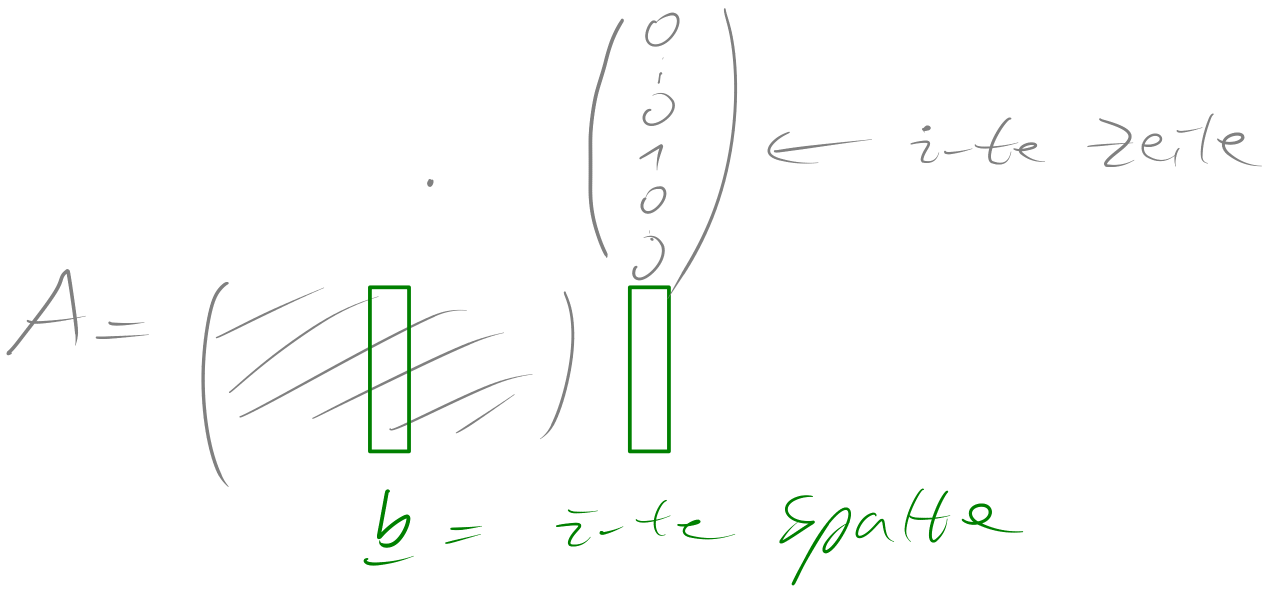


$$A \cdot \underline{e}_i = \underline{b}$$

A 2×2 – Matrix / $\mathbb{R}$

$$\text{Lös}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \emptyset$$

$$\text{Lös}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$$

$$\text{Lös}\left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) = \mathbb{R}^2$$

$$A \cdot \underline{x} = \underline{b} \quad \big| \; A^{-1}$$

$$\underline{x} = A^{-1} \cdot \underline{b}$$

$$\text{Lös}(A, \underline{b}) = \{A^{-1} \cdot \underline{b}\}$$

$F: V \longrightarrow ?$

$$\text{Rang}(F) + \dim \text{Ker}(F) = \dim V$$

$$\text{Rang}(A) + \dim \text{Lös}(A) = n$$

$\Downarrow$ also

$$\boxed{\dim(\text{Lös}(A)) = n - \text{Rang}(A)}$$

b:
$$\text{Rang}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$
$$\text{Lös}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2$$

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{\begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Rang 2 = n

$$\text{Lös}\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \emptyset$$

$$\text{Lös}(A, \underline{b}) = \begin{cases} \underline{x}_0 + \text{Lös}(A) \\ \emptyset \end{cases}$$

$$\dim \text{Lös}(A) = n - \text{Rang}(A)$$

1. $\underline{b} = \underline{0}$ $\quad \text{Lös}(A, \underline{b}) = \text{Lös}(A)$

$\dim \text{Lös}(A) \geq 2$, also

$$\text{Rang}(A) = n - \dim \text{Lös}(A) \leq n - 2$$

2. $\underline{b} \neq \underline{0}$

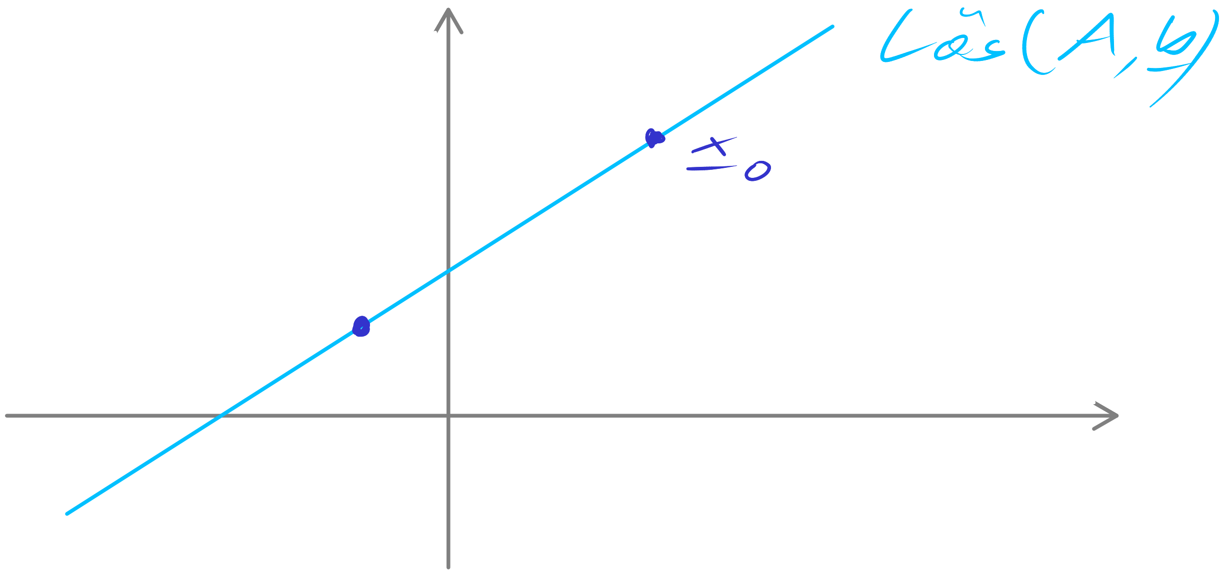
z.B. $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\text{Lös}\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \underbrace{\mathbb{R} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\text{Lös}(A)}$$

$\cup$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\text{Rang}(A) = 1 = n - 1.$$


Lös(A,b)
x_0

$V = \mathbb{R}^2$

$W = \mathbb{R}$

$$F : \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto y$$

$\mathcal{B} = \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$ $\qquad$ $\mathcal{C} = \left((2) \right)$

$\mathcal{B}' = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$ $\qquad$ $\mathcal{C}' = \left((3) \right)$

$$M^{\mathcal{B}}_{\mathcal{C}}(F) = M\left(\Phi_{\mathcal{C}}^{-1} \circ F \circ \Phi_{\mathcal{B}}\right)$$
$$= \left(\tfrac{1}{2}\right) \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \tfrac{1}{2} & 0 \end{pmatrix}$$

$$\Phi_{\mathcal{B}}\Big(\overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{e_1}\Big) = \overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{b_1} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \checkmark$$

$$\Phi_{\mathcal{B}}\Big(\overbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}^{e_2}\Big) = \overbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}^{b_2} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \checkmark$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[rrr, "\left(\frac{1}{2}\ 0\right)"] \arrow[dr, "\Phi_B"'] & & & \mathbb{R} \arrow[dl, "\Phi_C", "\cdot 2"'] \\
& \mathbb{R}^2 \arrow[r, "F", "(0\ 1)"'] & \mathbb{R} & \\
\mathbb{R}^2 \arrow[ur, "\Phi_{B'}"'] & & & \mathbb{R} \arrow[ul, "\Phi_{C'}"]
\end{tikzcd}
$$

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$C:\ (2) \qquad (3) = C'$$

$$B' = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \qquad C' = \left((3) \right)$$

$$\Phi_{B'}(e_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Phi_{B'}(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Schritt 2:

$$\left(\overbrace{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}^{B} \;\middle|\; \overbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}^{E_2} \right) \circlearrowleft \qquad (C')^{-1} = \left(\tfrac{1}{3} \right)$$

$$\rightsquigarrow \left(\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{E_2} \;\middle|\; \underbrace{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}}_{B^{-1}} \right)$$

$$L_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L_{C'} = \left(\tfrac{1}{3} \right)$$

$$
\begin{tikzcd}
\mathbb{R}^2 \arrow[rrr, "\begin{pmatrix} \frac{1}{2} & 0 \end{pmatrix}"] \arrow[dr, "\Phi_B"', "\cong"] & & & \mathbb{R} \arrow[dl, "\cdot 2"', "\Phi_{\mathcal{C}} = (2)"] \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \mathbb{R}^2 \arrow[ul, dashed, bend left, "L_B"] \arrow[r, "F"', "\begin{pmatrix} 0 & 1 \end{pmatrix}"] & \mathbb{R} \arrow[dr, dashed, bend right, "L_{\mathcal{C}'}"'] & \\
\mathbb{R}^2 \arrow[ur, "\Phi_{B'}"'] \arrow[rrr, "\begin{pmatrix} \frac{1}{3} & \frac{1}{3} \end{pmatrix}"'] & & & \mathbb{R} \arrow[ul, "\Phi_{\mathcal{C}'} = (3)"']
\end{tikzcd}
$$

$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} =$

$$M^{B'}_{\mathcal{C}'}(F) = \underbrace{\left(\tfrac{1}{3}\right) \cdot (2) \cdot \begin{pmatrix} \tfrac{1}{2} & 0 \end{pmatrix}} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} \tfrac{1}{3} & \tfrac{1}{3} \end{pmatrix}}}$$

Probe: $F \circ \Phi_{B'} \overset{?}{=} \Phi_{\mathcal{C}'} \circ F_{\left(\frac{1}{3} \; \frac{1}{3}\right)}$

$$\underbrace{\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}}_{\begin{pmatrix} 1 & 1 \end{pmatrix}} \overset{?}{=} \underbrace{(3) \cdot \begin{pmatrix} \tfrac{1}{3} & \tfrac{1}{3} \end{pmatrix}}_{\begin{pmatrix} 1 & 1 \end{pmatrix}}$$

$K^N \supset \text{Lös}(A)$

UVR der Dimension

Lösungsverfahren liefert Basis

$\mathcal{C} = (c_1, \dots, c_\ell)$ von $\text{Lös}(A)$

$\ell = n - \text{Rang}(A)$

$c_i \in K^N$

$$\begin{tikzcd}
K^\ell \arrow[r, "\cdot(c_1 \dots c_\ell)"] \arrow[dr, "\cong", "\Phi_{\mathcal{C}}"'] & K^N \\
& \text{Lös}(A) \arrow[u, hook]
\end{tikzcd}$$

$A \in M(n \times N; K)$

$\text{Lös}(A) \subset K^N \xrightarrow[\cdot A]{} K^n$

a: $S = E_m$
$T = E_n$

$$M = E_m^{-1} \cdot M \cdot E_n$$

b:

$$M' = S^{-1} M T$$

$$\Updownarrow$$

$$S \cdot M' \cdot (T^{-1}) = M$$

$$\parallel$$

$$(S^{-1})^{-1} \cdot M' \cdot (T^{-1})$$

c:

$$M' = S^{-1} M T$$

$$M'' = (S')^{-1} M' T'$$

$$= (S')^{-1} \cdot S^{-1} M T \cdot T'$$

$$= \underbrace{S'^{-1} \cdot S^{-1}}_{(S \cdot S')^{-1}} \cdot M \cdot (T T')$$

$$(S \cdot S')^{-1} \cdot M \cdot (T T')$$

$$(ab)^{-1} = b^{-1} \cdot a^{-1}$$ in allgemeinen Ring

- M äquivalent zu M'

$$\Leftrightarrow \text{Rang}\, M = \text{Rang}\, M'$$

- für $M \in M(m \times n; K)$ ist

Zeilenrang $\rightarrow$ $\text{Rang}(M) \leq m$

$\shortparallel$

Spaltenrang $\rightarrow$ $\text{Rang}(M) \leq n$,

$\shortparallel$

Rang

also $0 \leq \text{Rang}(M) \leq \min(m, n)$