Lambda Rings I

why care? ① The first ring you ever met was a λ-ring!
② exterior powers seem useful

Vector spaces over a field F can be
added $\oplus$
& multiplied $\otimes$
in a way that satisfies usual axioms of $+$ & $\cdot$, e.g.

$$V \oplus W \cong W \oplus V$$

$$V \otimes (W_1 \oplus W_2) \cong V \otimes W_1 \oplus V \otimes W_2$$

We also have
"exterior powers" Λ^i,
compatible with $\oplus$ & $\otimes$ in various ways. In particular:

① $\Lambda^0(V) \cong F$

② $\Lambda^1(V) = V$

③ $\Lambda^k(V \oplus W) \cong \bigoplus_{i=0}^{k} \Lambda^i(V) \otimes \Lambda^{k-i}(W)$

Consider

$$K(F) := \left(\begin{array}{c}\text{free abelian group on} \\ \text{iso-classes of f.d. } F\text{-vs}\end{array}\right) \Big/ \approx$$

with $[V] \approx [U] + [W]$ for any s.e.s. $0 \to U \to V \to W \to 0$.

(Any element in $K(F)$ has the form
$\sum a_i [V_i] - \sum b_i [W_i]$ for certain $a_i, b_i \in \mathbb{N}$, V_i, W_i f.d. vs $/F$

$\overset{ss}{=} [\bigoplus_i V_i^{\oplus a_i}] - [\bigoplus_i W_i^{\oplus b_i}]$,

so any element can be written as
$[V] - [W]$ for certain f.d. vs $V, W / F$.)

⊕ and ⊗ give $K(F)$ the structure of a ring, and Λ^i define some additional maps
$$\lambda^i : K(F) \longrightarrow K(F).$$

Note: $K(F) \cong \mathbb{Z}$ ← first ring you ever met

$$[V] \mapsto \dim V$$

$+$ usual $+$

$\cdot$ usual $\cdot$

λ^i $\quad n \mapsto \binom{n}{i} = \dfrac{n\cdot(n-1)\cdot\ldots\cdot(n-i)}{i!}$

$$\left(\dim \Lambda^i(V) = \binom{\dim V}{i}\right)$$

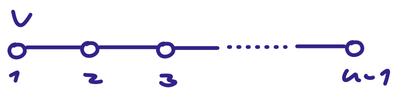
This example not so interesting, but many other examples in nature arise in precisely this way:

Eg 1: G (Lie) group

$$R(G) := \left(\begin{array}{c}\text{free abelian group on iso-classes}\\ \text{of (continuous) } G\text{-representations}\end{array}\right)\Big/\cong$$

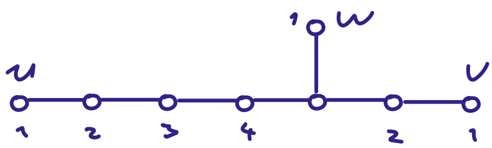
Subexamples

$$R(SL_n) = \mathbb{Z}[V, \Lambda^2 V, \Lambda^3 V, \ldots, \Lambda^{n-1} V]$$



$$R(E_8) = \mathbb{Z}[U, \Lambda^2 U, \Lambda^3 U, \Lambda^4 U,$$
$$V, \Lambda^2 V,$$
$$W, \; Z] \quad \text{with } Z := \Lambda^5 U \text{ or } \Lambda^3 V \text{ or } \Lambda^2 W$$

unique compact (simply connected & trivial centre)



(General statement: Conway/Adams/Guillot (2007))

Eg 2: X top. space

$$K^{top}(X) := \left(\text{free abelian group on iso-classes of } \mathbb{C}\text{-v.b.}/X\right)/\approx$$

Eg 3: X variety

$$K^{alg}(X) := \left(\text{free abelian group on iso-classes of v.b.}/X\right)/\approx$$

Subexample:

$$K^{top}(S^2) \cong K^{alg}(\mathbb{P}^1)$$

$\mathbb{C}P^1 =$

$L \downarrow \mathbb{C}\mathbb{P}^1$ tautological

$$= \mathbb{Z}[L] / \text{"}2 = L + L^{-1}\text{"}$$

$(1 \to L \to \mathbb{C}^2 \to L^\vee \to 1)$

$$= \mathbb{Z}[\alpha] / \alpha^2 \qquad \text{with } \alpha := L - 1$$

Here, $\Lambda^0(L) = \mathbb{C}$ i.e. $\lambda^0(L) = 1$

$\Lambda^1(L) = L$ $\qquad \lambda^1(L) = L$

$\Lambda^{\geq 2}(L) = 0$ $\qquad \lambda^{\geq 2}(L) = 0$

Theorem:

All examples above $(R(G), K(X), \dots)$ are λ-rings in the following sense.

Def: A λ-ring is a commutative ring R together with maps (of sets)

$$\lambda_i : R \longrightarrow R,$$

one for each $i \in \mathbb{N}_0$, that satisfy:

1. $\lambda_0 =$ constant 1
2. $\lambda_1 = \mathrm{id}$
3. $\lambda_k(a+b) = \sum_{i=0}^{k} \lambda_i(a)\lambda_{k-i}(b)$
4. $\lambda_k(a \cdot b) = P_k(\lambda_1(a), \dots, \lambda_k(b), \lambda_1(b), \dots, \lambda_k(b))$
5. $\lambda_\ell(\lambda_k(a)) = P_{k,\ell}(\lambda_1(a), \dots, \lambda_{k\cdot\ell}(a))$

for certain specific polynomials P_k and $P_{k,\ell}$ (determined by properties of Λ^i).

Property ③:

$$\lambda_t : (R,+) \longrightarrow (R[[t]]^*, \cdot)$$
$$a \longmapsto \sum_{i \geq 0} \lambda_i(a) t^i$$

is a group homorphism. (So $\lambda_t(0) = 1$.)

Example of 4: For a line bundle L,

$$\Lambda^k(L \underset{\otimes}{\cdot} V) = \underbrace{L^k}_{\Lambda^1(L)^k} \cdot \Lambda^k(V)$$

Example of 5: For line bundles L, M, N ist

$$\Lambda^2(\Lambda^2(L \underset{\oplus}{+} M + N)) = \Lambda^2(L\cdot M + L\cdot N + M \cdot N)$$
$$= \dots$$
$$= L^2 \cdot M \cdot N + L \cdot M^2 \cdot N + L \cdot M \cdot N^2$$
$$= \underbrace{L \cdot M \cdot N}_{\Lambda^3(L+M+N)} \cdot \underbrace{(L + M + N)}_{\Lambda^1(L+M+N)}$$

Rmk: The λ-structure on $R(G)$ and $K(X)$ is natural.

($f: X \longrightarrow Y$ induces $K(X) \xleftarrow{f^*} K(Y)$, and $f^*(\lambda_i y) = \lambda_i(f^* y)$ $\forall i$)

Theorem (Adams operations):

Any λ-ring R comes equipped with maps $\psi^s : R \longrightarrow R$ $(s \in \mathbb{N})$ s.t.

1. ψ^s is a ring homomorphism
2. $\psi^s \circ \psi^t = \psi^{s \cdot t}$
3. $\psi^p(a) \equiv a^p \mod p$ for all primes p.

Note: λ-rings have characteristic zero

Proof: $\lambda_t(0) \underset{(3)}{=} 1$

$\qquad \neq$

$\lambda_t(n) = \lambda_t(1)^n \underset{(4)\,[\dots]}{=} (1+t)^n = 1 + \dots + t^n$

So $0 \neq n$. $\square$

One brutal definition:

$$\sum_{j=1}^{\infty} \psi^j(a) t^j = -t \frac{d}{dt} \log \lambda_{-t}(a)$$

(eg: $\lambda_t(a) = 1 + \ell t$

$$\Rightarrow \sum_{j \geq 1} \psi^j(a) t^j = -t \frac{d}{dt} \log(1 - \ell t) = -t \frac{d}{dt}\Big(-\sum_{j \geq 1} \frac{1}{j} (\ell t)^j\Big)$$

$$= \sum_{j \geq 1} \ell^j \cdot t^j, \quad \text{i.e. } \psi^j(a) = \ell^j. \quad)$$

More explanations next time.

Rmk: In all examples above, ψ^s again natural.

Rmk: On line bundles / rank one representations L,

$$\psi^s(L) = L^{\otimes s}.$$

Example: $K(S^2) = K(\mathbb{P}^1) = \mathbb{Z}[\alpha]/\alpha^2$ with $\alpha = L - 1$.

$$\psi^s(\alpha) = L^s - 1 = (\alpha+1)^s - 1 = (s \cdot \alpha + 1) - 1$$

$$= s \cdot \alpha$$

More generally,

$K(S^{2k}) = \mathbb{Z}[\alpha]/\alpha^2$ with

$\psi^s(\alpha) = s^k \cdot \alpha$.

How are they useful?

Def.: A *division algebra* over $\mathbb{R}$ is a f.d. $\mathbb{R}$-vs V with a non-degenerate bilinear map
$$\mu : V \times V \longrightarrow V.$$

so $\mu(a,b) \neq 0$ if $a \neq 0$ and $b \neq 0$

eg:

$\mathbb{R} \times \mathbb{R} \xrightarrow{\cdot} \mathbb{R}$ $\quad(\dim \mathbb{R} = 1)$

$\mathbb{C} \times \mathbb{C} \xrightarrow{\cdot} \mathbb{C}$ $\quad(\dim \mathbb{C} = 2)$

$\mathbb{H} \times \mathbb{H} \xrightarrow{\cdot} \mathbb{H}$ $\quad(\dim \mathbb{H} = 4;$ not commutative$)$

$\mathbb{O} \times \mathbb{O} \xrightarrow{\cdot} \mathbb{O}$ $\quad(\dim \mathbb{O} = 8;$ not associative$)$

Theorem: These are the only division algebras $/\mathbb{R}$.

Def: A manifold is *parallelizable* if its tangent bundle is trivial.

(Any Lie group is parallelizable. $S^1 \times \dots \times S^1$ is parallelizable.)

Theorem: S^n parallelizable $\iff n \in \{0, 1, 3, 7\}$

Adams' proof:

$\mathbb{R}^n$ division algebra
(or S^{n-1} parallelizable)

$\Downarrow$ $\boxed{1}$

$\exists$ continuous
"multiplication" on S^{n-1}

n odd:

n odd

$\Downarrow$

certain ring structure on $K(S^{n-1})$

$\Downarrow$

$n = 1$

not for today

n even:

n even

$\Downarrow$ $\boxed{2}$

$\exists$ certain top. space X built from two spheres

$\Downarrow$ $\boxed{3}$ apply $K(-)$

$\exists$ λ-ring with certain properties

$\Downarrow$ $\boxed{4}$

$n \in \{2, 4, 8\}$

Ad 1: Define

continuous!

$$\hat{\mu} : S^{n-1} \times S^{n-1} \to S^{n-1}$$
$$(x, y) \mapsto \frac{\mu(x,y)}{\|\mu(x,y)\|}$$

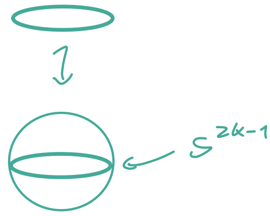
Can modify μ s.t. μ has a unit; then $\hat{\mu}$ also has a unit and defines an H-space structure on S^{n-1} (no associativity or inverses assumed).

Ad 2: Suppose $\boxed{n = 2k.}$

Extend $\hat{\mu}$:

$$\begin{tikzcd}
S^{2k-1} \times S^{2k-1} \arrow[r, "\hat{\mu}"] \arrow[d, hook] & S^{2k-1} \arrow[d, hook] \\
S^{4k-1} \arrow[r, "f"] & S^{2k} \\
\end{tikzcd}$$

$$S^{4k-1} = \partial(D^{4k}) \cong \partial(D^{2k} \times D^{2k})$$



$$S^{2k-1} \times D^{2k} \ni (x,y) \longmapsto \|x\| \cdot \hat{\mu}\left(x/\|x\|, y\right) \in D^{2k} \subset_{S^{2k-1}}$$
$$\cup$$
$$D^{2k} \times S^{2k-1} \ni (x,y) \longmapsto \|y\| \cdot \hat{\mu}\left(x, y/\|y\|\right) \in D^{2k}$$

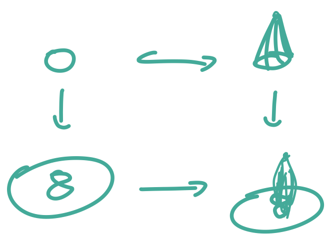
Fact: This is a very special map of spheres; surjective; $\{H\text{-unit}\} \times D^{2k} \xrightarrow{\cong}$ upper D^{2k}

$D^{2k} \times \{H\text{-unit}\} \xrightarrow{\cong}$ lower D^{2k}

$\rightsquigarrow$ Hopf invariant ± 1

Now build space X:

$$\begin{tikzcd}
S^{4k-1} \arrow[r, hook] \arrow[d, "f"'] & D^{4k} \arrow[d] \\
S^{2k} \arrow[r, "i"] \arrow[d, dotted] & X \arrow[d, dotted, "p"] \\
* \arrow[r, dotted] & D^{4k}/S^{4k-1} \cong S^{4k}
\end{tikzcd}$$



There are canonical maps

$$\begin{tikzcd}
S^{2k} \arrow[r, "i"] \arrow[rr, bend right, "\text{constant}"'] & X \arrow[r, "p"] & S^{4k}
\end{tikzcd}.$$

Ad 3:

$$\begin{tikzcd}
K(S^{2k}) \arrow[d, equal] & K(X) \arrow[l, "i^*"'] & K(S^{4k}) \arrow[l, "p^*"'] \arrow[ll, bend right, "0"'] \arrow[d, equal] \\
\mathbb{Z}[\alpha_{2k}]/\alpha_{2k}^2 & & \mathbb{Z}[\alpha_{4k}]/\alpha_{4k}^2
\end{tikzcd}$$

$\psi^s = s^k$ $\qquad\qquad\qquad\qquad$ $\psi^s = s^{2k}$

$$\beta \mapsfrom \alpha_{4k}$$

$$\alpha_{2k} \mapsfrom \alpha$$

Fact: $K(X) = \mathbb{Z} \oplus \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$

with $\alpha^2 = \pm\beta$ $\quad$ Hopf invariant

$\beta^2 = 0$

$\alpha \cdot \beta = 0$

$$\begin{tikzcd}
\tilde{K}(S^{2k}) \otimes \tilde{K}(S^{2k}) \arrow[r, "\cong", "\text{Bott iso}"'] & \tilde{K}(S^{4k}) \arrow[d] \\
\tilde{K}X \otimes \tilde{K}X \arrow[u] \arrow[r] & \tilde{K}X
\end{tikzcd}$$

$\beta := p^*(\alpha_{4k})$, so $\beta^2 = 0$ clear

$i^*(\alpha^2) = 0$, so $\alpha^2 = 0 \cdot \alpha + h \cdot \beta$ for <u>some</u> $h \in \mathbb{Z}$

7,5

Ad 4: $\psi^s(\beta) = \psi^s(p^* \alpha_{2k}) = s^{2k} \cdot \beta$

$\psi^s(\alpha) = s^k \cdot \alpha + m_s \cdot \beta$ für ein $m_s \in \mathbb{Z}$

Constraints on coefficients m_s:

1. $$\psi^2(\alpha) \overset{!}{\equiv} \alpha^2 \mod 2$$
$$\equiv \beta \mod 2$$
So m_2 must be odd.

2. $$\begin{aligned} \psi^t\psi^s(\alpha) &= \psi^t(s^k \cdot \alpha + m_s \beta) = s^k(t^k \alpha + m_t \beta) + t^{2k} m_s \beta \\ &= s^k t^k \alpha + \left(\boxed{s^k m_t + t^{2k} m_s}\right) \cdot \beta \\ \overset{!}{\Vert}\quad & \\ \psi^s\psi^t(\alpha) &= \ldots\ldots = s^k t^k \alpha + \left(\boxed{s^{2k} m_t + t^k m_s}\right) \cdot \beta \end{aligned}$$

Take $s=2, t=3$:

$$2^k m_3 + 3^{2k} m_2 = 2^{2k} m_3 + 3^k m_2,$$
$$3^k(3^k - 1) m_2 = 2^k(2^k - 1) m_3.$$

As m_2 is odd, this implies
$$2^k \mid 3^k - 1.$$

This is only true for $k \in \{1, 2, 4\}$

(i.e. for $n \in \{2, 4, 8\}$) $\square$