z. B. $V = M(2\times 2, \mathbb{R}) \cong \mathbb{R}^4$

$\mathcal{B} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$\mathcal{B}' = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

z. B.: $\text{Lös}(A) \subset K^n$

Beispiel:

① $V = K^n$, $W = K^m$,
$\mathcal{B}$, $\mathcal{C}$ Standardbasen.
Dann $\Phi_{\mathcal{B}} = \mathrm{id}_{K^n}$, $\Phi_{\mathcal{C}} = \mathrm{id}_{K^m}$,
$M^{\mathcal{B}}_{\mathcal{C}}(F) = M(F)$ die übliche F darstellende Matrix.

② $V = \mathbb{R}^2$ mit Basis $\mathcal{B} := \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

$W = \mathbb{R}$ mit Basis $\mathcal{C} := ((2))$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[dr, "\Phi_{\mathcal{B}} = F_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}"'] & & & \mathbb{R} \arrow[dl, "\Phi_{\mathcal{C}} = F_{(2)}"] \\
& \mathbb{R}^2 \arrow[r, "F"] & \mathbb{R} &
\end{tikzcd}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto y$$

$$M(F) = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$\begin{aligned}
M^{\mathcal{B}}_{\mathcal{C}}(F) &= M\left(\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}} \right) \\
&= M(\Phi_{\mathcal{C}}^{-1}) \cdot M(F) \cdot M(\Phi_{\mathcal{B}}) \\
&= (2)^{-1} \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \left(\tfrac{1}{2}\right) \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \tfrac{1}{2} & 0 \end{pmatrix}
\end{aligned}$$

Beweis: Per Def. m_{ij} eindeutig bestimmt durch

$$\left(\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}}\right)(e_j) = \sum_i m_{ij} e_i$$

(Beweis des Hauptsatz, Vorl. 16)

Also

$$F(\underbrace{\Phi_{\mathcal{B}}(e_j)}_{b_j}) = \sum_i m_{ij} \underbrace{\Phi_{\mathcal{C}}(e_i)}_{c_i}$$

$\square$

Beweis:

$$M(m \times n; K) \underset{\substack{\text{(Vorlesung} \\ 16)}}{\cong} \mathrm{Hom}_K(K^n, K^m) \cong \mathrm{Hom}_K(V, W)$$

$$\begin{array}{ccc} M(F) & \longleftarrow & F \\ M & \longmapsto & F_M \end{array} \qquad \begin{array}{ccc} \Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}} & \longleftarrow & F \\ F & \longmapsto & \Phi_{\mathcal{C}} \circ F \circ \Phi_{\mathcal{B}}^{-1} \end{array}$$

$$\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}} \longleftarrow F$$

$$\longmapsto \underbrace{\underbrace{\Phi_{\mathcal{C}} \circ \Phi_{\mathcal{C}}^{-1}}_{id} \circ F \circ \underbrace{\Phi_{\mathcal{B}} \circ \Phi_{\mathcal{B}}^{-1}}_{id}}_{F}$$

$$M_{\mathcal{C}}^{\mathcal{B}}(F) \longleftarrow F$$

$\square$

$$
\begin{tikzcd}
K^n \arrow[rrrrr, "M^{\mathcal{B}}_{\mathcal{C}}(F)"] \arrow[dr, "B"'] \arrow[drr, "\Phi_{\mathcal{B}}", "\cong"'] & & & & & K^m \arrow[dll, "\Phi_{\mathcal{C}}"', "\cong"] \arrow[dl, "C"] \\
& K^N \arrow[ul, dashed, bend left, "L_B"] & V \arrow[l, hook', "i"'] \arrow[r, "F"] & W \arrow[r, hook, "j"] & K^M \arrow[dr, dashed, bend left, "L_{C'}"] & \\
K^n \arrow[rrrrr, "M^{\mathcal{B}'}_{\mathcal{C}'}(F)"'] \arrow[ur, "B'"] \arrow[urr, "\Phi_{\mathcal{B}'}"', "\cong"] & & & & & K^m \arrow[ull, "\Phi_{\mathcal{C}'}", "\cong"'] \arrow[ul, "C'"']
\end{tikzcd}
$$

Beweis: Per Definition:

$$M^{\mathcal{B}}_{\mathcal{C}}(F) = M(\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}})$$

$$M^{\mathcal{B}'}_{\mathcal{C}'}(F) = M(\Phi_{\mathcal{C}'}^{-1} \circ F \circ \Phi_{\mathcal{B}'})$$

Daher

$$M^{\mathcal{B}'}_{\mathcal{C}'}(F) = M(\Phi_{\mathcal{C}'}^{-1} \circ \Phi_{\mathcal{C}} \circ \Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}} \circ \Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'})$$

$$= M(\Phi_{\mathcal{C}'}^{-1} \circ \Phi_{\mathcal{C}}) \cdot \underbrace{M(\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}})} \cdot M(\Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'})$$

$$= M(\Phi_{\mathcal{C}'}^{-1} \circ \Phi_{\mathcal{C}}) \cdot \underbrace{M^{\mathcal{B}}_{\mathcal{C}}(F)} \cdot \underbrace{M(\Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'})}_{L_{\mathcal{B}} \cdot \mathcal{B}'}$$

Es ist $L_{\mathcal{B}} \cdot \mathcal{B} = E_n$

also $F_{L_{\mathcal{B}}} \circ F_{\mathcal{B}} = id_{K^n}$

$$(F_{L_{\mathcal{B}}} \circ i) \circ \underset{\uparrow\ \text{Iso}}{\Phi_{\mathcal{B}}} = id_{K^n}$$

Daher

$$\Phi_{\mathcal{B}}^{-1} = F_{L_{\mathcal{B}}} \circ i$$

$$\Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'} = F_{L_{\mathcal{B}}} \circ \underbrace{i \circ \Phi_{\mathcal{B}'}}$$

$$= F_{L_{\mathcal{B}}} \circ F_{\mathcal{B}'}$$

$$= F_{L_{\mathcal{B}} \cdot \mathcal{B}'}$$

$$\boxed{M\left(\Phi_{\mathcal{B}}^{-1} \circ \Phi_{\mathcal{B}'}\right) = L_{\mathcal{B}} \cdot B'.}$$

Analog $\quad M\left(\Phi_{\mathcal{C}'}^{-1} \circ \Phi_{\mathcal{C}}\right) = L_{\mathcal{C}'} \cdot C.$ $\square$