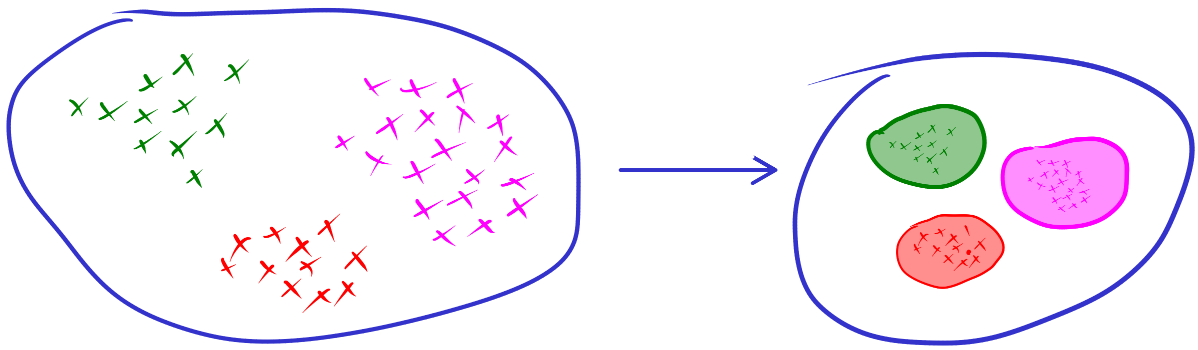
$$\begin{tikzcd}
V \arrow[r, "F"] \arrow[d] & W \\
V/\operatorname{Ker} F \arrow[r, phantom, "\cong"] & \operatorname{Im} F \arrow[u, hook]
\end{tikzcd}$$

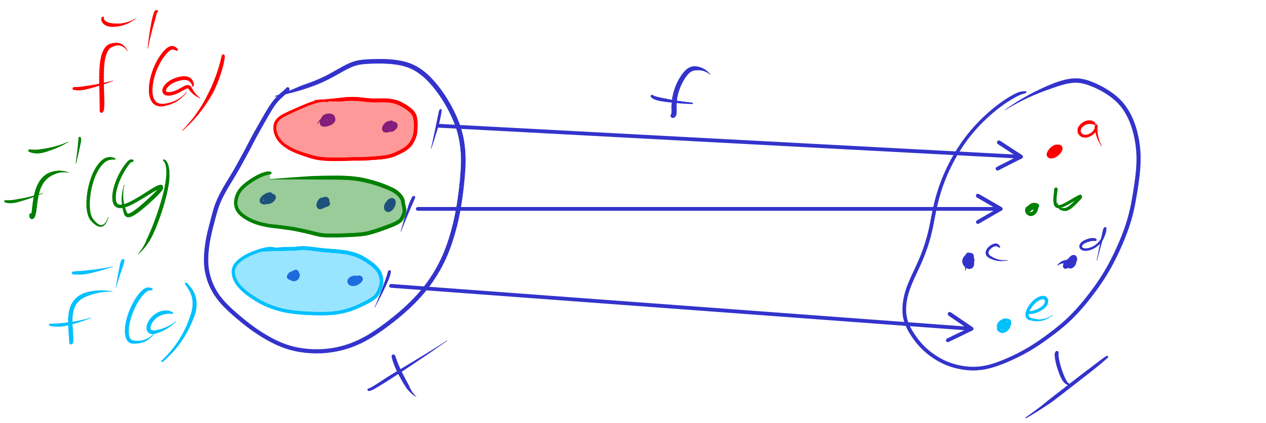


X $\qquad\qquad$ $X/\sim$

$\sim_f$ $\qquad$ $x \sim_f x$

$x_1 \sim_f x_2 \Rightarrow x_2 \sim_f x_1$

$x_1 \sim_f x_2 \sim_f x_3 \Rightarrow x_1 \sim_f x_3$



$\bar{f}^{-1}(c) = \emptyset$

$\bar{f}^{-1}(d) = \emptyset$

<u>Beweis:</u>

$\bar{f}$ wohldefiniert.

Sei $[x] = [x']$. Dann ist

$x \sim_f x'$, d.h. $f(x) = f(x')$.

$\bar{f}$ surjektiv: ✓

$\bar{f}$ injektiv:

Sei $\underbrace{\bar{f}([x])} = \underbrace{\bar{f}([x'])}$. Dann ist

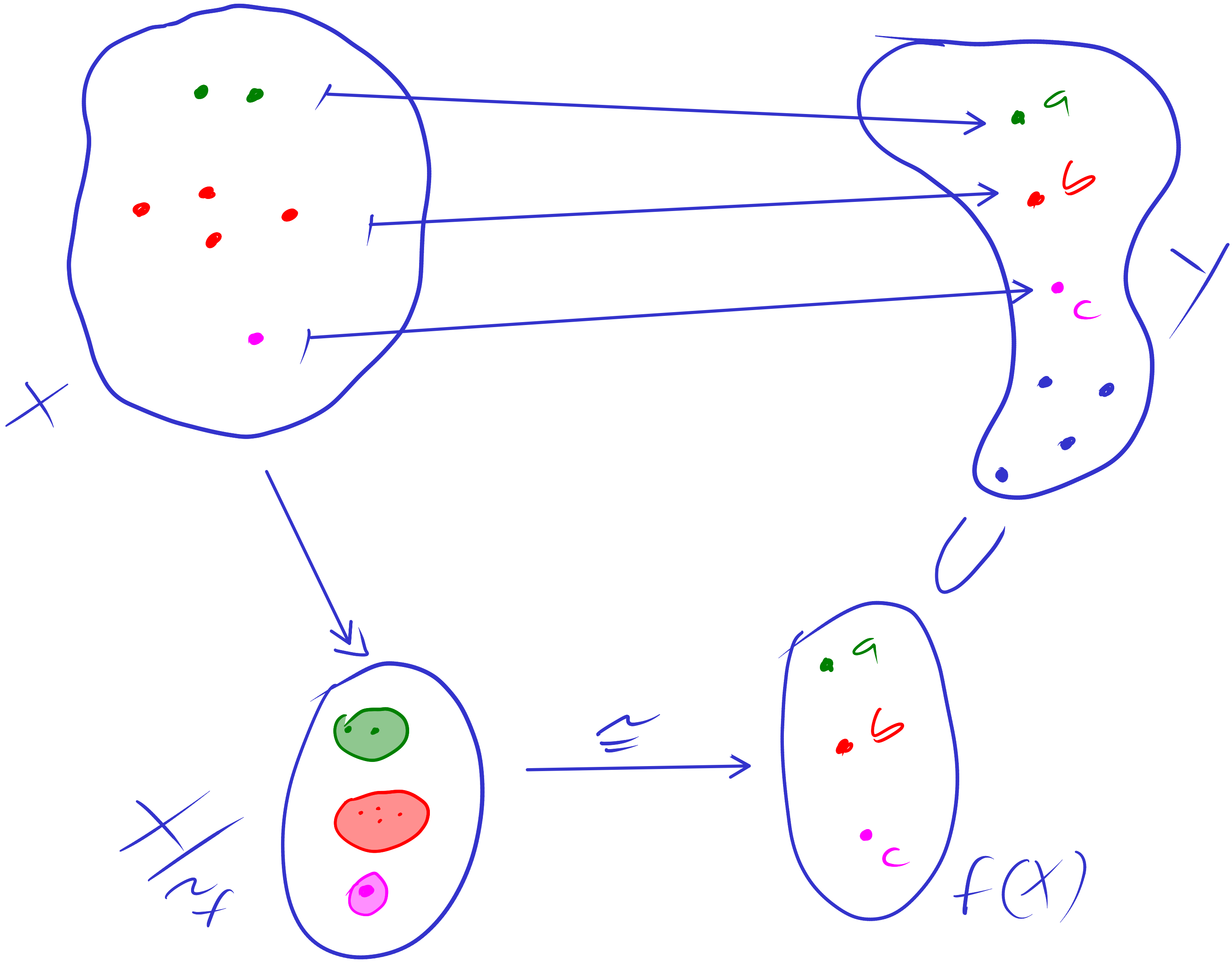
$f(x) = f(x')$,

also $x \sim_f x'$,

also $[x] = [x']$.

$i \circ \bar{f} \circ \pi_{\sim_f} = f$

$\square$



X
Y
a
b
c
X/~f
≅
f(X)

Beweis:

$$x \sim_H x \iff 0 \in H \quad ✓$$

$$x_1 \sim_H x_2 \iff x_1 - x_2 \in H$$

$$\Longrightarrow \underbrace{-(x_1 - x_2)}_{= \, x_2 - x_1} \in H \quad ✓$$

d.h. $x_2 \sim_H x_1$ ✓

$$x_1 \sim_H x_2 \sim_H x_3 : \quad \underbrace{(x_1 - x_2) + (x_2 - x_3)}_{= \, x_1 - x_3} \in H \quad ✓$$

d.h. $x_1 \sim_H x_3$

//

$g \in G$

$$[g] = \{ x \in G \mid x \sim_H g \}$$

$$= \{ x \in G \mid x - g \in H \} = g + H$$

$\square$

Beispiel:

(1) $G = \mathbb{Z}$

$H = m \cdot \mathbb{Z} \quad (m \geq 1)$

$$x_1 \sim_H x_2 \iff x_1 - x_2 \in m \cdot \mathbb{Z}$$

$$\iff m \mid (x_1 - x_2) \quad ✓$$

also $\sim_H = \sim_m$

$$\mathbb{Z}/m\mathbb{Z} = \mathbb{Z}/\sim_m$$

(2) $G = (\mathbb{R}^2, +)$

$$H = \left\{ \begin{pmatrix} x \\ 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\}$$

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \sim_H \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \iff \begin{pmatrix} x_1 - x_2 \\ y_1 - y_2 \end{pmatrix} \in H$$

$$\iff y_1 = y_2 .$$

Beweis:

(+) wohldef wie im Beweis, dass $\mathbb{Z}/m\mathbb{Z}$ Gruppe ist

neutrales Element $H = 0 + H$

$= [0]$

inverses Element: $-(x+H) = (-x) + H$

$= [-x]$. $\square$

Beweis

$$x_1 \sim_f x_2 \qquad\qquad x_1 \sim_{\mathrm{Ker} f} x_2$$
$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$
$$f(x_1) = f(x_2) \qquad\qquad x_1 - x_2 \in \mathrm{Ker} f$$
$$\Updownarrow \qquad\qquad\qquad \Updownarrow$$
$$\underline{f(x_1) - f(x_2)} = 0 \qquad\qquad \underline{f(x_1 - x_2)} = 0$$

f Gruppenhomom.

$\square$

Beweis

$\bar{f}$ Homomorphismus:

$$\bar{f}\big((x_1 + \mathrm{Ker} f) \oplus (x_2 + \mathrm{Ker} f)\big)$$
$$\underset{\text{Def. } \oplus}{=} \bar{f}\big((x_1 + x_2) \oplus \mathrm{Ker} f\big)$$
$$= f(x_1 + x_2)$$
$$\underset{\text{f Homom.}}{=} f(x_1) + f(x_2)$$
$$= \bar{f}(x_1 + \mathrm{Ker} f) + \bar{f}(x_2 + \mathrm{Ker} f)$$

$\square$

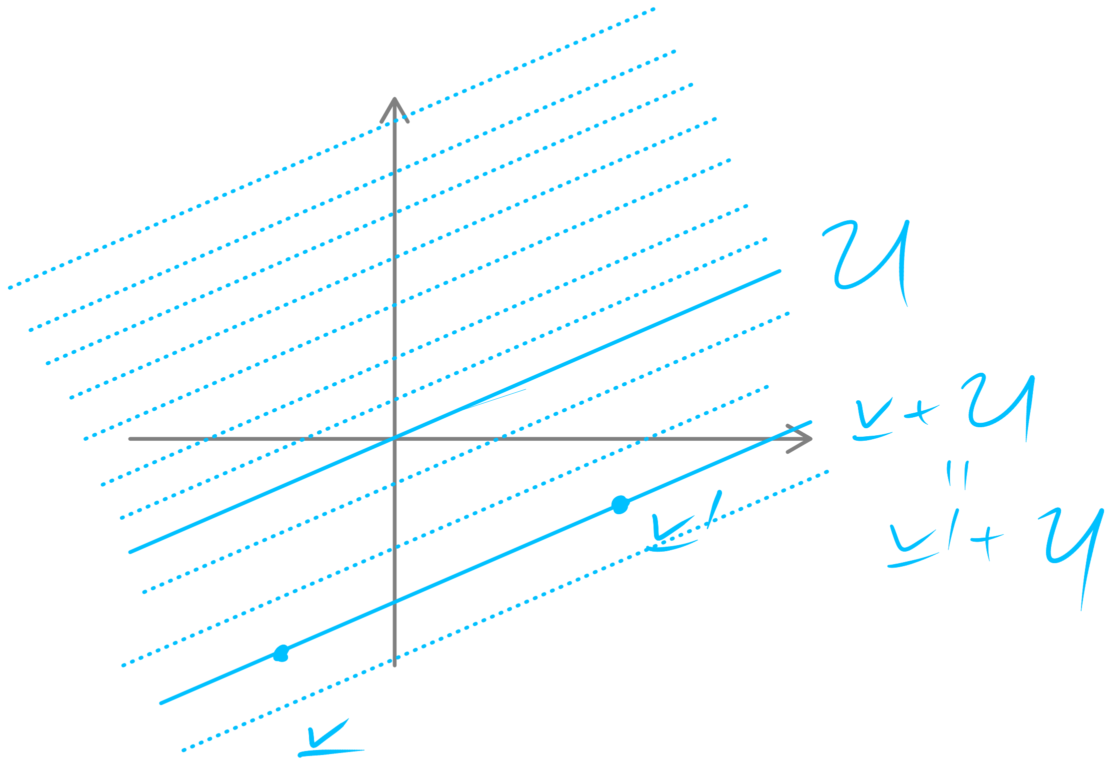
__Beispiel:__

$$\begin{tikzcd}
(\mathbb{Z}, +) \arrow[rr, "f"] \arrow[d] & & (\mathbb{R}^\times, \cdot) \\
\boxed{\begin{array}{c} (\mathbb{Z}/2\mathbb{Z}, +) \xrightarrow{\widehat{=}} (\{+1, -1\}, \cdot) \\ [z] \longmapsto (-1)^z \end{array}} & &
\end{tikzcd}$$

$$z \longmapsto (-1)^z$$

$$\begin{aligned}
\operatorname{Ker} f &= f^{-1}(1) \\
&= 2 \cdot \mathbb{Z}
\end{aligned}$$

Beispiel: $V = \mathbb{R}^2$



Beweis des Satzes:

wohldef von $\cdot$:

$[\underline{v}] = [\underline{v}']$, also $\underline{v} - \underline{v}' \in U$.

Dann auch $\lambda \cdot (\underline{v} - \underline{v}') \in U$

$$\lambda \cdot (\underline{v} - \underline{v}') = (\lambda \cdot \underline{v}) - (\lambda \cdot \underline{v}')$$

d.h. $[\lambda \cdot \underline{v}] = [\lambda \cdot \underline{v}']$ ✓

VR-Axiome für V/U folgen daraus, dass VR-Axiome für V erfüllt sind. $\square$

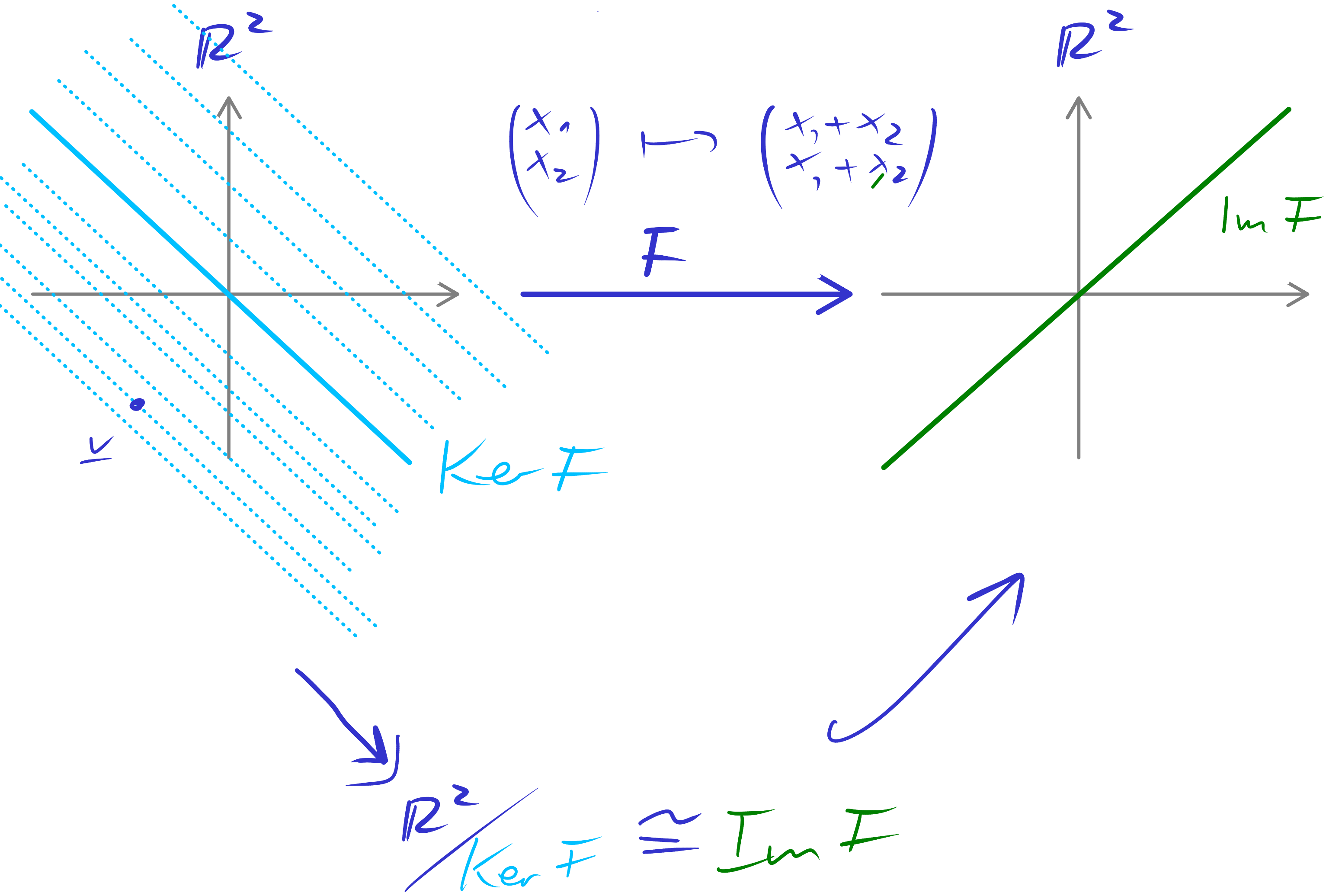
Beweis:

Nur noch z.z.:

$$\overline{F}(\underbrace{\lambda(\underline{v}+\operatorname{Ker} f)}_{\lambda\underline{v}+\operatorname{Ker} f}) \overset{!}{=} \lambda\cdot\overline{F}(\underbrace{\underline{v}+\operatorname{Ker} f}_{F(\underline{v})})$$

$$F(\lambda\cdot\underline{v}) = \lambda\cdot F(\underline{v})$$

$\square$



$\mathbb{R}^2$
$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_1 + x_2 \\ x_1 + x_2 \end{pmatrix}$
F
$\mathbb{R}^2$
Im F
$\underline{v}$
Ker F
$\mathbb{R}^2 / \mathrm{Ker}\, F \cong \mathrm{Im}\, F$