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A Structure Theorem for SI-Modules

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An associative ring R is called a *left SI-ring* if every singular left R-module is injective. In Goodearl [4] it is shown that these rings have a finite ring decomposition into a ring K with $K/Soc\,K$ left semisimple, and simple rings which are Morita equivalent to left SI-domains.

For an R-module M denote by $\sigma[M]$ the full subcategory of R-*Mod* subgenerated by M. Extending the definition of SI-rings, we call an R-module M an ***SI-module*** if every ***singular module in*** $\sigma[M]$ is M-injective. This also generalizes a similar notion in Yousif [11]. We obtain that every finitely generated, self-projective SI-module M has a decomposition

$$M = K \oplus V_1 \oplus \cdots \oplus V_n,$$

with fully invariant submodules K, V_i, such that $K/Soc\,K$ is a semisimple R-module, and, for $i = 1, \cdots, n$, $End_R(V_i)$ is a simple ring, and the category $\sigma[V_i]$ is equivalent to T_i-*Mod* for an SI-domain T_i.

1 Preliminary results

Let R be an associative ring with unit and R-*Mod* the category of unital left R-modules. For $M \in R$-*Mod* we denote by $\sigma[M]$ the full subcategory of R-*Mod* whose objects are submodules of M-generated modules. M is called ***self-projective*** if it is M-projective. $Soc\,M$ (resp. $Rad\,M$) denotes the ***socle*** (resp. the ***radical***) of the module M. An R-submodule of M is said to be ***fully invariant*** (or ***characteristic***) if it is invariant under any R-endomorphism of M.

The kernel of a homomorphism f is denoted by $Ke\,f$. Morphisms are written on the opposite side of the scalars. For basic notions see [10].

The following elementary observations will be useful:

1.1 Proposition. *Consider a self-projective R-module M with $S = End_R(M)$.*

(1) If $Rad\,M = 0$, then S has zero (Jacobson) radical.

(2) Assume M is finitely generated. Then M has no non-trivial fully invariant submodules if and only if S is a simple ring.

Proof: (1) This follows from the fact that, for any simple homomorphic image E of M, $Hom_R(M, E)$ is a simple left $End_R(M)$-module.

(2) For every ideal $I \subset S$, $MI \subset M$ is fully invariant. Since M is self-projective, $I = Hom_R(M, MI)$ by [10, 18.4] and hence $MI \neq M$ for $I \neq S$.

For every fully invariant submodule $U \subset M$, $Hom_R(M, U)$ is an ideal in S.

If $K \subset M$ is an essential submodule, we write $K \trianglelefteq M$.

Let M and N be R-modules. N is called ***singular in*** $\sigma[M]$ or ***M-singular*** if $N \simeq L/K$ for some $L \in \sigma[M]$ and $K \trianglelefteq L$ (see [9]).

By definition, every M-singular module belongs to $\sigma[M]$. For $M = R$ the notion ***R-singular*** is identical to the usual definition of ***singular*** for R-modules.

The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (e.g. [10, 17.3 and 17.4]). Hence every module $N \in \sigma[M]$ contains a ***largest M-singular submodule*** which we denote by $Z_M(N)$. The following properties of M-singular modules are shown in [9, 1.1] and [8, 2.4]:

1.2 Proposition. *Let M be an R-module.*

(1) A simple R-module E is M-singular or M-projective.

(2) If $Soc\,M = 0$, then every simple module in $\sigma[M]$ is M-singular.

(3) If M is self-projective and $Z_M(M) = 0$, then the M-singular modules form a hereditary torsion class in $\sigma[M]$.

We extend the definition of a *left SI-ring* (see [4]) to modules:

Definition: An R-module M is called an SI-module if every M-singular module is M-injective.

In Yousif [11], M is called *SI-module* if *every* singular module in *R-Mod* is M-injective. Since M-singular modules are singular in *R-Mod*, this is a stronger condition then the one given above.

Though for $M = R$ the two notions coincide, in general SI-modules in our sense need not be SI-modules in the sense of Yousif (compare the example after [9, 2.2]):

Let T be a left SI-ring which is not left semisimple (for examples see [4], [1]), and R the ring of lower triangular $(2,2)$-matrices over T. The map

$$R = \begin{pmatrix} T & 0 \\ T & T \end{pmatrix} \longrightarrow T, \qquad \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \longmapsto a,$$

is a surjective ring homomorphism whose kernel is essential as left ideal in R. Hence every left T-module is singular as R-module and all T-singular modules are T-injective, i.e. T is a SI-module over R. Since T is not left semisimple, not every R-singular module is T-injective. Hence T is not an SI-module over R in the sense of Yousif.

Every left module over a left SI-ring is an SI-module in the sense of Yousif and hence is an example of an SI-module in our sense.

In Smith [7], R is called a *left RIC-ring* if every *cyclic* singular left R-module is injective. It is observed in [5, Corollary 5] that RIC-rings are SI-rings. By [9, 3.8 and 3.10] and [2, Lemma 2], we have more general statements in our next Proposition which also include Proposition 3.1 and 3.6 in [4]. For this we recall two definitions:

An R-module M is called *hereditary in* $\sigma[M]$ if every submodule of M is projective in $\sigma[M]$ (see [10, 39.1]). M is a *GCO-module* (*generalized co-semisimple*) if every M-singular simple R-module is M-injective (see [9, 2.2]).

1.3 Proposition. *For a finitely generated, self-projective R-module M the following conditions are equivalent:*

(a) M is an SI-module;

(b) every cyclic M-singular module is M-injective;

(c) M/K is semisimple for every $K \trianglelefteq M$ and $Z_M(M) = 0$.

(d) M is hereditary in $\sigma[M]$ and M-singular modules are semisimple;

(e) M is a GCO-module, $M/Soc\,M$ is noetherian and $Soc(M/K) \neq 0$ for every $K \trianglelefteq M$.

We will need the following

1.4 Lemma. *Let M be a self-projective SI-module with finite uniform dimension and $Rad\, M = 0$. Then M contains no proper fully invariant submodule which is essential as R-submodule.*

Proof: Assume $V \subset M$ is a fully invariant submodule and $V \trianglelefteq {}_R M$. Since $Rad\, M = 0$, $S := End_R(M)$ has zero radical by Proposition 1.1. $Rad\, M = 0$ also implies that $Hom_R(M, U) \neq 0$ for non-zero $U \subset M$ (see [8], p. 1475, (iv)). With this knowledge we derive from Theorem 3.7 in [8] that there exists a monomorphism $f : M \to V$, and since M has finite uniform dimension, the image of every monomorphism in S is essential in M. Hence the image of f^2 is essential in M. Therefore M/Mf^2 is a semisimple module and the following exact sequence splits:

$$0 \to Mf/Mf^2 \to M/Mf^2 \to M/Mf \to 0.$$

Applying the functor $Hom_R(M, -)$ and the isomorphisms $Sg \simeq Hom(M, Mg)$ for any $g \in S$ (see [10], 18.4), we obtain that Sf/Sf^2 is a direct summand in the S-module S/Sf^2. Hence there exists a submodule $Sf^2 \subset U \subset S$ with $Sf + U = S$ and $Sf \cap U = Sf^2$. This yields $id = rf + u$ for some $r \in S$ and $u \in U$ and hence $f = frf + fu$. Since $fu \in Sf \cap U = Sf^2$ we finally have $f = frf + sf^2$ for some $s \in S$. Since f is monic this means $id = fr + sf$ and $M = Mfr + Msf \subset V$.

2 Structure Theorem

Let us first describe uniform SI-modules with zero socle:

2.1 Proposition. *For a finitely generated, self-projective R-module M, the following are equivalent:*

(a) M is a uniform SI-module with $Soc\, M = 0$;

(b) M is a self-generator and $End_R(M)$ is a left SI-domain which is not a division ring.

Under this condition, M has no fully invariant submodules and $End_R(M)$ is a simple ring.

Proof: $(a) \Rightarrow (b)$ If M is an SI-module with zero socle, all simple modules in $\sigma[M]$ are M-singular (by Proposition 1.2), hence M-injective and M-generated. Therefore M is a projective generator in $\sigma[M]$. This implies that $\sigma[M]$ is equivalent to S-*Mod* (see [10, 18.5 and 46.2]) and S is a left SI-ring.

Since $Z_M(M) = 0$, every $f \in End_R(M)$ is a monomorphism.

$(b) \Rightarrow (a)$ The functor $Hom_R(M, -)$ is an equivalence.

The last part follows from Lemma 1.4 and Proposition 1.1.

Now we investigate the decomposition of SI-modules with zero socles:

2.2 Theorem. *For a finitely generated, self-projective R-module M and* $S = End_R(M)$, *the following are equivalent:*

(a) M is an SI-module and $Soc\, M = 0$;

(b) M is a generator in $\sigma[M]$ *and S is a left SI-ring with zero left socle;*

(c) $M = M_1 \oplus \cdots \oplus M_n$, *with* M_i *minimal fully invariant submodules, and* $\sigma[M_i] = \sigma[L_i]$ *for some finitely generated, self-projective and uniform SI-module* L_i *with zero socle;*

(d) $M = M_1 \oplus \cdots \oplus M_n$, *with* M_i *fully invariant submodules,* $End_R(M_i)$ *simple rings and* $\sigma[M_i]$ *equivalent to* T_i*-Mod, for left SI-domains* T_i *which are not division rings.*

Proof: $(a) \Leftrightarrow (b)$ As observed in the proof of Proposition 2.1, (a) implies that M is a projective generator in $\sigma[M]$. Hence $\sigma[M]$ is equivalent to S-*Mod* (see [10, 18.5 and 46.2]) and M is an SI-module if and only if S is a left SI-ring.

$(a) \Rightarrow (c)$ As noted above, M is a generator in $\sigma[M]$, and by Proposition 1.3, M is noetherian and hereditary in $\sigma[M]$. Every essential submodule of M is an intersection of maximal submodules, and $Soc\, M = 0$ implies $Rad\, M = 0$ and S has zero radical by Proposition 1.1.

By Theorem 3.7 in [8], the endomorphism ring of the M-injective hull $\widehat{M}$ is semisimple artinian, i.e. $End_R(\widehat{M}) = T_1 \oplus \cdots \oplus T_n$ with simple artinian rings T_i. Denoting by e_i the unit in T_i, we have $e_1 + \cdots + e_n = id_{\widehat{M}}$ and, since the e_i are in the center of $End_R(\widehat{M})$,

$$\widehat{M} = \widehat{M}e_1 \oplus \cdots \oplus \widehat{M}e_n$$

is a decomposition into fully invariant submodules. The intersection $M_i := M \cap \widehat{M}e_i$ is a fully invariant submodule of M and $M_1 \oplus \cdots \oplus M_n \trianglelefteq {}_R M$. As we have seen in Lemma 1.4, this means

$$M_1 \oplus \cdots \oplus M_n = {}_RM.$$

Since $Hom_R(M_i, M_j) = 0$ for $i \neq j$, we observe that $\widehat{M}e_i$ is the injective hull of M_i in $\sigma[M_i]$. Moreover M_i is a self-projective self-generator with $Z_M(M) = 0$ and, again applying Theorem 3.7 in [8], we know that $End_R(\widehat{M}e_i) \simeq T_i$ is the classical left quotient ring of $End_R(M_i)$. Hence $End_R(M_i)$ has no non-trivial central idempotents and M_i has no non-trivial decomposition into fully invariant submodules.

To study properties of the summands M_i we may assume that M itself has no non-trivial decomposition into fully invariant submodules. We want to show that M has no proper fully invariant submodules.

Let $X \subset M$ be fully invariant. First consider a non-zero R-submodule $Y \subset M$ with $X \cap Y = 0$. We show that X and Y do not have isomorphic uniform submodules: Assume, for a uniform submodule $U \subset X$, there exists a monomorphism $g : U \to Y$. Since $Hom_R(M, U)$ is a non-zero left ideal in S and $Rad\ S = 0$, we can find $f : M \to U$ with $f^2 \neq 0$ and hence $(U)f \neq 0$. Then $(U)fg \subset Y$ and, by the invariance of X, also $(U)fg \subset X$ implying $(U)f = 0$, a contradiction.

Now let $\{Y_\lambda\}_\Lambda$ denote the family of all submodules of M, with no uniform submodules isomorphic to submodules of X, and put $Y = \sum_\Lambda Y_\lambda$.

Assume Y contains a uniform submodule U isomorphic to a submodule of X. Since M is hereditary in $\sigma[M]$, we may suppose $U \subset \bigoplus_\Lambda Y_\lambda$, and we conclude that U has an isomorphic copy in one of the Y_λ's (compare [10, 39.7]), a contradiction. Obviously, $X \cap Y = 0$ and, by the above observation, $X \oplus Y \trianglelefteq {}_RM$.

Hereditariness of M also implies that, for any $f \in S$, $(Y)f$ has no uniform submodules isomorphic to submodules in X. Hence $(Y)f \subset Y$, i.e. Y and $X \oplus Y$ are fully invariant in M. By Lemma 1.4, we have $X \oplus Y = M$. This means by assumption $X = M$.

Now choose a uniform submodule $U \subset M$ and a non-zero $f \in Hom_R(M, U)$. Then $L := (M)f$ is uniform and M-projective. The trace $Tr(L, M)$ of L in M is fully invariant and hence $Tr(L, M) = M$, implying $\sigma[M] = \sigma[L]$.

$(c) \Rightarrow (d)$ Each of the L_i is a progenerator in $\sigma[M_i] = \sigma[L_i]$ (see proof of $(a) \Leftrightarrow (b)$). Hence $\sigma[M_i]$ is equivalent to T_i-Mod where $T_i := End_R(L_i)$ is a left

SI-domain by Proposition 2.1.

According to Proposition 1.1, $End_R(M_i)$ is a simple ring.

$(d) \Rightarrow (b)$ By the given equivalences, every M_i is an SI-module and $\sigma[M_i]$ contains an M_i-projective generator L_i with zero socle. Then also M_i has zero socle and is a progenerator in $\sigma[M_i]$ (see proof of $(a) \Leftrightarrow (b)$), and $End_R(M_i)$ is a left SI-ring. As a product of these rings, $End_R(M)$ is also a left SI-ring.

Remark: For the proof of $(b) \Rightarrow (c)$ we could have used part of Goodearl's structure theorem for left SI-rings in [4, 3.11]. For $M = R$ our proof provides an alternative to Goodearl's proof of the corresponding part.

Finally we are ready to prove the following extension of Goodearl's characterization of SI-rings in [4, 3.11]:

2.3 Structure Theorem. *For a finitely generated, self-projective R-module M and $S = End_R(M)$, the following are equivalent:*

(a) M is an SI-module;

(b) $Z_M(M) = 0$ and M has a decomposition

$$M = K \oplus V_1 \oplus \cdots \oplus V_n$$

with fully invariant submodules K, V_i, such that $K/Soc\,K$ is a semisimple R-module, and, for $i = 1, \cdots, n$, $End_R(V_i)$ is a simple ring and the category $\sigma[V_i]$ is equivalent to T_i-Mod, for an SI-domain T_i which is not a division ring.

Under the given conditions, S is a left SI-ring.

Proof: $(a) \Rightarrow (b)$ Assume M is an SI-module. As already observed in Proposition 1.3, M is hereditary and $\overline{M} := M/Soc\,M$ is noetherian.

As noted in Proposition 1.2, the M-singular modules form a torsion class in $\sigma[M]$. Let K denote the R-submodule $Soc\,M \subset K \subset M$ such that $K/Soc\,M$ is the torsion submodule of $\overline{M}$ in this torsion theory. Since $Soc\,M$ is fully invariant in M and $K/Soc\,M$ is fully invariant in $M/Soc\,M$, K is fully invariant in M.

By construction, $Soc\,K = Soc\,M$. Also $K/Soc\,M$ is an SI-module and $Soc\,M \trianglelefteq K$ since K is projective in $\sigma[M]$ (M hereditary). Hence $K/Soc\,M$ is semisimple by 1.3 and M-injective by assumption. Therefore

$$\overline{M} = K/Soc\,M \oplus N/Soc\,M$$

for some R-submodule $N \subset M$ containing $Soc\, M$. Since $Soc\, M$ is a fully invariant submodule, $\overline{M}$ is self-projective. As M/L is semisimple for $L \trianglelefteq M$, and $Soc\, M$ is the intersection of all $L \trianglelefteq M$, we conclude $Rad\, \overline{M} = 0$.

Hence $M/K \simeq N/Soc\, M$ is a self-projective SI-module with zero radical. By definition of K, M/K contains no M-singular submodules. Therefore every simple submodule of M/K is M-projective by Proposition 1.2. Since $Soc\, M \subset K$, we conclude $Soc\,(M/K) = 0$.

Denote by $\{H_\lambda\}_\Lambda$ the family of all submodules of M with $Soc\, H_\lambda = 0$ and set $V = \sum_\Lambda H_\lambda$. Since all simple submodules of $V \subset M$ are M-projective (by Proposition 1.2) and $\bigoplus_\Lambda H_\lambda$ has zero socle, also $Soc\, V = 0$ and $K \cap V = 0$. The M-projectivity of simple submodules of M also implies that, for every $f \in S$, $(V)f$ has zero socle and hence $(V)f \subset V$, i.e. V is fully invariant. It is obvious from the definitions and the properties derived that $Soc\, M \oplus V \trianglelefteq K \oplus V \trianglelefteq {}_R M$ and that $K \oplus V$ is a fully invariant submodule of M.

Passing to the factor module, we have that $(K \oplus V)/K$ is a fully invariant submodule of M/K which is essential as R-submodule. Recalling the properties of M/K shown above, by Lemma 1.4, this implies $K \oplus V = M$.

The decomposition of V is now obtained from Theorem 2.2.

$(b) \Rightarrow (a)$ Obviously, for every essential submodule $U \subset M$, M/U is semisimple and hence M is an SI-module by Proposition 1.3.

It remains to show that S is a left SI-ring. Since M is hereditary in $\sigma[M]$, S is left semi-hereditary by [10, 39.14] and hence left non-singular.

From the exact sequence $0 \to Soc\, M \to M \to \overline{M} \to 0$, we derive the exact sequence

$$0 \to Hom(M, Soc\, M) \to S \to Hom(M, \overline{M}) \to 0.$$

Since $\overline{M} \simeq K/Soc\, K$ is semisimple, $Hom(M, \overline{M})$ is a semisimple left S-module. From $Hom(M, Soc\, M) \subset Soc\, S$ we conclude that $S/Soc\, S$ is left semisimple and S is a left SI-ring by Proposition 1.3.

Remark: For $M = R$, our Structure Theorem yields Goodearl's Structure Theorem for SI-rings (see [4, 3.11]), which was also proved in Theorem 2.7 of Baccella [1] in a different way.

Obviously any SI-module is a GCO-module (compare 1.3). By our Structure Theorem we obtain that self-projective GCO-modules with descending chain condition on essential submodules are SI-modules. Referring to [9, 3.11] we have:

2.4 Corollary. *For a finitely generated, self-projective R-module M, the following are equivalent:*

(a) M is an SI-module with dcc on essential submodules;

(b) M is a GCO-module with dcc on essential submodules;

(c) $M/\mathrm{Soc}\,M$ *is semisimple and* $Z_M(M) = 0$.

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