**Aufgabe 1. Hesse Normal Forms.**

(a) It is readily apparent that $\vec{v} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$ is orthogonal to $\begin{pmatrix} -2 \\ 5 \end{pmatrix}$. Scalar multiplication gives us

$$\langle \begin{pmatrix} 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \rangle = 25 + 4 = 29,$$

and so the normalized normal vector is $\frac{1}{\sqrt{29}} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$. The point $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ lies on the line, and we find that

$$\langle \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \end{pmatrix} \rangle = 15 + 8 = 23.$$

Hence the Hesse Normal Form is given by

$$\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \langle \begin{pmatrix} x \\ y \end{pmatrix}, \frac{1}{\sqrt{29}} \begin{pmatrix} 5 \\ 2 \end{pmatrix} \rangle = \frac{23}{\sqrt{29}} \right\}.$$

***Note to markers:*** *Give the students 1 point for finding the normal vector, 1 point for normalizing it correctly, 1 point for getting the scalar product with* $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ *right, and 1 point for writing down the correct Hesse Normal Form, i.e.* $1 + 1 + 1 + 1 = 4$ *points all in all.*

(b) First, the students need to find a vector orthogonal to both $\begin{pmatrix} 2 & -1 & 1 \end{pmatrix}^T$ and $\begin{pmatrix} 2 & 3 & 2 \end{pmatrix}^T$. If one does it by means of the cross product, one finds

$$\begin{vmatrix} \vec{x} & \vec{y} & \vec{z} \\ 2 & -1 & 1 \\ 2 & 3 & 2 \end{vmatrix} = ((-1) \cdot 2 - 1 \cdot 3)\vec{x} - (2 \cdot 2 - 1 \cdot 2)\vec{y} + (2 \cdot 3 - (-1 \cdot 2))\vec{z} = -5\vec{x} - 2\vec{y} + 8\vec{z}.$$

One may also solve the system of simultaneous equations

$$\begin{aligned} 2x - y + z &= 0, \\ 2x + 3y + 2z &= 0. \end{aligned}$$

Subtracting the first row from the second, we have

$$\begin{aligned} 2x - y + z &= 0, \\ 4y + z &= 0. \end{aligned}$$

From the second row, letting $y = t$, we find $z = -4t$. In the first row, we then find $2x = 5t$, or $x = -\frac{5}{2}t$. From this we may read out that a normal vector exists in the form $\begin{pmatrix} 5 & 2 & -8 \end{pmatrix}$.

Next, we normalize this,

$$\langle \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} \rangle = 25 + 4 + 64 = 93,$$

and so the normalized normal vector is $\frac{1}{\sqrt{93}} \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix}$. The point $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ lies on the line, and we find that

$$\langle \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} \rangle = 5 + 16 = 21.$$

Hence the Hesse Normal Form is given by

$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \frac{1}{\sqrt{93}} \begin{pmatrix} 5 \\ 2 \\ -8 \end{pmatrix} \right\rangle = \frac{21}{\sqrt{93}} \right\}.$$

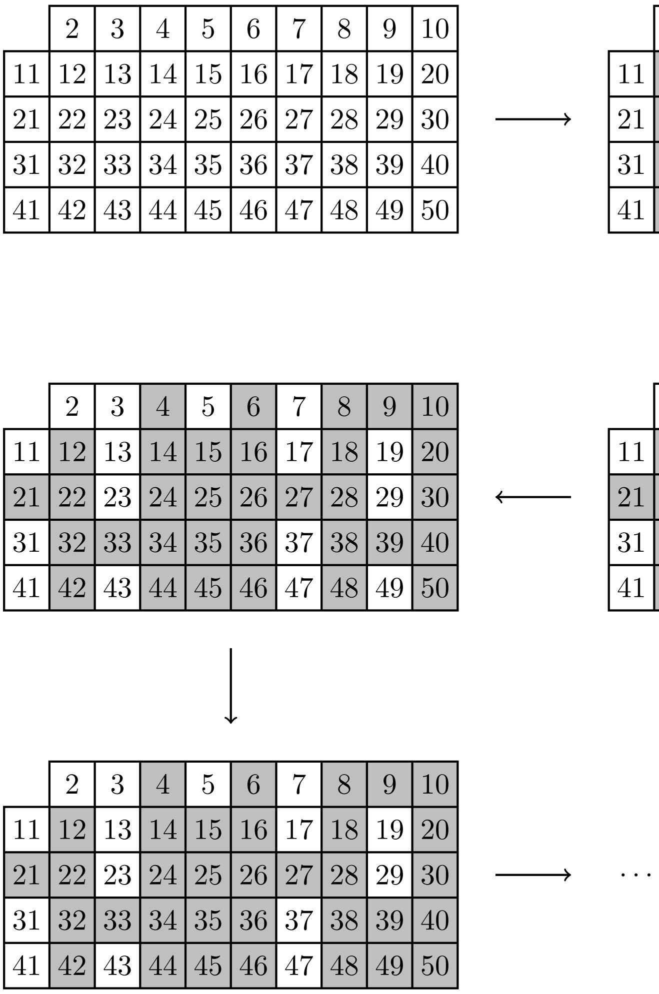
***Note to markers:*** *Give the students 3 points for finding the normal vector, 1 point for normalizing it correctly, 1 point for getting the scalar product with* $\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$ *right, and 1 point for writing down the correct Hesse Normal Form, i.e.* $3 + 1 + 1 + 1 = 6$ *points all in all.*

**Aufgabe 2. The Euclidean Algorithm.**

(a) Two elements $a, b$ in an integral domain are said to be *coprime* if the unit element is a greatest common divisor of the two.

***Note to markers:*** *Give the students 1 point for getting this definition right.*

(b) Personally, I would do this directly by prime factorization. $94 = 2 \cdot 47$, and 47 is easily seen to be a prime number by the sieve of Eratosthenes. $314 = 2 \cdot 157$.



The sieve of Eratosthenes gives 157 to be a prime number as well, but it isn't strictly speaking necessary to perform that algorithm again, since one only needs to check that neither 2 nor 47 is a divisor of 157. Hence it's clear that the greatest common divisor of 94 and 314 is 2.

More conventional however is the use of Euclid's algorithm[1]. We obtain

$$\begin{aligned} 314 &= 3 \times 94 + 32, \\ 94 &= 2 \times 32 + 30, \\ 32 &= 1 \times 30 + 2, \\ 30 &= 15 \times 2. \end{aligned}$$

[1] Some people are so *basic* in their choice of ancient Greek mathematicians.

Since 1 is not a greatest common divisor of 94 and 314, they are not coprime.

***Note to markers:*** *Give the students 2 points for determining that the greatest common divisor is 2, 1 point for noting that this means that 314 and 94 are not coprime. All in all,* $2 + 1 = 3$ *points.*

(c) This one will of course be a lot easier if you proceeded in the preceding subquestion with Euclid's algorithm. We find

$$\begin{aligned} 2 &= 32 - 30 \\ &= 32 - (94 - 2 \times 32) \\ &= 3 \times 32 - 94 \\ &= 3 \times (314 - 3 \times 94) - 94 \\ &= 3 \times 314 - 10 \times 94, \end{aligned}$$

giving us $p = 3$ and $q = -10$.

***Note to markers:*** *Give the students 2 points for completing this subquestion without any calculation error.*

(d) Cometh the dreaded polynomial long division subquestion. This time it's surprisingly easy, though! We find

$$X^4 + 7X^3 + 6X^2 + X + 6 = X(X^3 + 7X^2 + 6X) + (X + 6)$$

and

$$\begin{array}{r|l} & X^2 \quad + X \\ \hline X + 6 & X^3 + 7X^2 + 6X \\ & -X^3 - 6X^2 \\ \hline & \qquad X^2 + 6X \\ & \qquad -X^2 - 6X \\ \hline & \qquad\qquad 0 \end{array}$$

in other words,

$$X^3 + 7X^2 + 6X = (X^2 + X)(X + 6),$$

meaning that they have a greatest common divisor in $(X + 6)$, and thus are not coprime.

***Note to markers:*** *Give the students 2 points for completing the polynomial long division without error, and an additional point for remembering to jot down that this means that the two polynomials are not coprime. All in all,* $2 + 1 = 3$ *points.*

(e) We can easily rearrange

$$X^4 + 7X^3 + 6X^2 + X + 6 = X(X^3 + 7X^2 + 6X) + (X + 6)$$

into

$$X^4 + 7X^3 + 6X^2 + X + 6 - X(X^3 + 7X^2 + 6X) = (X + 6),$$

giving us $P = 1$ and $Q = -X$.

***Note to markers:*** *Give the students 1 point for getting that right.*

**Aufgabe 3. Jordan Normal Forms, Part I.**

(a) A matrix is diagonalizable if and only if its minimal polynomial splits and the roots thereof are all of multiplicity one.

***Note to markers:*** *Give the students 1.5 points for giving the correct definition.*

(b) For the first matrix, we have

$$\chi_A(\lambda) = \begin{vmatrix} -1-\lambda & 3 \\ -4 & 1-\lambda \end{vmatrix} = (\lambda+1)(\lambda-1)+12 = \lambda^2 - 1 + 12 = \lambda + 11,$$

and since this polynomial doesn't split over $\mathbb{R}$, we may conclude that $\mu_A(\lambda) = \lambda + 11$.

***Note to markers:*** *Give the students 0.5 points for finding the correct characteristic polynomial. Give the students 0.5 points for finding the correct minimal polynomial. Give the students 0.5 points for providing the right reasoning. All in all,* $0.5 + 0.5 + 0.5 = 1.5$ *points.*

For the second matrix, we have

$$\chi_B(\lambda) = \begin{vmatrix} 1-\lambda & 5 \\ 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda),$$

and we may then easily read out that $\mu_B(\lambda) = \chi_B(\lambda)$.

***Note to markers:*** *Give the students 0.5 points for finding the correct characteristic polynomial. Give the students 0.5 points for finding the correct minimal polynomial. Give the students 0.5 points for providing the right reasoning. All in all,* $0.5 + 0.5 + 0.5 = 1.5$ *points.*

For the third matrix, we have

$$\chi_C(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)^3.$$

The question then is, what is the minimal polynomial, $\mu_C(\lambda)$? We have three options. $(\lambda-3)$, $(\lambda-3)^2$, and $(\lambda-3)^3$. It is immediately apparent that $C - 3\mathbb{1} \neq 0$, ruling out the first option. However,

$$(C - 3\mathbb{1})^2 = \left( \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \right)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence $\mu_C(\lambda) = (\lambda-3)^2$.

***Note to markers:*** *Give the students 0.5 points for finding the correct characteristic polynomial. Give the students 1 points for finding the correct minimal polynomial. Give the students 1 points for providing the right reasoning. All in all,* $0.5 + 1 + 1 = 2.5$ *points.*

(c) It is possible to put a matrix in Jordan normal form if and only if its characteristic polynomial decomposes into linear factors (in the polynomial ring of the field over which we are working). Hence, $A$ cannot be put in Jordan normal form, but $B$ and $C$ can.

The multiplicity of a given root in the minimal polynomial of a matrix gives the size of the largest Jordan block of that root in the Jordan normal form of said matrix. Hence the Jordan normal forms of $B$ and $C$ respectively are

$$\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

***Note to markers:*** *In the case of matrix $A$, give the students 0.5 points for stating that the Jordan normal form doesn't exist, and 0.5 points for providing the correct reasoning for why it doesn't exist. In the case of matrix $B$, give the students 0.5 points for stating that the Jordan normal form exists, and 0.5 points for giving it. In the case of matrix $C$, give the students 0.5 points for stating that the Jordan normal form exists, and 0.5 points for giving it. All in all,* $(0.5 + 0.5) + (0.5 + 0.5) + (0.5 + 0.5) = 3$ *points.*

**Aufgabe 4. Jordan Normal Forms, Part II.**

(a) We find

$$\begin{aligned}
\chi_A(\lambda) &= \begin{vmatrix} -1-\lambda & 2 & 3 \\ -4 & 5-\lambda & 4 \\ -2 & 1 & 4-\lambda \end{vmatrix} \\
&= (-1-\lambda)((5-\lambda)(4-\lambda)-4) - 2(-4(4-\lambda)-(-2)\cdot 4) + 3(-4-(-2)(5-\lambda)) \\
&= (-\lambda^3+8\lambda^2-7\lambda-16)+(16-8\lambda)+(18-6\lambda) \\
&= -\lambda^3+8\lambda^2-21\lambda+18.
\end{aligned}$$

We find $\chi_A(2) = -8+32-42+18 = 0$, and hence $(\lambda-8)$ is a factor of $\chi_A$. Polynomial long division gives

$$\begin{array}{rl}
 & -X^2 \quad +6X \quad -9 \\ \hline
X-2) & -X^3+8X^2-21X+18 \\
 & X^3-2X^2 \\ \hline
 & 6X^2-21X \\
 & -6X^2+12X \\ \hline
 & -9X+18 \\
 & 9X-18 \\ \hline
 & 0
\end{array}$$

i.e. $\chi_A(\lambda) = -(\lambda-2)(\lambda^2-6\lambda+9)$. The second factor can easily be factorized once again, giving us $\chi_A(\lambda) = -(\lambda-2)(\lambda-3)^2$. Hence the minimal polynomial may be either $\mu_A(\lambda) = (\lambda-2)(\lambda-3)$ or $\mu_A(\lambda) = (\lambda-2)(\lambda-3)^2$. We find that

$$\left(\begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right)\left(\begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} -3 & 2 & 3 \\ -4 & 3 & 4 \\ -2 & 1 & 2 \end{pmatrix}\begin{pmatrix} -4 & 2 & 3 \\ -4 & 2 & 4 \\ -2 & 1 & 1 \end{pmatrix}.$$

Already for the first element in this $3 \times 3$ matrix, we obtain $12-8-6 = -2 \neq 0$, and hence $\mu_A(\lambda) = (\lambda-2)(\lambda-3)^2$.

***Note to markers:*** *Give the students 1 point for finding the characteristic polynomial $\chi_A(\lambda)$. Give them 1 point for giving the correct factorization $\chi_A(\lambda) = -(\lambda-2)(\lambda-3)^2$. Give them 1 point for correctly deducing (with full reasoning!) that $\mu_A(\lambda) = (\lambda-2)(\lambda-3)^2$. All in all, $1+1+1 = 3$ points.*

(b) This was all done as part of subquestion (a). The eigenvalues are $\lambda = 2$ and $\lambda = 3$.

***Note to markers:*** *Give the students 0.5 points for writing that down.*

(c) Remember the definition,

$$\mathrm{Hau}(A,a) := \{\vec{v} \in V \mid (A - a\mathbb{1})^n = 0 \quad \text{for some } n \in \mathbb{N}\},$$

and that the dimensionality of $\mathrm{Hau}(A,a)$ equals the multiplicity of $a$ as a root of the characteristic polynomial of $A$.

We have

$$\begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x+2y+3z \\ -4x+5y+4z \\ -2x+y+4z \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

giving us a system of simultaneous equations

$$\begin{aligned} -(1+\lambda)x + 2y + 3z &= 0, \\ -4x + (5-\lambda)y + 4z &= 0, \\ -2x + y + (4-\lambda)z &= 0. \end{aligned}$$

In the case of $\lambda = 2$, this reads

$$\begin{aligned} -3x + 2y + 3z &= 0, \\ -4x + 3y + 4z &= 0, \\ -2x + y + 2z &= 0. \end{aligned}$$

Subtracting the third row twice from the second row gives us $-y = 0$, which sorts out the value of that parameter. The second row then reads $-2x + 2z = 0$, from which we may conclude that if $x = t$, then $z = t$ as well. Hence we have obtained the eigenvector associated with $\lambda = 2$ in the form of $\vec{v} = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$. Since the root $\lambda = 2$ only appears with multiplicity one in the characteristic polynomial, it follows that $\mathrm{Hau}(A; 2) = \langle \vec{v} \rangle$.

In the case of $\lambda = 3$, since the factor $(\lambda - 3)$ appears with multiplicity 2 in the minimal polynomial, in the Jordan normal form, there is a $2 \times 2$ block associated with that eigenvalue, and hence we conclude that there may only exist one eigenvector associated to it. Thus rather than to go looking for the eigenvector, it is quickest and simplest to directly look for a basis for $\ker((A - 3\mathbb{1})^2)$.

Since

$$\left( \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix} - 3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right)^2 = \begin{pmatrix} -4 & 2 & 3 \\ -4 & 2 & 4 \\ -2 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix},$$

we must simply find a basis for the solution space to $2x - y - z = 0$. It is easy to read out (for instance) that a basis is given by $\vec{w} = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$ and $\vec{u} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$. Hence, $\mathrm{Hau}(A; 3) = \langle \vec{w}, \vec{u} \rangle$ (or equivalent).

***Note to markers:*** *Give the students 1.5 point for finding Hau$(A; 2)$. Give them 1.5 points for finding Hau$(A; 3)$. All in all,* $1.5 + 1.5 = 3$ *points.*

(d) We have our first basis vector in the eigenvector associated with $\lambda = 2$, noted above.

For the next two, we need to find $\ker(A - 3\mathbb{1})$ (i.e. the eigenvector associated to $\lambda = 3$) and a vector $\vec{b} \in \ker((A - 3\mathbb{1})^2)$ such that

$$\ker((A - 3\mathbb{1})^2) = \langle \vec{b} \rangle \oplus \ker(A - 3\mathbb{1}). \tag{4d.1}$$

Of course, if $\vec{c} \in \ker((A - 3\mathbb{1})^2)$, then it follows that $(A - 3\mathbb{1})\vec{c} \in \ker(A - 3\mathbb{1})$, so we just need to find $\vec{c} \in \ker((A - 3\mathbb{1})^2)$ such that $(A - 3\mathbb{1})\vec{c} \neq 0$. If we try with $\vec{c} = \vec{w} = \begin{pmatrix} 1 & 2 & 0 \end{pmatrix}^T$, we find

$$\begin{pmatrix} -4 & 2 & 3 \\ -4 & 2 & 4 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

so that doesn't work. However, if we instead try $\vec{c} = \vec{u} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$, we find

$$\begin{pmatrix} -4 & 2 & 3 \\ -4 & 2 & 4 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix},$$

Hence, the Jordan normal basis is given by

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, (A + \mathbb{1})\vec{u}, \vec{u} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}. \tag{4d.2}$$

***Note to markers:*** *Give the students 1 point for writing down formula* (4d.1) *and providing the right reasoning accompanying it. Give the students 0.5 points for picking a suitable vector* $\vec{b}$ *and a further 1 point for finding what the eigenvecctor associated with the eigenvalue 3 is. Finally, give them 1 point for writing down formula* (4d.2) *correctly. All in all,* $1 + 0.5 + 1 + 1 = 3.5$ *points.*

**Aufgabe 5**

Kreuzen Sie in den folgenden sieben Aufgabenteilen alle Aussagen an, die richtig sind.

*Es ist pro Aufgabenteil* mindestens eine *Aussage richtig. Manchmal sind* mehrere *Aussagen richtig. Als Gesamtpunktzahl erhalten Sie die Differenz aus der Anzahl aller richtig gesetzten Kreuze und aller falsch gesetzten Kreuze, mindestens aber 0 Punkte und höchstens 10 Punkte.*

---

(1) Ein Endomorphismus $f$ eines endlich-dimensionalen Vektorraums $V$ ist genau dann diagonalisierbar, wenn gilt:

- ■ Das Minimalpolynom von $f$ zerfällt in paarweise verschiedene Linearfaktoren.
- ☐ Das charakteristische Polynom von $f$ zerfällt in Linearfaktoren.
- ☐ Das Minimalpolynom von $f$ stimmt bis auf ein Vorzeichen mit dem charakteristischen Polynom von $f$ überein.

(2) Sei $f$ ein Endomorphismus eines endlich-dimensionalen Vektorraums $V$. Ist $a$ ein Eigenwert von $f$ mit **algebraischer Vielfachheit** $m$, so gilt:

- ■ Das charakteristische Polynom $\chi_f$ hat die Form $\chi_f(X) = (X-a)^m \cdot P(X)$, wobei $P$ ein zu $X-a$ teilerfremdes Polynom ist.
- ☐ Das Minimalpolynom $\mu_f$ hat die Form $\mu_f(X) = (X-a)^m \cdot P(X)$, wobei $P$ ein zu $X-a$ teilerfremdes Polynom ist.
- ☐ Der Eigenraum $\mathrm{Eig}(f;a)$ hat Dimension $\geq m$.

(3) Für jeden endlich-dimensionalen euklidischen Vektorraum $V$ gilt:

- ☐ Jede Basis von $V$ ist eine Orthonormalbasis.
- ☐ Es gibt, bis auf Reihenfolge der Vektoren, genau eine Orthonormalbasis von $V$.
- ■ Mindestens eine Basis von $V$ ist eine Orthonormalbasis.

(4) Für eine **Isometrie** $f\colon V \to V$ eines euklidischen Vektorraums $V$ gilt:

- ☐ Alle Eigenvektoren haben Länge 1.
- ■ $f$ ist injektiv.
- ☐ Der einzige Eigenwert von $f$ ist 1.

(5) Für jede reelle symmetrische Matrix $A \in \mathrm{Mat}_{\mathbb{R}}(n \times n)$ gilt:

- ☐ Für jeden Eigenwerte $a$ von $A$ ist auch $-a$ ein Eigenwert von $A$.
- ☐ Die durch $A$ definierte Bilinearform auf $\mathbb{R}^n$ ist ein Skalarprodukt.
- ■ Eigenvektoren zu verschiedenen Eigenwerten von $A$ stehen senkrecht zueinander (bezüglich des Standardskalarprodukt auf $\mathbb{R}^n$).

(6) Die folgenden Ringe sind Integritätsringe:

- ■ $\mathbb{Z}/11\mathbb{Z}$
- ☐ $\mathbb{R} \times \mathbb{R}$ (mit komponentenweiser Addition und Multiplikation)
- ■ $\mathbb{Z}[\sqrt{-5}] := \{a + b\sqrt{-5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$

**Aufgabe 6. Prof. Zibrowius's Signature "Clever Question."**

***Note to markers:*** *Seeing this is the signature "clever question," it is far from unlikely that correct answers given by the students will differ from the ones given here.*

(a) The elements of the basis $B^*$, are defined as

$$\mathbf{b}_i^* : V \to K, \qquad \mathbf{b}_j^* \mapsto \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

***Note to markers:*** *Award the students 1 points for knowing this.*

(b) A basis of $V^* \otimes W^*$ may be given by the vectors $(\mathbf{b}_1^* \otimes \mathbf{c}_1^*, \mathbf{b}_1^* \otimes \mathbf{c}_2^*, \dots, \mathbf{b}_n^* \otimes \mathbf{c}_m^*)$.

A basis of $V \otimes W$ may be given by the vectors $(\mathbf{b}_1 \otimes \mathbf{c}_1, \mathbf{b}_1 \otimes \mathbf{c}_2, \dots, \mathbf{b}_n \otimes \mathbf{c}_m)$. A basis of $(V \otimes W)^*$ may then by given by $((\mathbf{b}_1 \otimes \mathbf{c}_1)^*, (\mathbf{b}_1 \otimes \mathbf{c}_2)^*, \dots, (\mathbf{b}_n \otimes \mathbf{c}_m)^*)$, where $(\mathbf{b}_i \otimes \mathbf{c}_j)^*$ is the map $V \otimes W \to K$ which sends $\mathbf{b}_k \otimes \mathbf{c}_\ell$ to 1 if $k = i$ and $\ell = j$, and to zero otherwise.

***Note to markers:*** *Award the students 1 points for getting the basis of $V^* \otimes W^*$ right, and 1 point for getting the basis of $(V \otimes W)^*$ right. All in all, $1 + 1 = 2$ points.*

(c) It is readily apparent that the mapping $f : V \times W \to K$ defined by $\mathbf{v} \times \mathbf{w} \mapsto \varphi(\mathbf{v}) \cdot \psi(\mathbf{w})$ is bilinear. Specifically, in as far as $V$ is concerned, we have

$$f(k\mathbf{v}, \mathbf{w}) = \varphi(k\mathbf{v})\psi(\mathbf{w}) = k\varphi(\mathbf{v})\psi(\mathbf{w}) = kf(\mathbf{v}, \mathbf{w}),$$

and

$$f(\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}) = \varphi(\mathbf{v}_1 + \mathbf{v}_2)\psi(\mathbf{w}) = \varphi(\mathbf{v}_1)\psi(\mathbf{w}) + \varphi(\mathbf{v}_2)\psi(\mathbf{w}) = f(\mathbf{v}_1, \mathbf{w}) + f(\mathbf{v}_2, \mathbf{w}),$$

and by symmetry we have the same affair with $W$. We also of course have the canonical mapping $g : V \times W \to V \otimes W$ defined by $\mathbf{v} \times \mathbf{w} \mapsto \mathbf{v} \otimes \mathbf{w}$. Hence, by Satz 17.6, there exists a unique linear map $L_{\varphi,\psi} : V \otimes W \to K$ such that $\mathbf{v} \otimes \mathbf{w} \mapsto \varphi(\mathbf{v}) \cdot \psi(\mathbf{w})$.

$$\begin{tikzcd} V \times W \arrow[rr, "g"] \arrow[dr, "f"'] & & V \otimes W \arrow[dl, dotted, "\exists! L_{\varphi,\psi}"] \\ & K & \end{tikzcd}$$

***Note to markers:*** *Award the students 3.5 points for getting this correct.*

(d) We define a linear map $h : V^* \otimes W^* \to (V \otimes W)^*$ by $\varphi \otimes \psi \mapsto L_{\varphi,\psi}$.

To see that it is isomorphic, consider how it maps basis vectors of $V^* \otimes W^*$ to basis vectors of $(V \otimes W)^*$, these being, as established in (b), the sets $(\mathbf{b}_1^* \otimes \mathbf{c}_1^*, \mathbf{b}_1^* \otimes \mathbf{c}_2^*, \dots, \mathbf{b}_n^* \otimes \mathbf{c}_m^*)$ and $((\mathbf{b}_1 \otimes \mathbf{c}_1)^*, (\mathbf{b}_1 \otimes \mathbf{c}_2)^*, \dots, (\mathbf{b}_n \otimes \mathbf{c}_m)^*)$ respectively.

The basis vector $\mathbf{b}_i^* \otimes \mathbf{c}_j^*$ of $V^* \otimes W^*$ maps to $L_{\mathbf{b}_i^*, \mathbf{c}_j^*}$ in $(V \otimes W)^*$, this being, as established in (c), the unique map $V \otimes W \to K$ such that $\mathbf{v} \otimes \mathbf{w} \mapsto \mathbf{b}_i^*(\mathbf{v})\mathbf{c}_j^*(\mathbf{w})$. Seeing that this is the map which sends $\mathbf{b}_k \otimes \mathbf{c}_\ell$ to 1 if $k = i$ and $\ell = j$, then as noted in (b), $L_{\mathbf{b}_i^*,\mathbf{c}_j^*} = (\mathbf{b}_i \otimes \mathbf{c}_j)^*$, a basis vector of $(V \otimes W)^*$.

This establishes that $h$ maps the basis vectors of $V^* \otimes W^*$ to the basis vectors of $(V \otimes W)^*$ in a one-to-one fashion. Seeing further that $V^* \otimes W^*$ has $nm$ basis vectors, as does $(V \otimes W)^*$, it follows that the mapping is also onto, establishing $h$ as an isomorphism.

***Note to markers:*** *Award the students 3.5 points for getting this correct..*

That completes the examination.