

$$\begin{aligned}
 + &: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
 \cdot &: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
 * &: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R} \\
 & (x, y) \mapsto x * y := x^2 - y^3 + \frac{x-y}{5}
 \end{aligned}$$

$$\begin{aligned}
 * &: \{0, 1\} \times \{0, 1\} \longrightarrow \{0, 1\} \\
 & \begin{array}{l} (0, 0) \\ (0, 1) \\ (1, 0) \\ (1, 1) \end{array} \begin{array}{l} \mapsto 0 \\ \mapsto 0 \\ \mapsto 1 \\ \mapsto 0 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 \text{Abb}(X, X) \times \text{Abb}(X, X) &\longrightarrow \text{Abb}(X, X) \\
 (f, g) & \quad f \circ g
 \end{aligned}$$

$$\begin{aligned}
 +: \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R}: \\
 & \text{(abelsch)}
 \end{aligned}$$

(Ass.) ✓

(n. E.) $0 + a = a$ ✓

(Inv.) $(-a) + a = 0$ ✓

(Kommut.) ✓

$$\cdot: \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

(Ass.) ✓

(n. E.) $1 \cdot a = a$

~~(Inv.) $a' \cdot a = 1$~~

~~$\exists 0': 0' \cdot 0 = 1$~~

$$\mathbb{R}^+ := \mathbb{R} \setminus \{0\}$$

$$\begin{aligned}
 \cdot: \mathbb{R}^+ \times \mathbb{R}^+ &\longrightarrow \mathbb{R}^+ \quad \text{(Inv.) } a' := \frac{1}{a} \\
 & \text{(abelsch)}
 \end{aligned}$$

X Mengen

$$\circ: \text{Abb}(X, X) + \text{Abb}(X, X) \longrightarrow \text{Abb}(X, X)$$
$$(f, g) \mapsto f \circ g$$

(Ass.) ✓

(u.ε.) $e = \text{id}_X$

$$\text{id}_X \circ f = f$$

~~(Inv.)~~ $f' \circ f = \text{id}_X$

Beweis: $\circ: \mathcal{S}(X) + \mathcal{S}(X) \longrightarrow \mathcal{S}(X)$

$$(f, g) \mapsto f \circ g$$

Ist wohldefiniert:

$f \in \mathcal{S}(X)$ and $g \in \mathcal{S}(X)$ dann auch

$$f \circ g \in \mathcal{S}(X). \quad [\dots]$$

(Ass.) ✓

(u.ε.): $\text{id}_X \in \mathcal{S}(X)$

(Inv.): $f' := f^{-1}$

□

$$\mathcal{S}_3 \ni \text{id} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

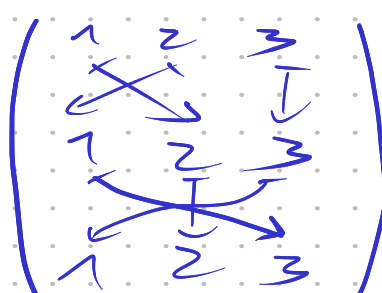
$$\ni \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

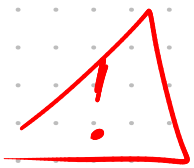
$$\ni \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$\ni \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\ni \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\ni \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$




\mathcal{S}_3 ist nicht abelsch.

Beweis:

i) $a * a' = e$

Wissen: $\exists (a')'$: $(a')' * a' = e$

Daher:

$$\begin{aligned} a * a' &= e * a * a' \\ &= (a')' * \underbrace{a' * a}_{e} * a' \\ &= (a')' * \underbrace{e * a'}_{e} \\ &= (a')' * a' \\ &= e \end{aligned}$$

ii) $a * e = a$

$$\begin{aligned} a * e &= a * \underbrace{a' * a}_{(i)} \\ &\stackrel{(Inv.)}{=} e * a \end{aligned}$$

$$\stackrel{(u.E.)}{=} a$$

iii) e eindeutig: Sei \tilde{e} weiteres
neutrales Element:

$\forall a \in G:$

$$e * a = a$$

$$\tilde{e} * a = a$$

$$a = \tilde{e}$$

Wegen (ii):

$$a * e = a$$

$$a * \tilde{e} = a$$

$$a = e$$

$$e * \tilde{e} = \tilde{e}$$

$$e * \tilde{e} = e$$

$$\tilde{e} = e$$

(iv) a' eindeutig Sei $a^\#$ weiteres
Inverses von a .

$$a' * a = e$$
$$a^\# * a = e$$

Wegen (i):

$$a * a' = e$$

$$a * a^\# = e$$

$$a' * a = a^\# * a$$

$$a' * a * a' = a^\# * a * a'$$

$$a' * e = a^\# * e$$

daher

Wegen ii)

$$a' = a^\#$$

$$(v) \quad \underline{(a')' = a}$$

$$\left. \begin{array}{l} (a')' * a' = e \quad (\text{Inv.}) \\ a * a' = e \quad (i) \end{array} \right\} (a')' = a \text{ wegen (iv).}$$

$$(vi) \quad \underline{(a * b)' = b' * a'}$$

$$\left. \begin{array}{l} (a * b)' * a * b = e \quad (\text{Inv.}) \\ (b' * a') * a * b \\ = b' * e * b \\ = b' * b = e \end{array} \right\} \begin{array}{l} \text{wegen (iv)} \\ (a * b)' \\ \parallel \\ b' * a' \quad \square \end{array}$$

in (G, \cdot) :

$$a \cdot 1 = 1 \cdot a = a$$

$$a \cdot a^{-1} = a^{-1} \cdot a = 1$$

$$(a \cdot b)^{-1} = b^{-1} \cdot a^{-1}$$

in $(G, +)$

$$a + 0 = 0 + a = a$$

$$a - a = -a + a = 0$$

$$-(a + b) = -b - a$$