Lambda Rings II – Coordinate-free approach following Tall & Wraith and Borger & Wieland

$\boxed{\text{Ring} := \text{commutative ring}}$

A plethora is to a ring what a ring is to an abelian group
(it controls the operations).

$\boxed{1a}$ **Prop:** For any abelian group,

$$\mathrm{Hom}_{Ab}(B, A)$$

is an abelian group in a unique natural way: in target A

$$(f+g)(b) = f(b) + g(b)$$ in target A

Equivalently:

$$f+g = \left(B \xrightarrow[m \longmapsto (m,m)]{\Delta} B \oplus B \xrightarrow{f \oplus g} A \right)$$

coproduct in Ab

Def: A *biring* is ring S together with a natural ring structure on

$$\mathrm{Hom}_{Rings}(S, R)$$

for any ring R. More explicitly, S is equipped with ring homomorphisms

coring structure

$$\left\{ \begin{array}{ll} \Delta^+ : S \to S \otimes S & \text{coproduct in Rings} \\ \Delta^\times : S \to S \otimes S & \\ \sigma^+ : S \to S & (\text{conegative}) \\ \varepsilon^+ : S \to \mathbb{Z} & (\text{cozero}) \\ \varepsilon^\times : S \to \mathbb{Z} & (\text{counit}) \end{array} \right.$$

satisfying "evident" relations such as

$$\begin{tikzcd} S & S \otimes S \arrow[l, "\varepsilon^+ \cdot id"'] \\ & S \arrow[ul, equal] \arrow[u, "\Delta^+"'] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} S & S \otimes S \arrow[l, "\sigma^+ \cdot id"'] \\ \mathbb{Z} \arrow[u, "\text{unique}"] & S \arrow[l, "\varepsilon^+"] \arrow[u, "\Delta^+"'] \end{tikzcd}$$

$\boxed{1b}$ eg: $\mathbb{Z}$: $\mathrm{Hom}_{Ab}(\mathbb{Z}, A) \cong A$

$\boxed{1c}$ eg: free abelian group on a set M

$$\mathbb{Z}M := \bigoplus_M \mathbb{Z}$$

$$\mathrm{Hom}_{Ab}(\mathbb{Z}M, A) \cong \prod_M A$$

$\boxed{*}$

eg: $\mathbb{Z}[e]$ with

$$\Delta^+(e) := e \otimes 1 + 1 \otimes e$$
$$\Delta^\times(e) := e \otimes e$$
$$\sigma^+(e) := -e$$
$$\varepsilon^+(e) = 0$$
$$\varepsilon^\times(e) = 1$$

$$\mathrm{Hom}_{Rings}(\mathbb{Z}[e], R) \cong R$$

eg: free ring on a set M

$$S(M) := \mathbb{Z}[x_m \mid m \in M]$$

is canonically a biring via

$$\Delta^+(x_m) = x_m \otimes 1 + 1 \otimes x_m$$
$$\Delta^\times(x_m) = x_m \otimes x_m$$
$$\sigma^+(x_m) = -x_m$$
$$\varepsilon^+(x_m) = 0$$
$$\varepsilon^\times(x_m) = 1$$

$$\mathrm{Hom}_{Rings}(S(M), R) = \prod_M R$$

eg: $S := \varprojlim_d \underbrace{\mathbb{Z}[x_1, \dots, x_d]^{S_d}}_{\text{graded ring}}$

$= \{$ power series in $x_1, x_2, x_3, \dots$ of bounded degree $\}$

$\ni \lambda_k := \sum_{i_1 < \dots < i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$ "elementary"

$\ni \sigma_k := \sum_{i_1 \le \dots \le i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$ "complete"

$\ni \psi_k := \sum_i x_i^k$

$$\Big(\$ \cong \mathbb{Z}[\lambda_1, \lambda_2, \lambda_3, \dots]$$

$$\cong \mathbb{Z}[\sigma_1, \sigma_2, \sigma_3, \dots]$$

$$\$ \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}[\psi_1, \psi_2, \psi_3, \dots] \text{ as a ring} \Big)$$

with
$$\left.\begin{aligned} \Delta^+(f) &= f(\dots, x_i \otimes 1, 1 \otimes x_i, \dots) \\ \Delta^\times(f) &= f(\dots, x_i \otimes x_j, \dots) \end{aligned}\right\} \text{order of variables irrelevant!}$$

$$\sigma^+(f) = f(-x_1, -x_2, -x_3, \dots)$$

$$\varepsilon^+(f) = f(0, 0, 0, \dots)$$

$$\varepsilon^-(f) = f(1, 0, 0, \dots)$$

$\mathrm{Hom}_{\mathrm{Rings}}(\$, R) =:$ "Big Witt ring of R"

$$\Big(\cong (1 + R[\![t]\!])^+ \text{ as group} \Big)$$

$$\Big(\cong \prod_{\mathbb{N}} R \text{ as set} \Big)$$

Note: $S(\mathbb{N}, \cdot) \hookrightarrow \$$ as biring

$$\Delta^+(\psi_k) = \psi_k \otimes 1 + 1 \otimes \psi_k$$

$$\Delta^\times(\psi_k) = \psi_k \otimes \psi_k$$

2a **Prop:** $\mathrm{Hom}_{Ab}(B,-)$ has a left adjoint:

$$\text{homom. } A \longrightarrow \mathrm{Hom}_{Ab}(B,A')$$
$$\hat{=} \ \text{homom. } B\otimes A \longrightarrow A'$$
$$\hat{=} \ \text{homom. } B \longrightarrow \mathrm{Hom}_{Ab}(A,A')$$

2b Explicitely: $B\otimes A = \mathbb{Z}\{b\otimes a\}/\ldots$

$b\otimes a \ \hat{=}$ linear operator b applied to a

$$b\otimes(a+a') = b\otimes a + b\otimes a'$$
$$(b+b')\otimes a = b\otimes a + b'\otimes a$$
$$\ldots$$
$$\ldots$$

2c **eg:** $\mathbb{Z}$ unit for $\otimes$

Prop: $\mathrm{Hom}_{Rings}(S,-)$ has a left adjoint, for any biring S:

$$\text{ring homom. } R \longrightarrow \mathrm{Hom}_{Rings}(S,R')$$
$$\hat{=} \ \text{ring homom. } S\odot R \longrightarrow R'$$
$$\cancel{S \longrightarrow \mathrm{Hom}_{Rings}(R,R')}$$

not a biring!

Explicitely: $S\odot R := \mathbb{Z}[s\odot r]/\ldots$

$s\odot r \ \hat{=}$ non-linear operator s applied to r

$$ss'\odot r = (s\odot r)\cdot(s'\odot r)$$

← operators multiplied pointwise

$$s\odot(r+r') = \text{"}\Delta^+(s)(r,r')\text{"}$$
$$:= \left(\textstyle\sum_i s_i^{(1)}\odot r\right)\cdot\left(\textstyle\sum_i s_i^{(2)}\odot r'\right)$$
$$\text{for } \Delta^+(s) = \textstyle\sum_i s_i^{(1)}\otimes s_i^{(2)}$$

↖ operators compatible with addition

$$\ldots$$
$$\ldots$$

eg: $\mathbb{Z}[e]$ unit for $\odot$

Prop: $S\odot -$ takes birings to birings (although $\mathrm{Hom}_{Rings}(S,-)$ does not).

3a **Def.:** A *ring* is an abelian group R together with $\cdot : R \otimes R \longrightarrow R$ and $1 : \mathbb{Z} \longrightarrow R$ such that ...

(T&W): biring triple

Def: Plethora := biring P together with $\circ : P \odot P \longrightarrow P$ and unit $\mathbb{Z}[e] \longrightarrow P$ [...]

3b **eg:** free ab. group on monoid M

$\mathbb{Z}M$ is ring in a canonical way

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Def.: A *plethora* (T&W: *biring triple*) is a biring P with

$$\circ : P \odot P \longrightarrow P$$

$$\varepsilon^\circ : \mathbb{Z}[e] \longrightarrow P$$

such that ...

eg: $\mathbb{Z}[e]$ with

$$\mathbb{Z}[e] \odot \mathbb{Z}[e] \longrightarrow \mathbb{Z}[e]$$

$$f(e) \odot g(e) \longmapsto f(g(e))$$

$$\left(\cong \mathbb{Z}[e] \longrightarrow \mathrm{Hom}_{\mathrm{Rings}}(\mathbb{Z}[e], \mathbb{Z}[e]) \right)$$

$$f \longmapsto (g \longmapsto f \circ g)$$

$$\varepsilon^\circ = \mathrm{id}$$

eg: free ring on monoid M

$$S(M) = \mathbb{Z}[x_m \mid m \in M]$$

is biring (above) and even plethora via

$$x_m \circ x_{m'} := x_{mm'}$$

monoid structure

subeg:

$$S(\mathbb{N}, \cdot) = \mathbb{Z}[\psi_1, \psi_2, \ldots]$$

with $\psi_m \circ \psi_{m'} = \psi_{mm'}$

eg: On $\mathbb{S}$, $\exists$ plethora structure s.t.

$$f \circ g = f(\ldots, \underbrace{x^\alpha, x^\alpha, \ldots, x^\alpha}_{n_\alpha \text{ times}}, \ldots) \text{ for } g = \sum n_\alpha \cdot x^\alpha$$

(extends uniquely to g with negative coefficients n_α)

4a $\boxed{R \text{ ring}}$

Def.: An R-module is an ab. group A together with homom. $R \otimes A \longrightarrow A$ s.t. ... ("associative" and compatible with 1)

4b eg: $\mathbb{Z}$-module = abelian group

4c eg: M monoid

$\mathbb{Z}M$-module = abelian group with M-action

$\boxed{*}$

Note: $S(\mathbb{N}, \cdot) \hookrightarrow \mathbb{S}$ even as plethora

$(\psi_n \circ \psi_\ell = \psi_{n\ell}$ in $\mathbb{S})$

$\boxed{P \text{ Plethora}}$

Def.: A P-ring (T&W: "module over biring triple," or also "P-ring") is a ring R with ring homom. $P \odot R \longrightarrow R$ s.t. ...

eg: $\mathbb{Z}[e]$-ring = ring

eg: M monoid

$S(M)$-ring = ring with G-action (by ring homos)

$x_m \odot (r+r') = \Delta^+(x_m)(r,r')$
$= (x_m \otimes 1 + 1 \otimes x_m)(r,r')$
$= x_m \odot r + x_m \odot r'$

$x_m \odot (rr') = \Delta^\times(x_m)(r,r')$
$= x_m \otimes x_m (r,r')$
$= x_m \odot r + x_m \odot r$

subeg: $S(\mathbb{N}, \cdot)$-ring = ring with Adams operations

eg./Def.: $\mathbb{S}$-ring = λ-ring

By note above, any λ-ring has Adams operations

4d **Prop:** R is the free R-module on one generator, i.e.

$$\mathrm{Hom}_{R\text{-mod}}(R, A) \cong \mathrm{Hom}_{Sets}(*, A) \cong A$$

$$\varphi \longmapsto \varphi(1)$$

$$(r \mapsto r\cdot m) \longleftarrow\!\shortmid\; m$$

5a **Def:** An *operation on R-modules* is a family of maps of sets $A \xrightarrow{\varphi_A} A$, one for each R-module A, s.t.

$$\begin{tikzcd} A \arrow[r, "\varphi_A"] \arrow[d, "f"'] & A \arrow[d, "f"] \\ A' \arrow[r, "\varphi_{A'}"'] & A' \end{tikzcd}$$

commutes for all R-linear f.

5b eg: Any $r \in R$ defines $A \xrightarrow{r\cdot} A$ for any A.

[T&W, Prop. 4.3]

Prop: P is the free P-ring on one generator

$$\mathrm{Hom}_{P\text{-rings}}(P, R) \cong \mathrm{Hom}_{Sets}(*, R) \cong R$$

$$\varphi \longmapsto \varphi(e)$$

$$(\alpha \mapsto \alpha \circ r) \longleftarrow\!\shortmid\; r$$

$\circ : P \otimes R \longrightarrow R$, $\alpha \otimes r \mapsto \alpha \circ r$

image of e under $e^{\circ}: \mathbb{Z}[e] \longrightarrow P$

(proof-sketch:

($\subset$ = id): clear $\qquad$ φ P-ring-homomorphism.

($\supset$ = id) $\alpha(\varphi(e)) = \alpha \circ \varphi(e) = \varphi(\alpha \circ e) = \varphi(\alpha)$)

Def.: An *operation on P-rings* is a family of maps of sets $R \xrightarrow{\varphi_R} R$, one for each P-ring R, s.t.

$$\begin{tikzcd} R \arrow[r, "\varphi_R"] \arrow[d, "f"'] & R \arrow[d, "f"] \\ R' \arrow[r, "\varphi_{R'}"'] & R' \end{tikzcd}$$

commutes for all morphisms of P-rings f.

eg: Any $p \in P$ defines $R \xrightarrow{p\circ} R$ for any R.

Subseq: Any $f \in \mathbb{S}$ defines an operation on λ-rings.

$\boxed{5c}$ **Thm:** $\left\{ \begin{matrix} \text{operations on} \\ R\text{-mod} \end{matrix} \right\} \cong R$

via $\varphi \longmapsto \varphi_R(1)$

multiplication with $r \longleftarrow r$

pointwise $+$ $\hat{=}$ $+$

pointwise $\cdot$ $\hat{=}$ $\cdot$

composition $\circ$ $\hat{=}$ also $\cdot$

In particular, all operations are R-linear!

Thm: $\left\{ \begin{matrix} \text{operations on} \\ P\text{-rings} \end{matrix} \right\} \cong P$

via $\varphi \longmapsto \varphi_P(e)$

$p \circ - \longleftarrow p$

pointwise $+$ $\hat{=}$ $+$

pointwise $\cdot$ $\hat{=}$ $\cdot$

composition $\circ$ $\hat{=}$ $\circ$

Operations here need not be linear (e.g. λ-operations)

(Theorem follows easily from previous proposition.)