — this is the universal approximation of the colimit construction by a construction that preserves weak equivalences. It results in a diagram

$$
\begin{array}{ccc}
C^{\mathbf{D}} & \xrightarrow{colim} & C \\
\downarrow & \nearrow & \downarrow \\
Ho(C^{\mathbf{D}}) & \xrightarrow{hocolim} & Ho(C)
\end{array}
$$

which is not commutative but instead filled in with a universal natural transformation. We emphasize that the homotopy colimit is usually not the colimit in the homotopy category; indeed in general $Ho(C^{\mathbf{D}})$ is not equivalent to $Ho(C)^{\mathbf{D}}$, so it does not even have the right domain category. The notion of *homotopy limit* is dual, with a left Kan extension instead of a right one.

In the category of simplicial sets all homotopy (co)limits in the sense just defined exist, and there have been developed many techniques for computing them. We will use this fact in the next section.

In [16] the reader can find an exposition of the basic notions and statements around relative categories in which the weak equivalences satisfy a very mild closure condition. A more powerful setting continuing this line of thought about homotopy (co)limits is the theory of derivators, see for example [26] and the references therein.

We note that these approaches to homotopy (co)limits allow us to stay in the framework of usual category theory. However, they have their limitations and a richer setting with a well developed theory is that of $(\infty, 1)$-categories, see the next sections.

3.3. Simplicial categories, simplicial localizations and homotopy (co)limits

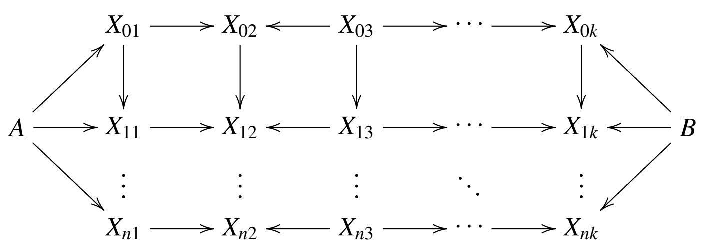
Lots of experience indicates that the category $C[W^{-1}]$ is in general too crude an object. If one wants to treat objects of C up to equivalence, the passage from C to $C[W^{-1}]$ forgets too much; morphisms from C get identified uncontrollably and equivalence preserving constructions in C can not be characterized or performed just in $C[W^{-1}]$ alone. This has already been visible in our above definition of homotopy (co)limits: For this we needed both the original relative category (C, W) and its localization $Ho(C)$.

Also note that the above definition of homotopy (co)limit only makes sense if the (co)limits in question exist in the category C, otherwise we have nothing along which we could take Kan extensions. Many categories of logics, however, lack (co)limits.

One can now pass to a refined variant of localization, called the simplicial localization, which results in a category enriched in simplicial sets, i.e. instead of sets of morphisms we get simplicial sets of morphisms, also called the *mapping spaces* of the simplicial category. Such a category is also called *simplicial category* (this should not be confused with simplicial objects in the category of categories). The basic intuition about a simplicial category is that the 0-simplices of the mapping spaces are morphisms, the 1-simplices are homotopies between morphisms, the higherdimensional simplices are homotopies between homotopies and so on.

We now give one possible construction of a simplicial localization: The *hammock localization* introduced in [17].

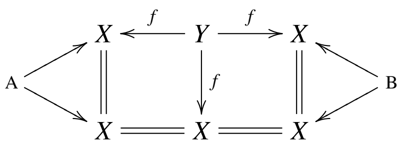
3.3.1. The Hammock localization. Given a relative category (C, W) we define its hammock localization $L^H(C, W)$ to be the following category enriched in simplicial sets: Again we take as objects those of C. For two objects A, B we now have to give a simplicial set $\mathrm{map}(A, B)$, also called the *mapping space* of A and B. We define the n-th set of this simplicial set to be the "set" of *reduced hammocks of width n*, i.e. commutative diagrams of the shape



in which the vertical arrows go downwards and are from W, in each column the horizontal arrows all point into the same direction, the horizontal arrows going from right to left are from W, no column consists only of identity arrows and the maps in adjacent columns go into different directions. The structure maps for the simplicial sets are given by composing downward pointing arrows, resp. duplicating a row and inserting identity arrows.

The composition maps $\mathrm{map}(A, B) \times \mathrm{map}(B, C) \to \mathrm{map}(A, C)$ are given by concatenation of zig-zags, followed by reducing the resulting hammocks (i.e. composing adjacent columns pointing into the same direction and deleting identity columns). For more on the hammock localization see [16].

3.3.2. Homotopy category of a simplicial category. From a simplicial category C one can obtain an ordinary category $Ho(C)$ by passing to the set of connected components of the Hom-simplicial sets, i.e. by keeping the same objects and defining $\mathrm{Hom}_{Ho(C)}(A, B) := \pi_0(\mathrm{map}_C(A, B))$. The resulting category is called the *homotopy category* of the simplicial category. If the simplicial category is the hammock localization $L^H(C, W)$ of a category with weak equivalences, the homotopy category is exactly the localization from above: $Ho(L^H(C, W)) \simeq C[W^{-1}]$. Indeed, the objects are the same, fullness is clear and the main point for faithfulness is that a zig-zag with two consecutive arrows from W pointing in opposite directions is in the same connected component of $\mathrm{map}(A, B)$ as the zig-zag with these arrows cancelled:



The details can be found in [17, Prop. 3.1].

3.3.3. Homotopy (co)limits revisited. A simplicial category C also has an underlying usual category $U(C)$ given by just remembering the 0-simplices of the mapping spaces, i.e. $\mathrm{Hom}_{U(C)}(A, B) := \mathrm{map}_C(A, B)_0$. There is a natural class W of weak equivalences on $U(C)$ given by those morphisms which become ismorphisms in $Ho(C)$ (alternatively one could take as weak equivalences morphisms $f \in \mathrm{map}(A, B)_0$ such that for all objects X the induced map $\mathrm{map}(X, A) \to \mathrm{map}(X, B)$ is a weak equivalence of simplicial sets; the two definitions coincide in good cases).

For those types of diagrams which have (co)limits in $U(C)$, we thus have the notion of homotopy (co)limit introduced above by Kan extensions. However, in a simplicial category one can also speak of homotopy (co)limits without supposing that the corresponding (co)limits are present in in the underlying category $U(C)$, by defining them through the mapping spaces. The idea is that in ordinary category theory one can define the limit of a diagram $\mathbf{D}$ as a representing object for the functor $X \mapsto \lim_{d\in\mathbf{D}} \mathrm{Hom}_C(X, d)$, i.e. by asking for a natural isomorphism of **Set**-valued functors $\lim_{d\in\mathbf{D}} \mathrm{Hom}_C(-, d) \simeq \mathrm{Hom}_C(-, \lim_{d\in\mathbf{D}} d)$. Now in a simplicial category one can instead ask for a natural weak equivalence of **sSet**-valued functors $\mathrm{holim}_{d\in\mathbf{D}} \mathrm{map}_C(-, d) \cong \mathrm{map}_C(-, \mathrm{holim}_{d\in\mathbf{D}} d)$, where the homotopy limit on the left hand side is taken in simplicial sets where we already know what it means by the definition through Kan extensions. For more details see [40, A.3.3.13].

A different formulation which easily relates to classical category theory is to ask for a final object in the simplicial category of "homotopy coherent" cones over the given diagram, but we will not go into this further and instead refer the reader to the exposition in [48].

3.3.4. Simplicial categories from 2-categories. Besides the hammock localization of a relative category there is a further source of simplicial categories relevant for us:

A 2-category is a category enriched in the category **Cat** of categories, i.e. for any two objects A, B there is a category $\underline{\mathrm{Hom}}(A, B)$, together with composition functors satisfying the usual axioms. In such a category there is a natural class of weak equivalences given by those morphisms $f\colon A \to B$ for which there exist a $g\colon B \to A$ and isomorphisms $f \circ g \simeq id_B \in \underline{\mathrm{Hom}}(B, B)$, $g \circ f \simeq id_A \in \underline{\mathrm{Hom}}(A, A)$. One way to get a simplicial category from this is to form the hammock localization with respect to the class of weak equivalences.

Another way is, for each pair of objects A, B, to take the maximal subgroupoid of the category $\underline{\mathrm{Hom}}(A, B)$, i.e. the subcategory of all objects and isomorphisms between them, and apply the nerve-functor to each of them (we could also apply the nerve functor to the whole category $\underline{\mathrm{Hom}}(A, B)$, but this would not capture the same class of weak equivalences).

These two constructions, though both very natural, do in general yield non-equivalent simplicial categories in the sense of Definition 3.3 below. The reason for this is that the hammock localization cannot distinguish between different automorphisms of the $\underline{\mathrm{Hom}}$-categories, while the nerve construction clearly captures them.

3.4. Equivalences of simplicial categories and $(\infty, 1)$-categories

Simplicial categories are objects of usual enriched category theory, as exposed in [34], where one considers categories which have, instead of sets of morphisms $\mathrm{Hom}(A, B)$, objects of morphisms $\underline{\mathrm{Hom}}(A, B)$ which live in some monoidal category $\mathcal{M}$ (here: the category of simplicial sets with the monoidal structure given by the product). An $\mathcal{M}$-enriched functor $F\colon C \to \mathcal{D}$ is a mapping of objects $Ob(C) \to Ob(\mathcal{D})$ and, for each pair A, B of objects a morphism $\underline{\mathrm{Hom}}_C(A, B) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(FA, FB)$ in $\mathcal{M}$, compatible with composition and identities. Example: Clearly a functor $F\colon (C, W) \to (\mathcal{D}, W')$ of relative categories such that $F(W) \subseteq W'$ induces a simplicial functor $L^H(F)\colon L^H(C, W) \to L^H(\mathcal{D}, W')$ between the hammock localizations as one sees easily from the definition of hammock localization.

In enriched category theory an $\mathcal{M}$-enriched functor is an equivalence if it is essentially surjective and fully faithful in the enriched sense, i.e. $\underline{\mathrm{Hom}}_C(A, B) \to \underline{\mathrm{Hom}}_{\mathcal{D}}(A, B)$ is required to be an isomorphism in $\mathcal{M}$ for all A, B. However, the role played by the simplicial enrichment in the example of the hammock localization of a relative category suggests that one is not interested in the Hom-simplicial-sets themselves, but only in the homotopy types of spaces they represent. Hence one defines:

Definition 3.3. A simplical functor $F\colon C \to \mathcal{D}$ of simplicial categories is an equivalence if it is

1. *essentially surjective*, i.e. every object in $\mathcal{D}$ is *equivalent*, i.e. isomorphic in $Ho(\mathcal{D})$, to an object in the image of F
2. *fully faithful*, i.e. the maps $\mathrm{map}_C(A, B) \to \mathrm{map}_{\mathcal{D}}(A, B)$ are *weak equivalences* (instead of isomorphisms) of simplicial sets

Note that in particular an equivalence $F\colon C \to \mathcal{D}$ of simplicial categories induces an equivalence $Ho(F)\colon Ho(C) \to Ho(\mathcal{D})$ of usual categories, but the condition of being an equivalence is stronger than that: For having an equivalence of homotopy categories it would be enough to demand that F induces isomorphisms $\pi_0\mathrm{map}_C(A, B) \to \pi_0\mathrm{map}_{\mathcal{D}}(FA, FB)$ between the sets of connected components $\pi_0\mathrm{map}_C(A, B)$, resp. $\pi_0\mathrm{map}_{\mathcal{D}}(FA, FB)$, of the mapping spaces, while for an equivalence of simplicial categories one asks for isomorphisms induced on π_0 and on all homotopy groups.

Definition 3.4. An $(\infty, 1)$-category is a simplicial category. A morphism of $(\infty, 1)$-categories is a simplicially enriched functor.

This definition allows to keep things simple and suffices for the purposes of this article. We insist however that with this definition the category of simplicial categories has to be seen as a relative category itself (with the class of equivalences just defined) – different simplicial categories may define equivalent $(\infty, 1)$-categories and it is only the simplicial categories up to equivalence that we are interested in. Roughly, we could also have defined an $(\infty, 1)$-category as an equivalence class of simplicial categories, with respect to the equivalence relation generated by the above notion of equivalence of simplicial categories. For more discussion on this see Section 3.5.1.

The name can be explained as follows: One thinks of the 0-simplices in the mapping spaces of a simplicial category as the morphisms (or "1-morphisms"), of the 1-simplices as homotopies between morphisms (or "2-morphisms"), of 2-simplices as homotopies between homotopies (or "3-morphisms") and so on, so that one has n-morphisms for every $n \in \mathbb{N}$. This explains the "∞" in the term "$(\infty, 1)$-category". Now homotopies between functions can always be inverted (if there is a homotopy from a function f to a function g, then there also is a homotopy from g to f and the composition of the two homotopies is homotopic to the identity), so that all n-morphisms for $n \geq 2$ are invertible — only the morphisms up to level 1 are actual directed morphisms, while the higher morphisms are (witnesses of) equivalences. This explains the "1" in the term "$(\infty, 1)$-category". More generally and (n, k)-category is a category which has higher morphisms up to level n, all of which are invertible from level $k + 1$ onwards. An $(\infty, 1)$-category is also sometimes called "a homotopy theory".

3.5. Models and computation of homotopy (co)limits

There can be very different looking but equivalent simplicial categories. One sees such simplicial categories as *presentations* of $(\infty, 1)$-categories, just like one can define a group by a presentation via generators and relations. A(n isomorphism class of a) group can be defined by very different looking presentations and it can in practice be very difficult to determine whether the groups given by two presentations are isomorphic. In our context one uses the term *model*, rather than "presentation".

Likewise one can see a relative category as a model (or presentation) of an $(\infty, 1)$-category, since it gives rise to a simplicial category via the hammock localization. This is the way in which $(\infty, 1)$-categories arise most commonly from usual mathematics. We have already seen an example of two different presentations of the same $(\infty, 1)$-category: The categories of simplical sets and of topological spaces, with their respective classes of weak equivalences, are presentations of the same $(\infty, 1)$-category, commonly called "the $(\infty, 1)$-category of spaces" (we gave the functors inducing this equivalence, but only hinted at the essential surjectivity part).

Now while the datum of a relative category allows to formulate the notions of homotopy (co)limits (as well as that of derived functors) it does not help in constructing them in concrete cases or to determine whether they exist. To this end one frequently employs models which are not just categories with a class of weak equivalences, but which are endowed with an additional auxiliary structure.

Probably the best kind of model one can ask for is a *model category*: A model category is a tuple (C, W, Fib, Cof) consisting of a category C with finite limits and colimits and three classes of morphisms, W (weak equivalences), Fib ("fibrations") and Cof ("cofibrations"), which are related by several axioms (often one asks for additional structure, such as "factorization functors"). This extra structure allows concrete constructions of homotopy limits and colimits and also the construction of adjunctions of $(\infty, 1)$-categories by the means of usual category theory.

A gentle introduction to model categories is [19], a more complete one is the book [29]. One downside of model categories is that it is hard to establish the existence of a model structure on a category. The category of logics and flexible morphisms of Chapter 2.2 for example has no chance of bearing the structure of a model category, simply because it lacks (co)limits.

A less demanding kind of model is given by the notion of *cofibration category*, see [46]. This is, roughly, half a model category structure, having only a class of cofibrations and weak equivalences, and only requiring the existence of special colimits. In a cofibration category homotopy colimits can be constructed explicitly and by the results of Szumiło in [49] every $(\infty, 1)$-category with all homotopy colimits has a presentation by a cofibration category. The category of logics and flexible morphisms of Chapter 2.2 carries such a structure.

There are several other kinds of models, such as Baues cofibration categories and semi-model categories, which meet different purposes and one has to see in each particular situation whether such a kind of model exists and is useful.

3.5.1. The homotopy theory of homotopy theories. Above we endowed the category of simplicial categories and simplicial functors with a class of weak equivalences. Thus the category of simplicial categories becomes itself a relative category. We also have a notion of weak equivalence of relative categories: This a functor $F\colon (C, W) \to (\mathcal{D}, W')$ satisfying $F(W) \subseteq W'$ and inducing an equivalence $L^H(C, W) \to L^H(\mathcal{D}, W')$ of simplicial categories on hammock localizations.

The resulting two relative categories, that of simplicial categories and that of relative categories, are again equivalent as relative categories. In fact on both, the category of simplicial categories and the category of relative categories, there are model structures and the equivalence can be given in a highly structured way (see [5], [4]).

There are several other ways of encoding $(\infty, 1)$-categories, where the emphasis is not on extra structure allowing constructions internal to an $(\infty, 1)$-category, but rather on relating $(\infty, 1)$-categories to each other, constructing new $(\infty, 1)$-categories from old ones and formulating and recognizing properties of $(\infty, 1)$-categories. For a survey of some such settings see [7]. The setting with the best developed theory is that of *quasicategories*, featuring, for example, $(\infty, 1)$-categorical notions of and theorems about fibrations, accessible and locally presentable categories, toposes, sketches and algebraic theories. A gentle introduction is [25], a good further introduction is the first chapter of [32] and the main references are the books [40] and [39].

4. $(\infty,1)$-categories of logics

In this section we consider the categories of logics from section 2.2 as $(\infty,1)$-categories. We will do this in the two ways given in Sections 3.3.1 and 3.3.4. For both we need to fix a notion of weak equivalences of logics.

4.1. Weak equivalences

We place ourselves in a category $\mathcal{L}og$ of logics as in section 2.2

Definition 4.1. 1. If L is a logic and $\varphi,\psi \in \mathrm{Fm}(L)$ are formulas satisfying $\{\varphi\} \vdash_L \psi$ and $\{\psi\} \vdash_L \varphi$, we write $\varphi \dashv\vdash_L \psi$ and call the formulas *logically equivalent*.

2. A morphism of logics $f: L = (S_L, \vdash_L) \to (S_{L'}, \vdash_{L'}) = L'$ is called a *homotopy equivalence* if there exists a morphism $g: L' \to L$ (a "homotopy inverse") such that for all $\varphi \in \mathrm{Fm}(L)$ we have $\varphi \dashv\vdash_L (g \circ f)(\varphi)$ and for all $\psi \in \mathrm{Fm}(L')$ we have $\psi \dashv\vdash_{L'} (f \circ g)(\psi)$.
3. A morphism of logics $f: L \to L'$ is called a *weak equivalence*, if $\Gamma \vdash_L \varphi \Leftrightarrow f(\Gamma) \vdash_{L'} f(\varphi)$ (i.e. it is a "conservative translation") and if for every formula φ in the target there exists a formula in the image of f with the same arity as φ which is logically equivalent to φ (it has "dense image").

Proposition 4.2. *1. In any category of logics with idempotent consequence relations, homotopy equivalences are weak equivalences.*

2. In any category of substitution invariant logics over $\mathcal{S}ig$ (i.e. with flexible morphisms), weak equivalences are homotopy equivalences.

Proof. 1. Let $f: L \to L'$ be a homotopy equivalence. Choose a homotopy inverse g. Any formula $\psi \in \mathrm{Fm}(L')$ is logically equivalent to $(f \circ g)(\psi)$, which is in the image, hence f has dense image. If $\Gamma \vdash \phi$ for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(L)$, then, since f is a translation, we have $\Gamma \vdash \varphi \Rightarrow f(\Gamma) \vdash f(\varphi)$. Conversely, since g is a translation we have $f(\Gamma) \vdash f(\varphi) \Rightarrow g(f(\Gamma)) \vdash g(f(\varphi))$. For every $\gamma \in \Gamma$ we have $\gamma \vdash g(f(\gamma))$ and also $g(f(\varphi)) \vdash \varphi$. Thus we have $\Gamma \vdash g(f(\Gamma)) \vdash g(f(\varphi)) \vdash \varphi$. Hence, by idempotence, $\Gamma \vdash \varphi$.

2. Let $f: L = (S_L, \vdash_L) \to (S_{L'}, \vdash_{L'}) = L'$ be a weak equivalence. To construct a homotopy inverse $g: L' \to L$, choose for every n-ary generating connective $c(x_1, \ldots, x_n)$ of L' a formula $\varphi \in \mathrm{Fm}(L)$ with $c(x_1, \ldots, x_n) \dashv\vdash_{L'} f(\varphi)$ (which exists since f has dense image). This defines a morphism of signatures $g: S_{L'} \to S_L$, which by construction satisfies $f(g(c(x_1, \ldots, x_n))) \dashv\vdash_{L'} c(x_1, \ldots, x_n)$ for all generating connectives. By substitution invariance it follows that $f(g(\psi)) \dashv\vdash_{L'} \psi$ for all formulas $\psi \in \mathrm{Fm}(L')$. In particular this holds for formulas of the form $\psi = f(\varphi)$ for any $\varphi \in \mathrm{Fm}(L)$, i.e. we have $f(g(f(\varphi))) \dashv\vdash_{L'} f(\varphi)$. By conservativity of f we conclude $g(f(\varphi)) \dashv\vdash_L \varphi$. □

Corollary 4.3. *In a category $\mathcal{L}og$ of of idempotent, substitution invariant logics the classes of weak equivalences and homotopy equivalences coincide.*

Remark 4.4. For Tarskian logics the homotopy equivalences have been characterized in [10, Prop. 4.3] as those (flexible) morphisms $f: L \to L'$ for which there exists a morphism $g: L' \to L$, such that f, g induce mutual inverse morphisms between the lattices of theories of L, L'.

Remark 4.5. The comparison of notions of weak equivalences and homotopy equivalences (those which have a morphism in the opposite direction that becomes an inverse in the homotopy category) is a standard theme in abstract homotopy theory. The coincidence of the two classes can be phrased as saying that every object is "fibrant" and "cofibrant". Indeed, the category $\mathcal{H}ilb$ bears the structure of a so-called cofibration category and it is true that every object of $\mathcal{H}ilb$ is fibrant and cofibrant in the sense of cofibration categories (see [46]).

Remark 4.6. Note that the two classes of weak equivalences and of homotopy equivalences no longer coincide, even for idempotent and substitution invariant logics, if we consider them on the category $\mathcal{L}og^{strict}$ of logics and strict morphisms. The reason is that to define the homotopy inverse in the proof of Proposition 4.2.2 we needed to map primitive connectives to derived connectives. Indeed, consider the two presentations of classical propositional logic $CPL_1 := \langle \wedge, \neg \mid \text{rules} \ldots \rangle$ and $CPL_2 := \langle \wedge, \neg, \vee, \rightarrow \mid \text{rules} \ldots \rangle$ of the introduction. Clearly we have a conservative inclusion $CPL_1 \to CPL_2$ which has moreover dense image, since every formula of classical propositional logic is equivalent to one built from just the connectives $\wedge$ and $\neg$. So this inclusion is a weak equivalence. A homotopy inverse, or indeed any translation from CPL_2 to CPL_1, in the category $\mathcal{L}og^{strict}$ would have to map the connective $\vee$ to $\wedge$, since the latter is the only primitive connective of CPL_1 of arity 2, but this is impossible since there are rules satisfied by $\vee$ but not by $\wedge$. We will come back to the relation of $\mathcal{L}og^{strict}$ and $\mathcal{L}og$ in section 4.3.1.

4.2. The hammock localization of $\mathcal{H}ilb$

With definition 4.1 we have made the category $\mathcal{H}ilb$ of Hilbert systems into a relative category, and hence gained an $(\infty, 1)$-category via hammock localization, which we will denote by $\mathcal{H}ilb_{hamm}$. We actually have more than just a relative category:

Theorem 4.7 ([1]). *In the category $\mathcal{H}ilb$ of Hilbert systems denote by W the class of weak equivalences of Definition 4.1 and by Cof the class of translations whose underlying signature morphism maps generating connectives injectively to generating connectives. Then the triple $(\mathcal{H}ilb, W, Cof)$ satisfies the axioms of a cofibration category in the sense of* [46].

The proof of this theorem is out of the scope of this article. We just give a hint of the kind of things one has to do: One of the axioms requires that every morphism factors as a cofibration followed by a weak equivalence. The proof of this for the category $\mathcal{H}ilb$ proceeds in close analogy to that for the category of topological spaces, namely by constructing "mapping cylinders":

Sketch of a proof of the factorization axiom: To factorize a translation $f\colon L = (S_L, \vdash_L) \to (S_{L'}, \vdash_{L'}) = L'$, define an intermediate logic $\widetilde{L}$ with the signature $S_{\widetilde{L}} := S_L \coprod S_{L'}$, so that the formulas of $\widetilde{L}$ are mixed from the connectives of L and L'. In particular we have the linguistic fragments $\mathrm{Fm}(L) \subseteq \mathrm{Fm}(\widetilde{L}) \supseteq \mathrm{Fm}(L')$. The consequence relation on $\mathrm{Fm}(\widetilde{L})$ is generated by the rules of $\vdash_L$ on $\mathrm{Fm}(L)$, the rules of $\vdash_{L'}$ on $\mathrm{Fm}(L')$ and by the rules $\varphi \dashv\vdash_{\widetilde{L}} \varphi'$ for every pair of formulas $\varphi, \varphi' \in \mathrm{Fm}(L)$ which arise from each other by replacing occurrences of connectives from L by their image under f or vice versa. This makes the linguistic fragment $\mathrm{Fm}(L) \subseteq \mathrm{Fm}(\widetilde{L})$ equivalent to its image under f in $\mathrm{Fm}(L')$.

Now we have an obvious cofibration $L \to \widetilde{L}$ which is just the inclusion $\mathrm{Fm}(L) \subseteq \mathrm{Fm}(\widetilde{L})$ and a translation $\widetilde{L} \to L'$ given by mapping the connectives from $\mathrm{Fm}(L') \subseteq \mathrm{Fm}(\widetilde{L})$ to themselves and those of $\mathrm{Fm}(L) \subseteq \mathrm{Fm}(\widetilde{L})$ to their image under f. The latter map is a homotopy equivalence, with a homotopy inverse given by the inclusion $\mathrm{Fm}(L') \subseteq \mathrm{Fm}(\widetilde{L})$.

Corollary 4.8. *The* $(\infty, 1)$*-category* $\mathcal{H}ilb_{hamm}$ *has all homotopy colimits of small diagrams.*

Proof. An $(\infty, 1)$-category for which there exists a presentation by a cofibration category, has all homotopy colimits, see [46], [49]. □

A proof of Theorem 4.7 will appear in [1], together with applications to the combination of logics via homotopy colimits: The extra structure of cofibrations gives an easy construction of homotopy colimits by means of those 1-categorical colimits which do exist in $\mathcal{H}ilb$. Also the concrete choice of cofibrations allows to transfer the usual preservation results for properties of logics under fibring (like existence of implicit connectives, position in the Leibniz hierarchy etc.) to the combination of logics through homotopy colimits.

4.3. The 2-categorical localization of $\mathcal{L}og$

Our categories of logics from Section 2.2 are naturally enriched in preorders: We can define a preorder on $\mathrm{Hom}_{\mathcal{L}og}(L, L')$ by

$$f \leq g \;:\Leftrightarrow\; \forall \varphi \in \mathrm{Fm}(L) \colon f(\varphi) \vdash g(\varphi)$$

Since preorders can be regarded as categories, this makes every category enriched in preorders into a 2-category.

Recall that we defined the equivalences of a 2-category to be those 1-morphisms $f \colon L \to L'$ for which there exists a 1-morphism $g \colon L' \to L$ and 2-isomorphisms $f \circ g \simeq id_{L'}$, $g \circ f \simeq id_L$. In our context the existence of these 2-isomorphisms simply means that $f(g(\psi)) \dashv\vdash_{L'} \psi$ and $g(f(\varphi)) \dashv\vdash_L \varphi$. Thus the notion of homotopy equivalence from Definition 4.1 is exactly the notion of equivalence coming from the structure of 2-category.

We can now perform the construction of a simplicial category from Section 3.3.4 with the 2-category just defined: Pass to the maximal subgroupoids of the hom-categories and then take their nerves. The maximal subgroupoids are simply the categories having the set of translations as objects and having a unique isomorphism between two translations f and g whenever $\forall \varphi \in \mathrm{Fm}(L) \colon f(\varphi) \dashv\vdash g(\varphi)$ holds. In this case we also say that f and g are *homotopic*. We will apply this to the categories $\mathcal{L}og$ and $\mathcal{L}og^{strict}$ and call the resulting simplicial categories $\mathcal{L}og_{2-cat}$ and $\mathcal{L}og^{strict}_{2-cat}$.

Proposition 4.9. *The simplicial categories* $\mathcal{L}og_{2-cat}$ *and* $\mathcal{L}og^{strict}_{2-cat}$ *are equivalent, in the sense of Section 3.4, to their respective homotopy categories.*

Proof. Since the hom-categories are preorders, the maximal subgroupoids are equivalent to discrete categories: Any two objects are either uniquely isomorphic or live in distinct connected components. Since the nerve functor sends equivalences of categories to weak equivalences of simplicial sets, we can replace the hom-categories by actual discrete categories, namely the set of connected components of the groupoids. By definition, the nerve

functor sends discrete categories to discrete simplicial sets, therefore $\mathcal{L}og_{2-cat}$ is equivalent to a simplicial category with discrete mapping spaces. Its homotopy category is constructed by taking π_0 (the set of connected components) of each mapping space. Since the mapping spaces are already weakly equivalent to discrete spaces, they are equivalent to their sets of conected components, i.e. the functor $\mathcal{L}og_{2-cat} \to Ho(\mathcal{L}og_{2-cat})$ is fully faithful in the sense of simplicial categories and hence an equivalence.
The same reasoning goes through for $\mathcal{L}og^{strict}_{2-cat}$. □

As we emphasized, homotopy (co)limits are almost never (co)limits in the homotopy category, but here this is the case.

Corollary 4.10. *Homotopy (co)limits in $\mathcal{L}og_{2-cat}$ (resp. $\mathcal{L}og^{strict}_{2-cat}$) are (co)limits in $Ho(\mathcal{L}og_{2-cat})$ (resp. $Ho(\mathcal{L}og^{strict}_{2-cat})$).*

Proof. Homotopy limits of discrete spaces, i.e. sets, are discrete again and are their limits in the category of sets: The inclusion of the sub-$(\infty, 1)$-category of discrete spaces into the $(\infty, 1)$-category of all spaces is right adjoint to the functor π_0 which takes a space to its set of connected components — therefore it preserves homotopy limits, see [40, 5.5.6.5].

Given a diagram $\mathbf{D}$ in $\mathcal{L}og_{2-cat}$, its homotopy limit was defined through the weak equivalence of mapping spaces $\operatorname{holim}_{\mathbf{D}} \operatorname{map}_{Hilb_{2-cat}}(-, d) \simeq \operatorname{map}_{\mathcal{L}og_{2-cat}}(-, \operatorname{holim}_{\mathbf{D}} d)$ where the left hand homotopy limit was taken in spaces. Now we have the chain of equivalences

$$\begin{aligned}
\lim_{\mathbf{D}} \operatorname{Hom}_{Ho(\mathcal{L}og_{2-cat})}(-, d) &\simeq \operatorname{holim}_{\mathbf{D}} \operatorname{Hom}_{Ho(\mathcal{L}og_{2-cat})}(-, d)\\
&\simeq \operatorname{holim}_{\mathbf{D}} \pi_0 \operatorname{map}_{\mathcal{L}og_{2-cat}}(-, d)\\
&\simeq \operatorname{holim}_{\mathbf{D}} \operatorname{map}_{\mathcal{L}og_{2-cat}}(-, d)\\
&\simeq \operatorname{map}_{\mathcal{L}og_{2-cat}}(-, \operatorname{holim}_{\mathbf{D}} d)\\
&\simeq \pi_0 \operatorname{map}_{\mathcal{L}og_{2-cat}}(-, \operatorname{holim}_{\mathbf{D}} d)\\
&= \operatorname{Hom}_{Ho(\mathcal{L}og_{2-cat})}(-, \operatorname{holim}_{\mathbf{D}} d)
\end{aligned}$$

Since this is a weak equivalence of discrete spaces, it is a bijection of sets and hence identifies $\operatorname{holim}_{\mathbf{D}} d$ as the limit of the diagram in $Ho(\mathcal{L}og_{2-cat})$.

The proof for homotopy colimits is completely analogous (note that homotopy *colimits* in $\mathcal{L}og_{2-cat}$ turn to homotopy *limits* of mapping spaces when mapping out of them, so the same remarks about homotopy limits of discrete spaces apply). □

In the following two statements we notice that homotopy equivalences in $\mathcal{L}og$ and $\mathcal{L}og^{strict}$ are characterized by the fact that they induce equivalences of mapping spaces.

Lemma 4.11. *A homotopy equivalence $z\colon X \to Y$ in $\mathcal{L}og$ (resp. $\mathcal{L}og^{strict}$) induces an equivalence of mapping spaces $z_*\colon map(A, X) \to map(A, Y)$*

Proof. The homotopy equivalence z has a homotopy inverse $z'\colon Y \to X$ with $z \circ z' \dashv\vdash id$ and $z' \circ z \dashv\vdash id$; this means literally that z'_* becomes an inverse after applying π_0. Since the mapping spaces of $\mathcal{L}og_{2-cat}$ (resp. $\mathcal{L}og^{strict}$) are homotopy discrete, this already means that the map z_* is a weak equivalence. □

Proposition 4.12. *Let $f\colon L \to L'$ be a morphism of logics such that for all logics H the induced map $f_* := (f \circ -)\colon map(H, L) \to map(H, L')$ is a weak equivalence of mapping spaces. Then f is a homotopy equivalence of logics.*

Proof. From the hypothesis in particular we get a weak equivalence $f_*\colon \mathrm{map}(L', L) \to \mathrm{map}(L', L')$, hence the identity morphism $id_{L'}$ is in the connected component of some morphism in the image, i.e. there is a morphism $g\colon L' \to L$ such that $f \circ g\colon L' \to L \to L'$ satisfies $(f \circ g) \dashv\vdash id_{L'}$, i.e. we get a left inverse up to homotopy g. We will show that we also have $(g \circ f) \dashv\vdash id_L$ and hence f is a homotopy equivalence.

For every logic H there is the diagram

$$\begin{array}{ccc} \mathrm{map}(H, L') & \xrightarrow{g_*} & \mathrm{map}(H, L) \\ & \underset{(f\circ g)_*}{\overset{\sim}{\searrow}} & \Big\downarrow{\scriptstyle \sim}\, f_* \\ & & \mathrm{map}(H, L') \end{array}$$

Since two of the three arrows are weak equivalences, so is the third, hence $g\colon L' \to L$ satisfies the hypothesis of the proposition and by what we already proved we get a left inverse $h\colon L \to L'$ to g with $g \circ h \dashv\vdash id_L$.

Now we know $f = f \circ id_L \dashv\vdash f \circ g \circ h \dashv\vdash id_{L'} \circ h = h$ and hence f is a left and right homotopy inverse. □

For a category $\mathcal{L}og$ of idempotent, substitution invariant logics we know that weak equivalences and homotopy equivalences coincide, and that hence weak equivalences can be detected on mapping spaces. In homotopy theoretical terms this can be seen as another incarnation of the fact that all objects of such a category $\mathcal{L}og$ are fibrant and cofibrant.

Remark 4.13. On the category $\mathcal{L}og^{strict}$ the 2-categorical notion of equivalence is that of homotopy equivalence. We also have the notion of weak equivalence and there are strictly more weak equivalences than homotopy equivalences. Weak equivalences do in general not induce equivalences of mapping spaces as we can once again see from the example of the two presentations of classical propositional logic CPL_1 with underlying signature $\{\wedge, \neg\}$ and CPL_2 with underlying signature $\{\wedge, \neg, \vee, \to\}$: $\mathrm{map}^{strict}(CPL_2, CPL_1) \to \mathrm{map}(CPL_2, CPL_1)$ is not surjective on connected components, since there is no strict translation equivalent to the flexible translations that are equivalences $CPL_2 \to CPL_1$. However, every logic L is weakly equivalent in $\mathcal{L}og^{strict}$ to a logic $Q(L)$ such that weak equivalences into $Q(L)$ induce equivalences of mapping spaces, see Lemma 4.19 below.

4.3.1. $\mathcal{L}og^{strict}_{2-cat}$ versus $\mathcal{L}og_{2-cat}$.

Convention 4.14. From now on we will suppose that the logics of $\mathcal{L}og$ have the property of idempotence.

We will relate the simplicial categories $\mathcal{L}og^{strict}_{2-cat}$ and $\mathcal{L}og_{2-cat}$ via the adjunction of Proposition 2.6. First we need to extend these functors from signatures to the corresponding categories of logics.

Let $L = (S_L, \vdash_L)$ be a logic. Recall from Definition 2.5 that the signature $Q(S_L)$ is defined by $Q(S_L)_n := \{c_\varphi \mid \varphi \in \mathrm{Fm}(S_L)[x_1, \ldots, x_n]\} \cong \mathrm{Fm}(S_L)[x_1, \ldots, x_n]$. We have an inclusion of signatures $s\colon S_L \to Q(S_L)$, $g \mapsto g(x_1, \dots x_n)$ given by considering the old n-ary generating connectives g of S_L as formulas $g(x_1, \dots, x_n)$ in $\mathrm{Fm}(S_L)[x_1, \dots, x_n]$. This signature morphism induces an inclusion of sets of formulas $s\colon \mathrm{Fm}(S_L) \to \mathrm{Fm}(Q(S_L))$.

Definition 4.15. Let $L = (S_L, \vdash_L)$ be a logic. We define $Q(L)$ to be the logic over the signature $Q(S_L)$ with the consequence relation generated by $s_*(\vdash_L)$ (i.e. the rules of L imported via s) and the rules $\{\psi \dashv\vdash \varphi\}$ for every pair of formulas which arise from each other by replacing connectives of the form c_φ with the corresponding formulas φ or vice versa.

Remark 4.16. If the logics in our category $\mathcal{L}og$ are substitution invariant and congruential, then it is enough to take the consequence relation generated by the rules $s_*(\vdash_L)$ and $\{c_\varphi(x_1, \dots, x_n) \dashv\vdash \varphi(x_1, \dots, x_n)\}$. The rules for more complex formulas from Definition 4.15 then become derivable by substitution invariance and congruentiality.

Lemma 4.17. *In $Q(L)$ every formula $\varphi(x_1, \dots, x_n)$ is equivalent to a formula $c(x_1, \dots, x_n)$ where c is a generating connective.*

Proof. By definition of the consequence relation on $Q(L)$, every formula of $Q(L)$ is logically equivalent to a formula φ of $s(\mathrm{Fm}(L)) \subseteq \mathrm{Fm}(Q(S_L))$, obtained by replacing all occurrences of the new connectives c_ψ with ψ. But this formula $\varphi \in \mathrm{Fm}(L)$ is itself logically equivalent to the generating connective c_φ of $Q(L)$. □

Lemma 4.18. *There are flexible homotopy equivalences $r : Q(L) \leftrightarrows L : s$.*

Proof. The inclusion $s\colon L \to Q(L)$ is a homotopy equivalence with homotopy inverse $r\colon Q(L) \to L$ given by sending the connectives of S_L to themselves and the connective c_φ to φ. Thus r takes a formula and replaces every ocurrence of a connective c_φ by the corresponding formula φ. Clearly s respects the consequence relation. To see the same for r, note that, for $\Gamma \cup \{\varphi\} \subseteq \mathrm{Fm}(Q(L))$ satisfying $\Gamma \vdash \gamma$, we have $r(\Gamma) \dashv\vdash \Gamma \vdash \varphi \dashv\vdash r(\varphi)$, hence by idempotence $r(\Gamma) \vdash r(\varphi)$. The composition $r \circ s$ is the identity and $(s \circ r)(\varphi)$ is logically equivalent to φ by definition of the consequence relation on $Q(L)$. □

Lemma 4.19. *Let H, L be logics. The inclusion $map^{strict}(H, Q(L)) \to map(H, Q(L))$ is an equivalence of simplicial sets.*

Proof. By Lemma 4.17 every n-ary formula φ of $Q(L)$ is equivalent to the formula $c_\varphi(x_1, \dots, x_n)$. Hence a (flexible) morphism $f\colon H \to Q(L)$ is homotopic to the morphism $\tilde{f}\colon H \to Q(L)$ defined on generating connectives by $c \mapsto c_{f(c)}(x_1, \dots, x_n)$, which is a strict morphism. Therefore the map in question becomes a bijection after applying π_0. This is enough since the mapping spaces are homotopy discrete. □

Our aim in this subsection is to show Theorem 4.26, saying that logics and flexible morphisms form a reflective sub-$(\infty, 1)$-category of logics and strict morphisms. A crucial step is the following theorem of Mariano and Mendes:

Theorem 4.20 ([42, Thm 2.12, Mariano/Mendes])**.** *The adjunction of signatures of Theorem 2.6 lifts to an adjunction $i : \mathcal{L}og^{strict} \rightleftarrows \mathcal{L}og : Q$ of logics.*

Since this adjunction respects homotopies between logics, it lifts further to an adjunction of $(\infty, 1)$-categories:

Proposition 4.21. *The functors* $i : \mathcal{L}og^{strict}_{2-cat} \rightleftarrows \mathcal{L}og_{2-cat} : Q$ *form an adjunction of* $(\infty, 1)$*-categories.*

Proof. It is clear that the maps of sets $\mathrm{Hom}_{\mathcal{S}ig^{strict}}(S, S') \to \mathrm{Hom}_{\mathcal{S}ig}(iS, iS')$ and $\mathrm{Hom}_{\mathcal{S}ig}(S, S') \to \mathrm{Hom}_{\mathcal{S}ig^{strict}}(Q(S), Q(S'))$ obtained from the functors i and Q of Definition 2.5 map translations to translations.

From Theorem 2.6 we have an adjunction on the level of signatures and from this the natural isomorphism $\mathrm{Hom}_{\mathcal{S}ig}(i(S), S') \to \mathrm{Hom}_{\mathcal{S}ig^{strict}}(S, Q(S'))$ which sends a flexible morphism $(f_n : S_n \to \mathrm{Fm}(S')[x_1, \ldots, x_n])_{n \in \mathbb{N}}$ to itself (but becoming a strict morphism to $Q(S')$), in other words it is given by postcomposition with the map $S' \to Q(S')$.

By Theorem 4.20 this lifts to an adjunction $i : \mathcal{L}og^{strict} \rightleftarrows \mathcal{L}og : Q$ of logics, i.e. the isomorphism restricts to an isomorphism $\mathrm{Hom}_{\mathcal{L}og}(i(L), L') \to \mathrm{Hom}_{\mathcal{L}og^{strict}}(L, Q(L'))$ between the sets of translations: Indeed, by Lemma 4.18 the morphism of logics $L \to Q(L)$ is a homotopy equivalence in $\mathcal{L}og$, hence by idempotence and Lemma 4.2 a weak equivalence and in particular conservative, and so a morphism of signatures $iS \to S'$ is a translation if and only if $iS \to S' \to Q(S')$ is.

Since homotopic translations get mapped to homotopic translations, this extends to a morphism $\mathrm{map}(iL, L') \to \mathrm{map}^{strict}(L, Q(L'))$ of mapping spaces which can be seen to be an equivalence from the diagram

$$\begin{array}{ccc}
\mathrm{map}(iL, L') & \xrightarrow{s_*} & \mathrm{map}^{strict}(L, Q(L')) \\
\| & & \downarrow \sim \\
\mathrm{map}(L, L') & \xrightarrow[s_*]{\sim} & \mathrm{map}(L, Q(L'))
\end{array}$$

Here the right vertical arrow is the equivalence of Lemma 4.19 and the lower horizontal arrow is an equivalence since the equivalence $s : L \to Q(L)$ of Lemma 4.18 induces an equivalence on mapping spaces by Lemma 4.11. Therefore the upper horizontal arrow must be an equivalence, too. □

Lemma 4.22. *The map of simplicial sets* $map(Q(H), Q(L)) \to map(H, Q(L))$*,* $f \mapsto f \circ (H \to Q(H))$ *is an equivalence.*

Proof. Again it is enough to show that applying π_0 induces a bijection. Injectivity: A morphism $Q(H) \to Q(L)$ is, up to equivalence, determined by its restriction to $H \subseteq Q(H)$, since every formula of $Q(H)$ is equivalent to one of H. Surjectivity: Any morphism $f : H \to Q(L)$ can be extended to $Q(H)$ by sending the additional generating connectives c_φ ($\varphi \in \mathrm{Fm}(H)$) to $f(\varphi)$. □

We denote by $Q(\mathcal{L}og)$ the image of the functor Q in $\mathcal{L}og^{strict}$ and we will for the rest of the subsection suppress the subscript "$2 - cat$".

Lemma 4.23. *The functor* $Q \circ i\,|_{Q(\mathcal{L}og)} : Q(\mathcal{L}og) \to Q(\mathcal{L}og)$ *is an equivalence of* $(\infty, 1)$*-categories.*

Proof. Essential surjectivity: We have to show that any object in the target category $Q(\mathcal{L}og)$ is equivalent to one in the image, i.e. one of the form $Q(Q(L))$. This is the case because the unit map $Q(L) \to Q(Q(L))$ is a weak equivalence of logics in $\mathcal{L}og^{strict}$ by Lemma 4.18, idempotence and Lemma 4.2.

Fully faithfulness: We have to show that the morphism of simplicial sets $\mathrm{map}^{strict}(Q(H), Q(L)) \to \mathrm{map}^{strict}(QQ(H), QQ(L))$ is an equivalence. We know from Lemma 4.19 that this is equivalent to showing that we have an equivalence of flexible mapping spaces $\mathrm{map}(Q(H), Q(L)) \to \mathrm{map}(QQ(H), QQ(L))$. We have a commutative diagram

$$\begin{array}{ccc} \mathrm{map}(Q(H), Q(L)) & \longrightarrow & \mathrm{map}(QQ(H), QQ(L)) \\ & \searrow^{\sim} & \downarrow{\sim} \\ & & \mathrm{map}(Q(H), QQ(L)) \end{array}$$

where the vertical arrow is the equivalence of Lemma 4.22 and the diagonal arrow is induced by the homotopy equivalence s of Lemma 4.18 and hence an equivalence by 4.11. Since two of the three arrows are equivalences, so is the third. □

Proposition 4.24. *The inclusion functor* $i\mid_{Q(\mathcal{L}og)}\colon Q(\mathcal{L}og) \to \mathcal{L}og$ *is an equivalence of simplicial categories.*

Proof. Essential surjectivity: This is the fact (Lemma 4.18) that $L \to Q(L)$ is an equivalence of logics. Fully faithfulness: This is Lemma 4.19. □

Lemma 4.25. *The category* $Q(\mathcal{L}og)$ *is a reflective* $(\infty, 1)$*-subcategory of* $\mathcal{L}og^{strict}$.

Proof. We know from Proposition 4.21 combined with Proposition 4.24 that the inclusion of $Q(\mathcal{L}og)$ into $\mathcal{L}og^{strict}$ has a left adjoint, and from Lemma 4.23 that their composition is an idempotent functor. This is what defines a reflective subcategory. □

Theorem 4.26. $\mathcal{L}og$ *is (equivalent to) a reflective sub-*$(\infty, 1)$*-category of* $\mathcal{L}og^{strict}$ *via the reflection functor* Q.

Proof. This is Lemma 4.25 combined with Proposition 4.24. □

4.3.2. Homotopy limits in $\mathcal{L}og_{2-cat}$. We will now show how homotopical thinking can lead one to the construction of homotopy limits in $\mathcal{L}og_{2-cat}$. By Corollary 4.10, homotopy limits are limits in $Ho(\mathcal{L}og_{2-cat})$ and the existence of these (for congruential Hilbert systems) has been established by Mariano and Mendes in [42, Thm 2.33].

Here we will give direct constructions of several kinds of homotopy limits to show some of the homotopy theoretical flavour. For example in our construction of homotopy equalizers there occur logics resembling the path spaces of the corresponding constructions in topology. This shows how signatures can be tailored to fullfill the needs of particular constructions while keeping them as small as possible. A more general approach would be to use always the construction $Q(-)$ of Definition 2.5.

We start with the easiest case.

Homotopy terminal objects. We have seen in Example 2.7 that the category $\mathcal{S}ig$ has no terminal object, hence by Proposition 2.24 neither does $\mathcal{L}og_{2-cat}$. In the homotopical world things look better:

Proposition 4.27. *The category $\mathcal{L}og_{2-cat}$ has homotopy terminal objects.*

Proof. By Corollary 4.10 it is enough to show that there is a terminal object in $Ho(\mathcal{L}og_{2-cat})$. For this take the signature S which has one generating connective of each arity (or any other signature which produces formulas of any arity) and endow it with the maximal consequence relation. Clearly for any logic L there is a translation $L \to (S, \vdash_{max})$ given on signatures by mapping each n-ary generating connective to the n-ary generating connective of S. Since in $(S, \vdash_{max})$ any two formulas are logically equivalent, all morphisms into $(S, \vdash_{max})$ are equal in the homotopy category. □

The reader who wishes to test this statement for another category of logics satisfying the assumptions of Remark 2.20, should make sure that for the maximal consequence relation actually all formulas are equivalent. This is certainly true for those of Convention 2.19.

Homotopy equalizers. Given two parallel arrows $f, g\colon L \rightrightarrows M$ in $\mathcal{L}og_{2-cat}$, their homotopy equalizer should be a morphism $e\colon E \to L$ such that $(f \circ e)(\varphi) \dashv\vdash_M (g \circ e)(\varphi)$ (i.e. instead of asking for equality, we ask for pointwise logical equivalence) and such that each $h\colon H \to L$ with $(f \circ h)(\varphi) \dashv\vdash_M (g \circ h)(\varphi)$ factorizes uniquely up to homotopy through e (this is usually not equivalent to the homotopy limit condition expressed through mapping spaces, but here it is since the mapping spaces are discrete).

Thus we would like to simply take as equalizer the set of formulas $\{\varphi \in \mathrm{Fm}(L) \mid f(\varphi) \dashv\vdash_M g(\varphi)\}$ and endow it with the consequence relation of L restricted to this subset. But this set of formulas is in general not a free algebra over some signature. On the other hand if one tries to take as generating connectives of the homotopy equalizer just those c which satisfy $f(c(x_1, \dots, x_n)) \dashv\vdash_M g(c(x_1, \dots, x_n))$ then one might miss formulas φ with $f(\varphi) \dashv\vdash_M g(\varphi)$. However, since homotopy equalizers are invariant under equivalences, we can substitute L by an equivalent logic for which the second solution works.

Definition 4.28. Let $L = (S_L, \vdash_L)$ be a logic, $f, g\colon L \rightrightarrows M$ be two morphisms in $\mathcal{L}og_{2-cat}$.
1. The signature $S^{(f,g)}$ is defined by

$$S_n^{(f,g)} := (S_L)_n \coprod \{c_\varphi \mid \varphi \in \mathrm{Fm}(L),\ f(\varphi) \dashv\vdash_M g(\varphi)\}$$

i.e. by taking as generating connectives those of S_L plus one extra n-ary connective c_φ for every $\varphi \in \mathrm{Fm}(L)[x_1, \dots, x_n]$ whose images under f and g are logically equivalent.
2. The logic $L^{(f,g)}$ is the logic over the signature $S^{(f,g)}$, defined by endowing $\mathrm{Fm}(S^{(f,g)})$ with the consequence relation generated by $\vdash_L$ (for the formulas of the linguistic fragment generated by S_L) and by the rules $\{\psi \dashv\vdash \varphi\}$ for every pair of formulas which arise from each other by replacing connectives of the form c_φ with the corresponding formulas φ or vice versa.

Thus the logic $L^{(f,g)}$ contains a copy of L as well as lots of new generating connectives which are equivalent to formulas of L which become logically equivalent under f, resp. g. As before, for substitution invariant, congruential logics it is enough to demand as generating rules the rules $\{c_\varphi \dashv\vdash \varphi \mid \varphi \in \mathrm{Fm}(S_L)\}$.

Lemma 4.29. *The inclusion $i\colon L \to L^{(f,g)}$ is a homotopy equivalence with homotopy inverse $r\colon L^{(f,g)} \to L$ given by sending the connectives of S_L to themselves and the connective c_φ to φ.*

Proof. This is precisely parallel to Lemma 4.18 where the inclusion $L \to Q(L)$ was shown to be a homotopy equivalence. Both morphisms respect the consequence relation by definition of the consequence relation on $L^{(f,g)}$. The composition $r \circ i$ is the identity and $i \circ r$ is the identity on $\mathrm{Fm}(L) \subseteq \mathrm{Fm}(L^{(f,g)})$ and maps the formula $c_\varphi(x_1, \ldots, x_n)$ to φ, which is logically equivalent. □

Definition 4.30. Define the signature S_E as the subsignature of $S^{(f,g)}$ given by the connectives $\{c_\varphi \mid \varphi \in \mathrm{Fm}(L),\ f(\varphi) \dashv\vdash_M g(\varphi)\}$. Then the logic $E^{(f,g)}$ is the logic over the signature S_E endowed with the strongest consequence relation such that the inclusion $S_E \to S^{(f,g)}$ becomes a translation. We denote the inclusion by $e\colon E^{(f,g)} \to L^{(f,g)}$.

Proposition 4.31. *The diagram*

$$E^{(f,g)} \xrightarrow{r \circ e} L \underset{g}{\overset{f}{\rightrightarrows}} M$$

is a homotopy equalizer diagram in $\mathcal{L}og_{2-cat}$.

Proof. We will directly show the criterion for mapping spaces, i.e. for any logic H the map of (homotopy discrete) simplicial sets

$$\mathrm{map}(H, E^{(f,g)}) \xrightarrow{(r \circ e)_*} \mathit{hoequ}\Big(\mathrm{map}(H, L) \underset{g_*}{\overset{f_*}{\rightrightarrows}} \mathrm{map}(H, M)\Big)$$

is a weak equivalence.

Since by Lemma 4.11 a homotopy equivalence $z\colon X \to Y$ in $\mathcal{L}og$ induces an equivalence of mapping spaces, by Lemma 4.29 we have a weak equivalence $\mathrm{map}(H, L) \leftrightarrows \mathrm{map}(H, L^{(f,g)})$. Also by Proposition 4.9 we have equivalences $\mathrm{map}(H, L^{(f,g)}) \to \pi_0 \mathrm{map}(H, L^{(f,g)})$, and hence the following equivalences of diagrams of the shape $\bullet \rightrightarrows \bullet$:

$$\begin{array}{ccc}
\mathrm{map}(H, L) & \underset{g_*}{\overset{f_*}{\rightrightarrows}} & \mathrm{map}(H, M) \\
\simeq\downarrow & & \downarrow\simeq \\
\mathrm{map}(H, L^{(f,g)}) & \underset{g_*}{\overset{f_*}{\rightrightarrows}} & \mathrm{map}(H, M) \\
\simeq\downarrow & & \downarrow\simeq \\
\pi_0\, \mathrm{map}(H, L^{(f,g)}) & \underset{g_*}{\overset{f_*}{\rightrightarrows}} & \pi_0\, \mathrm{map}(H, M)
\end{array}$$

Since homotopy equalizers do not change up to equivalence upon replacing a diagram by an equivalent one, and the homotopy equalizer of a diagram of sets is just the equalizer, we have the following equivalences:

$$\begin{aligned} hoequ\Big(\,\mathrm{map}(H,L) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \mathrm{map}(H,M)\Big) &\simeq equ\Big(\,\pi_0\mathrm{map}(H,L^{(f,g)}) \overset{f_*}{\underset{g_*}{\rightrightarrows}} \pi_0\mathrm{map}(H,M)\Big) \\ &\simeq \{\overline{h} \in \pi_0\mathrm{map}(H,L^{(f,g)}) \mid \overline{f} \circ \overline{h} = \overline{f} \circ \overline{h}\} \\ &\simeq \{\overline{h} \in \pi_0\mathrm{map}(H,L^{(f,g)}) \mid f \circ h \dashv\vdash g \circ h\} \end{aligned}$$

where $\overline{(-)}$ denotes the equivalence class of a morphism in the quotient $\pi_0\,\mathrm{map}(-,-)$.

The map $\mathrm{map}(H,E^{(f,g)}) \to \{\overline{h} \in \pi_0\,\mathrm{map}(H,L^{(f,g)}) \mid f \circ h \dashv\vdash g \circ h\}$, since it goes into a set, factors through $\pi_0\mathrm{map}(H,E^{(f,g)})$ and we have to show that $\pi_0\,\mathrm{map}(H,E^{(f,g)}) \to \{\overline{h} \in \pi_0\,\mathrm{map}(H,L^{(f,g)}) \mid f \circ h \dashv\vdash g \circ h\}$ is a bijection.

Surjectivity: Every $h \in \mathrm{Hom}_{\mathcal{L}og}(H,L^{(f,g)})$ such that $f \circ h \dashv\vdash g \circ h$ is homotopic to a morphism going into the linguistic fragment of E: For each connective $c \in S_H$ we have $(f \circ h)(c(x_1,\dots,x_n)) \dashv\vdash (g \circ h)(c(x_1,\dots,x_n))$ and thus h is homotopic to $h'\colon H \to L^{(f,g)}$ defined by $c \mapsto c_{h(c)} \in S_{E^{(f,g)}} \subseteq S_{L^{(f,g)}}$.

Injectivity: If $h,h'\colon H \to E^{(f,g)}$ go after composition with e to the same morphism $(H \to E^{(f,g)} \to L^{(f,g)}) \in \pi_0\,\mathrm{map}(H,L^{(f,g)})$, then $(e \circ h)(\varphi) \dashv\vdash_{L^{(f,g)}} (e \circ h')(\varphi)$ for all $\varphi \in \mathrm{Fm}(H)$. The fact that $\vdash_{E^{(f,g)}}$ is the restriction of $\vdash_{L^{(f,g)}}$ to $\mathrm{Fm}(S_{E^{(f,g)}})$ means that e is conservative, so $h(\varphi) \dashv\vdash_{E^{(f,g)}} h'(\varphi)$ for all $\varphi \in \mathrm{Fm}(H)$, hence $\overline{h} = \overline{h'} \in \pi_0\,\mathrm{map}(H,E^{(f,g)})$. □

We still note that the homotopy equalizer can be characterized by $\mathrm{map}(H,E^{(f,g)}) \simeq \pi_0\,\mathrm{map}(H,E^{(f,g)}) \cong \{\overline{h} \in \pi_0\,\mathrm{map}(H,L^{(f,g)}) \mid f \circ h \dashv\vdash g \circ h\} \simeq \{\overline{h} \in \pi_0\,\mathrm{map}(H,L) \mid f \circ h \dashv\vdash g \circ h\}$, where the latter bijection comes from the fact that every translation $H \to L^{(f,g)}$ is equivalent to one going into L, seen as a sublogic of $L^{(f,g)}$.

In the construction of homotopy equalizers we could have used $Q(L)$ instead of $L^{(f,g)}$ with almost the same proofs, but we thought it to be instructive to show how one can construct a logic tailored to the problem at hand. If we demand more properties from the logics of the category $\mathcal{L}og$, these specialized logics can become much smaller than the universal solution via $Q(L)$.

One can construct homotopy pullbacks in an entirely similar way to the construction of homotopy equalizers given here. We leave this to reader, but the existence of homotopy pullbacks follows from Theorem 4.39 below which asserts the homotopy cocompleteness of $\mathcal{L}og_{2-cat}$.

Homotopy products. Given a family of logics $(L_i = (S_i, \vdash_i) \mid i \in I)$ a first tentative construction of the product logic might be as follows: Take as signature $S := \prod S_i$, so that the generating connectives are tuples of generating connectives from the S_i. This signature has projection maps $pr_i\colon S \to S_i$, defined on generating connectives by $(c_i)_{i\in I} \mapsto c_i$. Then define a logic $\prod_i L_i$ by endowing $\mathrm{Fm}(S)$ with the strongest consequence relation such that all these projection maps are translations — this is the consequence relation given by $(\Gamma_i) \vdash (\varphi_i) \Leftrightarrow \forall i \in I\, \Gamma_i \vdash_{L_i} \varphi_i$.

We would now have to show that $\langle pr_i \mid i \in I\rangle\colon \pi_0\,\mathrm{map}(H,\prod_i L_i) \to \prod_i \pi_0\,\mathrm{map}(H,L_i)$ is a bijection, but surjectivity in general fails, as seen by the following example:

Example 4.32. Take all the L_i to be the same logic $L = (S, \vdash)$ with signature S generated by a single unary connective $\Box$ and the rule $\Box x \vdash x$. Then $\mathrm{Fm}(L) = \{\Box^i x_k \mid i, k \in \mathbb{N}\}$, where $\Box^i := \Box\Box\ldots\Box$ (i times). This logic has the feature that no two formulas are equivalent, unless one can be obtained from the other by a substitution of variables with other variables. The product $\prod_{n\in\mathbb{N}} \pi_0\,\mathrm{map}(L, L)$ contains the family (f_n) with $f_n\colon L \to L$ given by mapping $\Box \mapsto \Box^n$. Now if $\langle pr_i \mid i \in I\rangle\colon \pi_0\,\mathrm{map}(H, \prod_i L_i) \to \prod_i \pi_0\,\mathrm{map}(H, L_i)$ were surjective, there would have to be (up to equivalence) a formula in $\prod_{n\in\mathbb{N}} L$ of the form $(\Box^n x)_{n\in\mathbb{N}}$. This can not be the case since our tentatively defined product signature is generated by the single connective $(\Box)_{n\in\mathbb{N}}$. Even if we allow a unary "identity connective" which does not change a formula when it is applied, our product signature would be generated by $\{(\triangle_n)_{n\in\mathbb{N}} \mid \triangle_n \in \{id, \Box\}\}$, i.e. by tuples which in each place have either the identity connective or $\Box$. Formulas of this signature would, as usual, be finite combinations of the generating connectives and any such connective would have a highest n such that a $\Box^n$ occurs in some place of the tuple.

The reason behind this failure is that products of free algebras are not in general free again, as already remarked in Chapter 2. The solution to the problem, much as in the case of homotopy equalizers, is to substitute the logics L_i whose homotopy product we want to form, by the equivalent logics $Q(L_i)$ which have the feature that every formula $\varphi(x_1, \ldots, x_n)$ is equivalent to a formula $c(x_1, \ldots, x_n)$ where c is a generating connective.

Definition 4.33. Let $L_i = (S_{L_i}, \vdash_{L_i}), i \in I$ be a family of logics. We define $\prod S_{L_i}$ to be the signature with $(\prod S_{L_i})_n := \{(c_i)_{i\in I} \mid c_i \in (S_{L_i})_n\}$ i.e. the signature whose n-ary generating connectives are tuples of n-ary generating connectives of the S_{L_i}. There are obvious projection maps $pr_i\colon \prod S_{L_i} \to S_{L_i}$. We define the consequence relation $\vdash_{\prod L_i}$ over $\prod S_{L_i}$ to be the biggest consequence relation such that all projection morphisms become translations. Finally we define the product logic to be $\prod L_i = (\prod S_{L_i}, \vdash_{\prod L_i})$

Remark 4.34. The logic $\prod L_i$ is the product of the logics L_i in the category $\mathcal{L}og^{strict}$, by the construction recipe given in Proposition 2.24.

Temporary convention 4.35. In the next two statements we establish the existence of homotopy products in special categories $\mathcal{L}og$. We make the distinction between the following two cases:

A. $\mathcal{L}og$ is a category of logics where infima of consequence relations are given by intersection. This means: If $\vdash_i$ are consequence relations over $\mathrm{Fm}(S)$ which are admissible for $\mathcal{L}og$ (i.e. $(S, \vdash_i)$ are objects of $\mathcal{L}og$), then their intersection $(\bigcap_i \vdash_i) \in \mathscr{P}(\mathscr{P}(\mathrm{Fm}(S)) \times \mathrm{Fm}(S))$ is also a consequence relation admissible for $\mathcal{L}og$.
B. $\mathcal{L}og$ is the full subcategory of one of the categories from A, given by logics which additionally are finitary.

Thus categories $\mathcal{L}og$ of type A include for example $\mathcal{L}og^{(Tarsk)}$ and $\mathcal{L}og^{(subst,Tarsk)}$ and categories $\mathcal{L}og$ of type B include $\mathcal{L}og^{(fin,Tarsk)}$ and $\mathcal{H}ilb$. We insert this digression for special categories, because this admits a nice concrete construction of homotopy products (only finite ones in case B).

The impatient reader can skip to Proposition 4.38 which guarantees the existence of homotopy products for general categories $\mathcal{L}og$, using results from abstract homotopy theory which will be sketched in Section 4.3.3.

Lemma 4.36. *Suppose that either the objects of $\mathcal{L}og$ are finitary logics (case B) and the family L_i of Definition 4.33 is finite, or that finitariness is not required for the objects of $\mathcal{L}og$ (case A). Then the consequence relation $\vdash_{\prod L_i}$ is given by $\Gamma \vdash_{\prod L_i} \varphi \ :\Leftrightarrow \forall i\ pr_i(\Gamma) \vdash_{L_i} pr_i(\varphi)$*

Proof. In the case (B) of a category of finitary logics $\mathcal{L}og$, the consequence relation was defined as the infimum in the complete lattice of finitary (and possibly substitution invariant, Tarskian, etc.) consequence relations of the inverse images $pr_i^{-1}(\vdash_{L_i})$. The inverse image relations are given by $\Gamma\, pr_i^{-1}(\vdash_{L_i})\varphi \Leftrightarrow pr_i(\Gamma) \vdash pr_i(\varphi)$.

By the proof of [2, Fact 4] this finitary infimum $[inf(\vdash_i)]$ of a family of consequence relations $\vdash_i$ is given by $\Gamma\,[inf(\vdash_i)]\,\varphi \Leftrightarrow \exists\,\Gamma' \subseteq_{finite} \Gamma$ such that $\forall i\colon\ \Gamma' \vdash_i \varphi$.

Since all the $\vdash_i$ are finitary, there exist $\Gamma_i \subseteq_{finite} \Gamma$ such that $\Gamma_i \vdash_i \varphi$. If the family is finite, then the union of these Γ_i is still finite, so that the existence of the $\Gamma' \subseteq_{finite} \Gamma$ is always ensured. This gives the claimed description of the product consequence relation.

In the case (A) where finitariness is not demanded from the objects of $\mathcal{L}og$, the infimum of a family of consequence relations is given by $\Gamma\,[inf(\vdash_i)]\,\varphi \Leftrightarrow \forall i\colon\ \Gamma' \vdash_i \varphi$. □

Proposition 4.37. *The category $\mathcal{L}og_{2-cat}$ has finite homotopy products, if the objects are demanded to be finitary (case B) and all homotopy products if the objects are not demanded to be finitary (case A).*

Proof. Let $L_i = (S_{L_i}, \vdash_{L_i}), i \in I$ be a family of logics, which we suppose to be finite in the first case. We claim that the strict product $\prod Q(L_i)$ of the replaced logics $Q(L_i)$ is a homotopy product of this family. For this we need to show that for any H the map

$$\pi_0\mathrm{map}(H, \prod Q(L_i)) \to \prod_i \pi_0\mathrm{map}(H, L_i), \quad \overline{f} \mapsto (\overline{pr_i \circ f})_{i\in I}$$

is a bijection.

Surjectivity: From Lemma 4.18 we know that $\pi_0\mathrm{map}(H, L_i) \cong \pi_0\mathrm{map}(H, Q(L_i))$, so we can replace the target by $\pi_0\mathrm{map}(H, Q(L_i))$. Given a family $(\overline{f_i})_{i\in I} \in \prod_i \pi_0\mathrm{map}(H, Q(L_i))$ we know from Lemma 4.19 that the $\overline{f_i}$ have representatives f_i given by strict morphisms. Now we can define a preimage f of the family (f_i): If f_i sends a generating connective c of H to a generating connective c_i of $Q(L_i)$, then define f by $f(c) := (c_i)_{i\in I} \in \prod S_{L_i}$. This is clearly a translation and a preimage of the family (f_i).

Injectivity: Suppose we have $\overline{f}, \overline{f'}$ such that $\overline{pr_i \circ f} = \overline{pr_i \circ f'}$. The latter condition means that $(pr_i \circ f)(\varphi) \dashv\vdash (pr_i \circ f')(\varphi)\ \forall\,\varphi$. By Lemma 4.36 this implies $f(\varphi) \dashv\vdash f'(\varphi)\ \forall\,\varphi$, i.e. $\overline{f} = \overline{f'}$. □

End of temporary convention. Now we return to our convention of denoting by $\mathcal{L}og$ any category of idempotent logics named in 2.19, or any category of idempotent logics satisfying the assumptions of 2.20 and admitting a homotopy terminal object (see the remark after Proposition 4.27).

Proposition 4.38. *The category $\mathcal{L}og_{2-cat}$ has all homotopy products.*

Proof. This follows from the fact that $\mathcal{L}og_{2-cat}$ is a reflective $(\infty, 1)$-subcategory of $\mathcal{L}og^{strict}_{2-cat}$ and that $\mathcal{L}og^{strict}_{2-cat}$ has all homotopy products – the latter will be established in the next section by means of a model structure on $\mathcal{L}og^{strict}$. Indeed, our adjoint functors i and Q induce an adjunction on the homotopy categories and by Corollary 4.10 it is enough to establish the existence of products in the homotopy category $Ho(\mathcal{L}og_{2-cat})$. Now it is an exercise in usual category theory that a reflective subcategory of a category with products has products itself. These products are given by forming the product in the ambient category and then applying the reflection functor. Thus the homotopy product of a family L_i is given by $Q(\prod_i L_i)$. □

Theorem 4.39. *The simplicial category $\mathcal{L}og_{2-cat}$ has all homotopy limits.*

Proof. We have constructed homotopy products (including a homotopy terminal object) and homotopy equalizers. By the dual of [40, Prop. 4.4.3.2] one can build homotopy limits from homotopy equalizers and homotopy products, analogously to the non-homotopical statement in classical category theory. That the limits of the cited proposition really correspond to homotopy limits as explained in section 3 is the content of [40, Prop. 4.2.4.1]. □

4.3.3. Homotopy colimits in $\mathcal{L}og_{2-cat}$. It would be possible to construct homotopy colimits by hand as we did for homotopy limits. Instead we sketch a proof of the existence of homotopy colimits by different means, to give a feeling for how homotopy theoretical machinery can be brought into play for solving such questions. For this we use Theorem 4.26, saying that $\mathcal{L}og_{2-cat}$ is a reflective $(\infty, 1)$-subcategory of $\mathcal{L}og^{strict}_{2-cat}$, and show that $\mathcal{L}og^{strict}_{2-cat}$ has all homotopy colimits. First we invoke the following theorem:

Theorem 4.40 ([35, Thm. 3.3, S. Lack])**.** *Let C be a finitely complete and finitely cocomplete category enriched in the category* **Cat** *of categories. Then there is a model structure on C, such that the weak equivalences are precisely the 2-categorical equivalences in the sense of Section 3.3.4. This model structure has the feature that every object is fibrant and cofibrant and is compatible with the enrichment.*

We can apply this to the category $\mathcal{L}og^{strict}$ enriched in the maximal subgroupoids of the usual Hom-preorders. This is not entirely trivial, as the (co)completeness condition of the theorem is to be understood in the enriched sense: Apart from the completeness and cocompleteness of $\mathcal{L}og^{strict}$, which we know from Proposition 2.24 (or [2, Prop. 2.11]) one has to show that $\mathcal{L}og^{strict}$ is tensored and cotensored, in the sense of [34], over the category **Cat** of categories. This can be done by techniques similar to those we used in the construction of homotopy limits in $\mathcal{L}og^{strict}_{2-cat}$.

From the theorem we then get a **Cat**-enriched model category in the sense of [35, Section 2.2]. Recall that the $(\infty, 1)$-category corresponding to a model category is given by taking the subcategory of fibrant and cofibrant objects and applying the hammock localization. Here, since all objects are fibrant and cofibrant this is simply the hammock localization of the whole category $\mathcal{L}og^{strict}$ with respect to the homotopy equivalences. As the $(\infty, 1)$-category corresponding to a model category has all homotopy limits and colimits, we have established the homotopy (co)completeness of this hammock localization.

However, we need to know that the *2-categorical* localization $\mathcal{L}og^{strict}_{2-cat}$ is cocomplete, so we still have to relate this to the hammock localization. The notion of **Cat**-enriched model category is such that we get, when we apply the nerve functor to the hom-groupoids (and here it is important that they are groupoids), a *simplicial model category* in the sense of Quillen, see [29, Def. 4.2.18]. Now by [18, Prop. 4.8] for a simplicial model category the simplicial subcategory of fibrant and cofibrant objects – which here is exactly $\mathcal{L}og^{strict}_{2-cat}$ – is equivalent to the hammock localization considered before. Hence we conclude that the $(\infty, 1)$-category $\mathcal{L}og^{strict}_{2-cat}$ has all homotopy limits and colimits.

Finally now we can now use Theorem 4.26, saying that $\mathcal{L}og_{2-cat}$ is a reflective $(\infty, 1)$-subcategory of $\mathcal{L}og^{strict}_{2-cat}$: The homotopy colimit of a diagram in $\mathcal{L}og_{2-cat}$ can be constructed by seeing it, via the functor Q, as a diagram in $\mathcal{L}og^{strict}_{2-cat}$, forming its homotopy colimit there, and then applying the reflection functor, which as a left adjoint preserves colimits.

A detailed elaboration of these arguments, and further exploration of the Lack model structure on $\mathcal{L}og^{strict}_{2-cat}$ will be given in a future work.

5. Vista

5.1. Further studies of categories of logics

In the previous chapter we gave two natural constructions of $(\infty, 1)$-categories of logics, and showed how to explore some of their properties with the examples of Hilbert systems. Many natural questions about $\mathcal{L}og_{2-cat}$ and $\mathcal{L}og_{hamm}$, and their strict versions, remain to be pursued. Most of these questions should not be hard to tackle, as we find ourselves in a rather easy region of the realm of abstract homotopy theory. The most immediate question is the following:

Question 5.1. Are the simplicial categories $\mathcal{L}og_{2-cat}$ and $\mathcal{L}og_{hamm}$ equivalent?

One approach to proving an equivalence is to simply write down an enriched functor between the two simplicial categories and prove it to be an equivalence. This would involve an analysis of the mapping spaces of the hammock localization, which would be interesting in its own right, as it might reveal criteria for determining whether two translations are homotopic. Other approaches could proceed by constructing appropriate models of the two $(\infty, 1)$-categories, which are more easily comparable than the simplicial categories. For example one could try to find model categories presenting both $(\infty, 1)$-categories and produce a Quillen equivalence between them. This allows to stay in the realm of usual categories. Candidates for such models appear below.

On the category $\mathcal{L}og^{strict}$ we have the two different notions of homotopy equivalence and weak equivalence and we can form the hammock localizations with respect to both of these notions of equivalence. From Lemma 4.18 and Lemma 4.19 we know that every logic is weakly equivalent to one such that every translation into it is homotopic to a strict translation. So it is natural to ask:

Question 5.2. Are $L^H(\mathcal{L}og^{strict}, W \cap \mathcal{L}og^{strict})$ and $L^H(\mathcal{L}og, W) = \mathcal{L}og_{hamm}$ equivalent $(\infty, 1)$-categories (where W denotes the class of weak equivalences)?

Again one can either directly try to construct an equivalence of simplicial categories or approach the question via models. The second approach is related to the next question.

Recall from Section 4.3.3 that there is a model structure on $\mathcal{L}og^{strict}$ whose equivalences are the homotopy equivalences. One may ask if this model structure admits a *Bousfield localization*: A Bousfield localization of a model category is a new model structure on the same underlying category which has additional weak equivalences and fewer fibrations, see [40, A.3.7]. The $(\infty, 1)$-category presented by the Bousfield localization is a reflective $(\infty, 1)$-subcategory of the $(\infty, 1)$-category presented by the original model structure.

Question 5.3. Is there a Bousfield localization of the Lack model structure on the category $\mathcal{H}ilb^{strict}$ whose weak equivalences are the weak equivalences of logics from Definition 4.1.3 ?

The construction of Bousfield localizations is for example available for so-called combinatorial model categories by a general theorem of J. Smith, see [3]. These are model categories which are "cofibrantly generated" (see [40, Def. A.2.6.1(2-3)]) and whose underlying category is locally presentable. The latter is the case for $\mathcal{H}ilb^{strict}$ by [2, Thm. 2.16].

Here is a further candidate category for a model for the $(\infty, 1)$-category $\mathcal{L}og_{hamm}$. In several places in the literature there have appeared proposals to consider logics not just on absolutely free algebras, but to instead consider consequence relations on arbitrary algebras – note that the definition of consequence relation, Definition 2.11, does not have to be altered for this to make sense. These gadgets have been called *abstract logics* in [9]. An advantage of allowing non-free algebras is that one has a better behaved, e.g. complete and cocomplete, category, which is, for example, the reason that they appeared in the context of fibring of institutions in [11]. A disadvantage is that one introduces objects which one would not commonly perceive as logics.

Question 5.4. Is there a model structure on abstract logics presenting the $(\infty, 1)$-category $\mathcal{L}og_{hamm}$?

The (co)completeness makes it possible in the first place to hope for such a model structure. The weak equivalences would have to be chosen such that every general logic would be weakly equivalent to a logic in the traditional sense and this would make the disadvantage of unusual objects disappear (up to equivalence). One approach: Every algebra is a quotient of an absolutely free algebra and for an algebra with consequence relation $(A, \vdash)$ one can choose such an absolutely free algebra $F \twoheadrightarrow A$ and endow it with the biggest consequence relation such that the quotient map becomes a translation (then terms which become equal in A are logically equivalent in F). A different candidate for the underlying category of a model category presenting $\mathcal{L}og_{hamm}$ would be the category of operads proposed in [2, Section 5].

We know from [2, Thm. 2.16] that the 1-category $\mathcal{H}ilb^{strict}$ of Hilbert systems and strict translations is locally finitely presentable. A locally finitely presentable category is a complete category with a small subcategory of *finitely presentable* objects (i.e. the functors corepresented by them commute with filtered colimits), such that every object is a filtered colimit of these. By [2, Prop. 2.15] finitely presentable objects in the case of $\mathcal{H}ilb^{strict}$ are the logics with finitely many generating connectives whose consequence relation is generated by finitely many rules. There is a corresponding notion of presentability for $(\infty, 1)$-categories, see [40, Def. 5.5.0.1].

Conjecture 5.5. The $(\infty, 1)$-category $\mathcal{H}ilb_{hamm}$ is presentable.

From [1] and [2] one can deduce that every logic is a filtered homotopy colimit of the finitely presentable objects in the sense defined above. It would remain to show that these finitely presentable objects are also finitely presentable in the sense of $(\infty, 1)$-categories, i.e. that the mapping space functors corepresented by them commute with filtered homotopy colimits, see [40, Def. 5.3.4.5]. We note that the corresponding conjecture about $\mathcal{H}ilb^{(con)}_{2-cat}$ true by work of Mariano and Mendes: In [42, Thm 2.33] they show that the homotopy category of this category is locally finitely presentable.

All these questions are still centered around our guiding examples of Tarski style abstract logic. Similar investigations to those carried out in the previous sections and proposed in the above questions make sense and would be interesting in other settings of Abstract Logic.

There is, for example, the category $\mathcal{A}lg$ of algebraizable logics in the sense of Blok-Pigozzi [8]. Morphisms of algebraizable logics are translations preserving the so-called algebraizing pairs that come with algebraizable logics, so this is a non-full subcategory of $\mathcal{H}ilb$. Jánossy, Kurucz and Eiben in [31, Def. 3.1.3] define an equivalence relation on the set $\mathrm{Hom}_{\mathcal{A}lg}(A, B)$ of morphisms of algebraizable logics in terms of algebraizing pairs. As for $\mathcal{H}ilb_{2-cat}$, this equivalence relation gives rise to a simplicial category $\mathcal{A}lg_{2-cat}$ with homotopy discrete mapping spaces. By Corollary 4.10, homotopy (co)limits in this simplicial category are precisely (co)limits in the homotopy category. The authors investigate this homotopy category, defined (without mention of a simplicial category) in [31, Def. 3.3]. They show that it is equivalent to a certain category of quasivarieties and that this category has non-empty colimits (the restriction to non-empty diagrams may be necessary; by [2, Remark 3.10] one has to be careful with initial objects). Thus we know that we have homotopy colimits of non-empty diagrams in $\mathcal{A}lg_{2-cat}$. Of course there is also a hammock localization $\mathcal{A}lg_{hamm}$ and one may ask about the relationship between the two. In [2, Thm 3.12] it is shown that the category $\mathcal{A}lg$ is finitely accessible, and one may ask if the same is true in the $(\infty, 1)$-categorical sense [40, Def. 5.4.2.1] for $\mathcal{A}lg_{2-cat}$.

In a similar vein one can explore the categories corresponding to the various levels of the Leibniz hierarchy, see [14] for some of these. The results of [20] on preservation of the position in the Leibniz hierarchy under the formation of strict colimits should prove useful here. A pioneering work in this direction is Mariano and Mendes' study of the category of congruential Hilbert systems [42].

A close variant of the logics treated in Chapters 2 and 4, for which the proofs should go through with almost no modifications is obtained by admitting typed signatures. This allows for example a natural treatment of first order logic by allowing a type of propositions, as we had before, and types of terms, as well as operations like "=" going from pairs of terms to propositions. Note however that first order logics can also be encoded into propositional logics as done in the appendix of [8].

A variant that still studies Tarski style logics, but with a possibly coarser notion of equivalence, is the program of Mariano and Pinto of representation theory for logics [43]. Indeed, [43, Thm. 3.5] shows that their notion of left Morita equivalence is coarser than the notion of weak equivalence studied here and one can expect an interesting relationship between the two coresponding homotopical categories.

Many other notions of logic, translation and equivalence have been proposed, like those for metafibring [13], those of institutions [15] and π-institutions [21], model-theoretic abstract logics [37], logical spaces [23], type theories [30], and many more, and they all give rise to homotopical categories to be explored and compared.

5.2. Invariants of logics

Given the fact that logics live naturally in homotopy theoretical universes, we can ask which other ideas of homotopy theory might apply. Classical homotopy theory studies homotopy invariants of topological spaces. One use of invariants is to discover if two topological spaces are *not* weakly equivalent. One often also tries to compute invariants because of specific information that they contain about a space or a map, not just to merely distinguish them.

Many of these invariants are given by mapping into, or out of, some test object. Let us review the example of singular homology: In Section 3.1 we gave a cosimplicial object $\Delta^\bullet$ in the category **Top** of topological spaces and defined the simplicial set $Sing(X)$ by $Sing(X)_n := Hom_{\mathbf{Top}}(\Delta^n, X)$. Applying the "free abelian group" functor to each of the sets $Sing(X)_n$ one gets an abelian group object in simplicial sets. The homotopy groups of this new simplicial set are the singular homology groups $H_n(X;\mathbb{Z})$. Alternatively one can build a chain complex from a simplicial abelian group by forming alternating sums of the face maps and take the homology of this chain complex.

Simplicial sets. The process of the formation of the simplicial set $Sing(X)$ works for any cosimplicial object in a category, but nothing guarantees that this construction has the good properties of singular homology, like the long exact sequences which make computations feasible. But we do indeed have a natural similar construction for logics.

Definition 5.6. 1. The category of *general logics*, $\mathcal{G}enLog$, is the category whose objects are pairs $(X,\vdash)$, where X is a set and $\vdash$ a consequence relation on it, and whose morphisms are consequence preserving maps of sets.

2. The notions of *homotopy equivalence* and *weak equivalence* of general logics are defined to be morphisms of general logics satisfying the conditions of Definition 4.1. Note that Proposition 4.2 holds here as well.
3. The *homotopy category of general logics* $Ho(\mathcal{G}enLog)$ is defined to be the homotopy category of the 2-categorical localization (with respect to the obvious preorder enrichment, parallel to the one from the beginning of Section 4.3), of $\mathcal{G}enLog$.
4. Denote by $U\colon \mathcal{L}og^{Tarsk} \to \mathcal{G}enLog$ the forgetful functor from Tarskian logics to general logics given by $(S, \vdash) \mapsto (\mathrm{Fm}(S), \vdash)$

Note that the functor U induces a functor between the homotopy categories which we will also denote by $U\colon Ho(\mathcal{L}og^{Tarsk}) \to Ho(\mathcal{G}enLog)$. We introduced the category $\mathcal{G}enLog$, because it contains a natural cosimplicial object:

Definition 5.7. Let D_n ($n \in \mathbb{N}_0$) be the general logic whose underlying set is the n-element set $\{\varphi_0, \dots, \varphi_n\}$ and whose consequence relation is the idempotent, increasing consequence relation generated by the rules $\varphi_i \vdash \varphi_{i+1}$.

Since the consequence relations are required to be idempotent and increasing, the general logics D_n clearly form a cosimplicial object, i.e. the obvious maps of ordered sets which leave out or duplicate a proposition become morphisms of general logics:

Definition 5.8. The cosimplicial object $D^\bullet\colon \Delta \to \mathcal{G}enLog$ is defined on objects by $[n] \mapsto D_n$ and by sending a morphism $f\colon [n] \to [k]$ of ordered sets to the map $D_n \to D_k, \varphi_i \mapsto \varphi_{f(i)}$.

Mapping out of a cosimplicial object produces a simplicial set. We use this to define a tentative invariant of logics:

Definition 5.9. The simplicial set of inferences $Inf_\bullet(L)$ of a Tarskian logic L is defined to be the simplicial set $\mathrm{Hom}_{\mathcal{G}enLog}(D^\bullet, U(L))$. This defines a functor $Inf_\bullet(L)\colon \mathcal{L}og^{Tarsk} \to$ **sSet**.

This simplicial set encodes the implication relations between the formulas of L. Indeed, we have

$$\begin{aligned} &Inf_\bullet(L)_0 \cong \mathrm{Fm}(S_L),\\ &Inf_\bullet(L)_1 \cong \{(\varphi_0, \varphi_1) \in \mathrm{Fm}(S_L) \times \mathrm{Fm}(S_L) \mid \varphi_0 \vdash \varphi_1\},\\ &Inf_\bullet(L)_2 \cong \{(\varphi_0, \varphi_1, \varphi_2) \in \mathrm{Fm}(S_L)^3 \mid \varphi_0 \vdash \varphi_1 \vdash \varphi_2\} \end{aligned}$$

and so on. One might now have the idea of applying homotopy theoretic invariants like homotopy and homology groups to this simplicial set, but this would contain very limited information about the logic: A simplicial set $X_\bullet$ remembers the direction of the edges; the two structure maps $X_1 \rightrightarrows X_0$ can be seen as source and target maps (and indeed this is what they are if X is the nerve of a category) and they are not interchangeable.

The topological invariants of simplicial sets, however, do not distinguish the direction of the edges, for example the nerve of a category and the nerve of its opposite are weakly equivalent simplicial sets. Thus applying topological invariants to $Inf_\bullet(L)$ would amount to only remembering whether formulas are connected by some inference, without distinguishing into which direction it goes. Of course it could still be possible to distinguish non-equivalent logics by these data but a lot of information would be lost.

Instead one should see the simplicial set $Inf_\bullet(L)$ as an object of *directed homotopy theory*. That is, one should see it not as an object of the standard model category of simplicial sets, where simplicial sets are taken to model topological spaces and in which paths have no preferred direction, but rather as an object of the model category of quasicategories where the objects are representing $(\infty, 1)$-categories.

In general, directed homotopy theory is a field where one studies spaces (for example modelled by simplicial sets) with directed paths, which cannot be gone backwards. For such a directed space, instead of the fundamental groupoid, one has a fundamental category, instead of homology groups one has preordered homology groups. These seem to be more promising invariants of logics[2]. A good introductory survey of these ideas is [24].

Dendroidal sets. The simplicial set $Inf_\bullet(L)$ encodes implication relations between single formulas. For finitary logics with a conjunction this should give fairly complete information. In general, however, the consequence relation is between sets of formulas and single formulas and such inferences with many hypotheses are not captured by the invariant $Inf_\bullet(L)$. For finitary logics we can produce a *dendroidal set* capturing such inferences. We will consider the definition of dendroidal sets via broad posets, as in [50] and [51].

Definition 5.10. A *commutative broad poset* is an idempotent, increasing general logic with the following properties:

1. If $\Gamma \vdash \varphi$, then Γ is finite.
2. If $\gamma \in \Gamma$, $\varphi \in \Phi$, $\Gamma \vdash \varphi$ and $\Phi \vdash \gamma$, then $\varphi = \gamma$

A commutative broad poset is *finite*, if the relation $\vdash \subseteq \mathscr{P}(A) \times A$ is a finite set. Note that by increasingness this implies that A is finite.

Definition 5.11. Let $(X, \vdash)$ be a commutative broad poset.
An element $x \in X$ is called a *root*, if there do not exist any $y, x_1, \ldots, x_n \in A$, $y \neq x$ such that $\{x, x_1 \ldots, x_n\} \vdash y$.
An element $x \in X$ is called a *leaf*, if there is no $\Gamma \neq \{x\}$ such that $\Gamma \vdash x$.

Definition 5.12. A commutative broad poset $(X, \vdash)$ is a *dendroidally ordered set*, if

1. it is finite
2. it has a root
3. If $x \in X$ is not a leaf, then there exists a unique $\Gamma \subseteq X$ such that $\Gamma \vdash x$ and with the following property: There exists no $\Gamma' = \{\gamma_1, \ldots, \gamma_n\} \subseteq X$ such that there is a partition $\Gamma = \coprod_i \Gamma_i$ with $\Gamma_i \vdash \gamma_i \, \forall i$

The full subcategory of general logics whose objects are the dendroidally ordered sets is called Ω.

Definition 5.13. A dendroidal set is a functor $\Omega^{op} \to \mathbf{Set}$.

[2] Cubical sets are more common than simplicial sets in directed algebraic topology, but it is also easy to write down a cocubical object in *GenLog* to produce a cubical variant of $Inf_\bullet(L)$

Remark 5.14. Our definition of Ω is a reformulation of [50, Def. 4.1.1] and [51, Def. 3.2] (where no logics are mentioned). The definition of Ω in the literature on dendroidal sets is not usually given in terms of broad posets, but rather in terms of trees and operads, see e.g. [44]. The equivalence between these two definitions is the content of [50, Thm. 4.1.15].

We admit that our definition of Ω is somewhat opaque, but the only thing that matters for now, is that it is a subcategory of the category of general logics.

Definition 5.15. The dendroidal set of inferences $Inf_\Omega(L)$ of a Tarskian logic L is defined to be the dendroidal set $\mathrm{Hom}_{GenLog}(\Omega, U(L))$. This defines a functor $Inf_\Omega(L)\colon \mathcal{L}og^{Tarsk} \to \mathbf{Set}^{\Omega^{op}}$.

Like the category of simplicial sets, the category of dendroidal sets bears several model structures. There is the Cisinski-Moerdijk model structure [12], with the fewest weak equivalences, which, roughly, sees dendroidal sets as encoding families of n-ary operations, closed under composition (operads). This will be the viewpoint that retains the most information about the logic. Further there are the covariant model structure of Heuts [28] which sees dendroidal sets as E_∞-spaces and the stable model structure of Bašić/Nikolaus [6] which sees dendroidal sets as connective spectra.

All of these model structures can be used to associate invariants to logics. In particular the last one can be used to define the algebraic K-theory of a logic, see [45]. It remains to be seen how computable these invariants are, whether e.g. there are long exact sequences induced by cofibrations of logics, and what exactly they capture about a logic. They are likely to get much more interesting, if one enriches the categories of logics via proof theory as in Section 5.3.

Galois style invariants. A further type of invariant of a logic L can be constructed by considering the category $L \downarrow \mathcal{L}og$ of logics receiving a translation from L and associating to it the group of automorphisms of the forgetful functor to $\mathcal{L}og$ or other categories as that of indexed frames, see Section 5.4. The homotopical viewpoint suggests, however, that one should take autoequivalences instead of automorphisms. This can be seen as a version of Galois theory for logics.

In a similar spirit are the Morita style invariants of Mariano and Pinto [43] who, roughly, associate to a logic the categories of algebraizable logics over/under the logic.

5.3. Refined categories of logics from proof theory

The natural enrichment of the categories of logics in preorders lead us to the 2-categorical localizations of Section 4.3. As much as this was an improvement of the corresponding usual categories, the homotopy discrete mapping spaces are not very interesting objects. However, they can be seen as the shadows of richer structures. We just give some sketches and ideas here.

The analogy between proof theory and homotopy theory is as follows: A logic is a space, formulas are points, proofs are paths between the points, transformations of proofs are homotopies between the paths.

The somewhat degenerate situation of homotopy discrete mapping spaces in logic can be seen from this angle. We said that two translations f, g should be declared equivalent if for all φ of the domain, we have $f(\varphi) \dashv\vdash g(\varphi)$. This only asks for *provability* of $f(\varphi)$ from $g(\varphi)$, it does not distinguish different *proofs*. This is like asking, on the topological side, whether two points $f(\varphi)$ and $g(\varphi)$ are in the same path component, while ignoring the different paths. Distinguishing different proofs is a way to get to more interesting enrichments of the categories of logics.

Recall that categories (actually multicategories) of proofs were introduced by Lambek [36]. Given a Hilbert system, presented by a set of deduction rules, one can define a category whose objects are the formulas and whose morphisms are proofs. Let us say for the moment that a proof from a hypothesis φ to a conclusion ψ is a sequence of formulas such that the final formula is ψ and any intermediate formula is either φ or follows from the preceding ones through one of the deduction rules.

We wish to use this for an enrichment of the category of Hilbert systems. We could start by saying that $\mathrm{Hom}_{\mathcal{H}ilb}(L, L')$ is the category whose objects are translations from L to L' and where a morphism $f \to g$ is given by a collection of proofs $\{f(\varphi) \to g(\varphi) \mid \varphi \in \mathrm{Fm}(L)\}$, i.e. we could ask pointwise for morphisms in the proof category of L'. Categorical thinking would, however, demand some kind of coherence between the different morphisms. If we think of a morphism from f to g as something like a natural transformation between functors, then we would ask for the commutativity of a certain diagram: A proof $\varphi \to \psi$ in the proof category of L would be mapped to two proofs $f(\varphi) \to f(\psi)$ and $g(\varphi) \to g(\psi)$ and we could ask that the naturality diagram commutes:

$$\begin{array}{ccc} f(\varphi) & \longrightarrow & g(\varphi) \\ \downarrow & & \downarrow \\ f(\psi) & \longrightarrow & g(\psi) \end{array}$$

This means asking for an equality of proofs, which again seems quite restrictive. On a middle ground we could ask that the two proofs are comparable in some sense, or transformable into each other.

This leaves us with lots of interesting options to choose from, all resulting in different categories. There are directed and symmetric versions of relations between proofs. Choosing the symmetric versions results in a $(2, 1)$-category of logics, while choosing the directed versions results in a $(2, 2)$-category of logics. Here are some examples:

1. Length of the proofs:
   - symmetric: equal length
   - directed: one is longer than the other
2. Normal forms:
   - symmetric: both proofs have the same normal form of some kind
   - directed: One is closer to normal form than the other (e.g. elimination rules occur before introduction rules)
3. Degree of generalizability (Lambek): Suppose we have a proof $p : A \to B$ and the formulas A and B arise by substitution of terms $t_1, \ldots, t_n$ into other formulas

φ, ψ, i.e. $A = \varphi(t_1, \ldots, t_n)$, $B = \psi(t_1, \ldots, t_n)$. Then one can ask whether the proof carries over to the more general situation, i.e. whether there is a proof $\hat{p}(x_1, \ldots, x_n)$: $\varphi(x_1, \ldots, x_n) \to \psi(x_1, \ldots, x_n)$ such that $p = \hat{p}(t_1, \ldots, t_n)$.
   - symmetric: p and q have the same degree of generalizability, i.e. for every "generalization" of A and B the proof p carries over if and only if q carries over
   - directed: For every generalization to which p carries over, q also carries over
4. Required strength of the logic: One proof $p : A \to B$ might use e.g. Modus Ponens, $\varphi \wedge \psi \vdash \varphi$ and $\neg\neg\phi \vdash \phi$ while another proof $q : A \to B$ only uses Modus Ponens. Or one proof might be constructive the other not.
   - symmetric: Both proofs use the same logical strength
   - directed: One uses less than the other.

The list could easily go on, but the point is made, that there are many interesting relations between proofs that one might want to study and they all lead to different categories with more interesting enrichments than before.

Going further, one could try to get a 3-category of logics, i.e. an enrichment in 2-categories. For this consider the example of normal forms: Often a proof can be brought into some normal form by a sequence of elementary steps. These sequences of steps can be seen as 2-morphisms in the proof category. Again one has to ask when two sequences of steps can be considered equivalent and get a 2-category of translations by pointwise application.

In usual logic nothing seems to be naturally coming after that. In Martin-Löf type theory on the other hand, one has identity types and can iterate up to arbitrary levels, which is exactly what inspired the homotopy theoretical semantics used in homotopy type theory. This should lead to the richest, least truncated, higher categories of logics.

We still remark that one can adapt the invariants of logics from the previous section to the setting of these less truncated categories of logics involving some proof theory and that here they might get more of a homotopical flavour than their more truncated companions. We also remark that other natural enrichments can be found, for example an enrichment in multicategories, via the provability relation between sets of formulas and single formulas.

5.4. Comparing paradigms of logic

Another question that gets an interesting twist, once we have turned our categories of logics into higher categories, is that of how the categories corresponding to different formalizations of abstract logic, like institution theory and Tarski style consequence relations, relate to each other. With the extra flexibility of $(\infty, 1)$-categories it seems easier to get adjunctions or equivalences between different such settings.

In another direction there is the work [27] where the authors note that for several formalizations of the notion of logic the resulting category has a forgetful functor to the category of *indexed frames*. A typical way to assess a category with such a forgetful functor is to consider the automorphisms of this functor. In good situations one can reconstruct the category from the base category and the knowledge of these automorphisms, but in any case one can associate in this way a group to a category. In a world of $(\infty, 1)$-categories one should instead consider *autoequivalences*.

6. Conclusion

Our aim was to show that a homotopy theoretical point of view is very natural and appropriate in logic. One is lead to natural constructions and questions and this should be the main reason to adopt such a point of view. A good side effect is, as exemplified by the (co)completeness results of Chapter 4, that things look better than they might otherwise through the lens of usual category theory.

We believe that the rich homotopy theoretical landscape of logics, of which we have unveiled a bit, gives ample confirmation of how fundamental and fruitful Jean-Yves' questions from the beginning of this article really are.

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