

THE BIGGER BRAUER GROUP AND TWISTED SHEAVES

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ABSTRACT. Given an algebraic stack with quasiaffine diagonal, we show that each \mathbb{G}_m -gerbe comes from a central separable algebra. In other words, Taylor's bigger Brauer group equals the étale cohomology in degree two with coefficients in \mathbb{G}_m . This gives new results also for schemes. We use the method of twisted sheaves explored by Lieblich and de Jong.

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INTRODUCTION

Let X be a scheme. About forty years ago, Grothendieck [8] posed the problem whether the inclusion $\mathrm{Br}(X) \subset H^2(X, \mathbb{G}_m)$ of the Brauer group of Azumaya algebras coincides with the torsion part of the étale cohomology group. It is known that this fails for certain nonseparated schemes [4]. On the other hand, there are strong positive results. Gabber [8] proved equality for affine schemes, and also had an unpublished proof for schemes carrying ample line bundles. Recently, de Jong [3] gave a new proof for the latter statement, based on the notion of *twisted sheaves*, that is, sheaves on gerbes. This method already turned out to be rich and powerful in the work of Lieblich [14], [15].

In this paper we shall prove that there is, for arbitrary noetherian schemes, an equality $\widetilde{\mathrm{Br}}(X) = H^2(X, \mathbb{G}_m)$, where $\widetilde{\mathrm{Br}}(X)$ is the *bigger Brauer group*. This group is defined in terms of so-called *central separable algebras*, and was introduced by Taylor [22] (Caenepeel and Grandjean [2] later fixed some technical problem in the original definition). Such algebras are defined and behave very similar to Azumaya algebras, but do not necessarily contain a unit. Raeburn and Taylor [17] constructed an inclusion $\widetilde{\mathrm{Br}}(X) \subset H^2(X, \mathbb{G}_m)$ using methods from nonabelian cohomology, and showed that this inclusion actually is an equality provided X carries ample line bundles. To remove this assumption, we shall use de Jong's insight [3] and work with a gerbe \mathcal{G} defining the cohomology class $\alpha \in H^2(X, \mathbb{G}_m)$.

The basic observation is that \mathcal{G} may be viewed as an algebraic stack (= Artin stack), and that the existence of the desired central separable algebra on X is equivalent to the existence of certain coherent sheaves on \mathcal{G} . A key ingredient in our arguments is the result of Laumon and Moret-Bailly that quasicoherent sheaves on noetherian algebraic stacks are direct limits of coherent sheaves [13].

This stack-theoretic approach suggests a generalization of the problem at hand: Why not replace the scheme X by an algebraic stack \mathcal{X} ? Our investigation actually takes place in the setting. Here, however, one has to make an additional assumption. Our main result is that $\widetilde{\text{Br}}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{G}_m)$ holds for noetherian algebraic stacks whose diagonal morphism $\mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is quasiaffine. Deligne–Mumford stacks, and in particular algebraic spaces and schemes, automatically satisfy this assumption. In contrast, there are algebraic stacks with $\widetilde{\text{Br}}(\mathcal{X}) \subsetneq H^2(\mathcal{X}, \mathbb{G}_m)$. We discuss an example based on observations of Totaro [21].

Working with sheaves and cohomology on algebraic stacks \mathcal{X} requires some care. A convenient setting is the so-called *lisse-étale site* $\text{Lis-et}(\mathcal{X})$. For our purposes, it is useful to work with a larger site as well, which we call the *big-étale site* $\text{Big-et}(\mathcal{X})$. The relation between the associated topoi $\mathcal{X}_{\text{lisse-ét}}$ and $\mathcal{X}_{\text{big-ét}}$ is not so straightforward as one might expect at first glance. The problem is, roughly speaking, that they are not related by a map of topoi. Such phenomena gained notoriety in the theory of algebraic stacks, and were explored by Behrend [1] and Olsson [16]. However, in the Appendix we recall that an abelian big-étale sheaf and its restriction to the lisse-étale site have the same cohomology.

1. GERBES ON ALGEBRAIC STACKS

In this section we recall some basic facts on gerbes over algebraic stacks. Throughout, we closely follow the book of Laumon and Moret-Bailly [13] in terminology and notation.

Fix a base scheme S , and let (Aff/S) be the category of affine schemes endowed with a morphism to S . Let \mathcal{X} be an algebraic S -stack. A *lisse-étale sheaf* on \mathcal{X} is, by definition, a sheaf on the *lisse-étale site* $\text{Lis-et}(\mathcal{X})$. The objects of the latter are pairs (U, u) , where U is an algebraic space, and $u : U \rightarrow \mathcal{X}$ is a smooth morphism. The morphisms from (U_1, u_1) to another object (U_2, u_2) are pairs (f, α) , where $f : U_1 \rightarrow U_2$ is a morphism of algebraic spaces, and α is a natural transformation between the functors $u_1, u_2 \circ f : U_1 \rightarrow \mathcal{X}$, such that we have a 2-commutative diagram

$$\begin{array}{ccc} & & u_1 \\ & \curvearrowright & \\ U_1 & & \mathcal{X} \\ & \curvearrowleft & \\ & & u_2 \circ f \end{array}$$

The Grothendieck topology is generated by those (f, α) with $f : U_1 \rightarrow U_2$ étale and surjective. We denote by $\mathcal{X}_{\text{lisse-ét}}$ the associated *lisse-étale topos*, that is, the category of lisse-étale sheaves on \mathcal{X} .

For the applications we have in mind, it is natural to work with a larger site as well. It resembles the big site of a topological space, so we call it the *big-étale site* $\text{Big-et}(\mathcal{X})$. Here the objects are pairs (U, u) , where again U is an algebraic space, but now the morphism $u : U \rightarrow \mathcal{X}$ is arbitrary. Morphisms and Grothendieck

topology are defined as for the lisse-étale site. The associated big-étale topoi is denoted by $\mathcal{X}_{\text{big-et}}$. Following [13], we write $\underline{\text{Big-et}}(\mathcal{X}) \subset \text{Big-et}(\mathcal{X})$ for the subsites of objects (U, u) with U affine. According to the Comparison Lemma [12], Exposé III, Theorem 4.1, this inclusion induces an equivalence between the corresponding topoi of sheaves.

The relation between the big-étale and the lisse-étale site is not as straightforward as one might expect. However, given a big-étale sheaf \mathcal{F} , there actually is a canonical map

$$H^i(\mathcal{X}_{\text{big-et}}, \mathcal{F}) \longrightarrow H^i(\mathcal{X}_{\text{lis-et}}, \mathcal{F}|_{\text{Lis-et}(\mathcal{X})}),$$

and we shall see in the appendix that this map is bijective (Proposition 4.1). We therefore write $H^i(\mathcal{X}, \mathcal{F})$ for the cohomology of big-étale sheaves, provided there is no risk of confusion.

Next we recall some facts on gerbes on \mathcal{X} . Let $G \rightarrow S$ be an algebraic space over S endowed with the structure of a commutative group. (For our applications we merely use the case $G = \mathbb{G}_{m,S}$.) This yields an abelian big-étale sheaf denoted $G_{\mathcal{X}}$, whose groups of local sections $\Gamma((U, u), G_{\mathcal{X}})$ is the set of S -morphisms $g : U \rightarrow G$. In turn, we have cohomology groups $H^i(\mathcal{X}, G_{\mathcal{X}})$. According to the previous paragraph, it does not matter whether we compute cohomology on the lisse-étale or big-étale site.

As explained in the Giraud's treatise [5], Chapter IV, §3.4, cohomology classes from $H^2(\mathcal{X}, G_{\mathcal{X}})$ correspond to equivalence classes of $G_{\mathcal{X}}$ -gerbes $\mathcal{G} \rightarrow \underline{\text{Big-et}}(\mathcal{X})$; equivalently, a gerbe on the lisse-étale site. By composing with $(U, u) \mapsto U$, we obtain a functor $\mathcal{G} \rightarrow (\text{Aff}/S)$. As Lieblich explains in [15], Proposition 2.4.3, this makes \mathcal{G} into an S -stack, endowed with a 1-morphism of S -stacks $F : \mathcal{G} \rightarrow \mathcal{X}$. Under fairly general assumptions, this S -stack is algebraic; the following criterion generalizes a result of de Jong [3] and Lieblich ([15], Corollary 2.4.4):

Proposition 1.1. *Notation as above. Suppose that the structure morphism $G \rightarrow S$ is smooth, separated and of finite presentation. Then the S -stack \mathcal{G} is algebraic.*

Proof. Choose a smooth surjection $P : X \rightarrow \mathcal{X}$ for some scheme X , with $\mathcal{G}_{X,P}$ nonempty. Then the projection $\mathcal{G} \times_{\mathcal{X}} X \rightarrow X$ has a section, and by [13], Lemma 3.21, there is a 1-isomorphism $\mathcal{G} \times_{\mathcal{X}} X \rightarrow B(G_X)$ into the S -stack of G_X -torsors and this stack is algebraic.

Choose a smooth surjection $Y \rightarrow \mathcal{G} \times_{\mathcal{X}} X$ from some scheme Y . Composing with the first projection, we obtain a smooth, surjective, representable morphism $H : Y \rightarrow \mathcal{G}$. In light of loc. cit., Proposition 4.3.2, it remains to check that the canonical morphism $Y \times_{\mathcal{G}} Y \rightarrow Y \times Y$ is quasicompact and separated. Note that $Y \times Y$ and $Y \times_{\mathcal{G}} Y$ are associated to a scheme and an algebraic space, respectively. Our map factors over $Y \times_{\mathcal{X}} Y$, and the morphism of schemes $Y \times_{\mathcal{X}} Y \rightarrow Y \times Y$ is quasicompact and separated, because the S -stack \mathcal{X} is algebraic. Whence it suffices to check that $Y \times_{\mathcal{G}} Y \rightarrow Y \times_{\mathcal{X}} Y$ is quasicompact and separated.

To verify this, consider the following G -action on the objects of the S -stack $Y \times_{\mathcal{G}} Y$: Given some $U \in (\text{Aff}/S)$, the objects in $Y \times_{\mathcal{G}} Y$ over U are, by definition, triples (u_1, u_2, φ) , where $u_i : U \rightarrow Y$ are S -morphisms, and $\varphi : H(u_1) \rightarrow H(u_2)$ is an isomorphism in \mathcal{G}_U . Then the S -morphisms $g : U \rightarrow G$ act on such triples via $g \cdot (u_1, u_2, \varphi) = (u_1, u_2, g\varphi)$. Using that $\mathcal{G} \rightarrow \underline{\text{Big-et}}(\mathcal{X})$ is a $G_{\mathcal{X}}$ -gerbe, we infer that our morphism $Y \times_{\mathcal{G}} Y \rightarrow Y \times_{\mathcal{X}} Y$, viewed as a morphism of algebraic spaces, is a G -principal bundle with respect to the étale topology. Our assumptions on

the structure morphism $G \rightarrow S$ ensure that it is quasicompact and separated. By descent, the same holds for $Y \times_{\mathcal{G}} Y \rightarrow Y \times_{\mathcal{X}} Y$, see [9], Exposé V, Corollary 4.6 and 4.8. \square

Remark 1.2. Using Artin's theorem [13] Proposition 10.31.1, the above proof generalizes to the case that $G \rightarrow S$ is a flat, separated group scheme of finite presentation if one considers gerbes in the fppf-topology.

Now suppose that $G \rightarrow S$ is smooth, separated, and of finite presentation. It is then easy to see that the resulting morphism $F : \mathcal{G} \rightarrow \mathcal{X}$ of algebraic S -stacks is smooth as well, compare [13], Remark 10.13.2. Given a quasicoherent sheaf \mathcal{H} on \mathcal{X} , we obtain functorially a quasicoherent sheaf $F^*(\mathcal{H})$ on \mathcal{G} , defined by

$$F^*(\mathcal{H})_{U,u} = \mathcal{H}_{U,Fu}, \quad (U, u) \in \text{Lis-et}(\mathcal{G}).$$

We now describe those quasicoherent sheaves on \mathcal{G} that are isomorphic to pullbacks $F^*(\mathcal{H})$. Let \mathcal{F} be a quasicoherent sheaf on \mathcal{G} , and $(U, u) \in \text{Lis-et}(\mathcal{G})$. Any local section $g \in \Gamma((U, Fu), G_{\mathcal{X}})$ induces an automorphism $(\text{id}_U, g) : (U, u) \rightarrow (U, u)$ in the lisse-étale site. In turn, it acts bijectively on local sections

$$(1) \quad (\text{id}_U, g)^* : \Gamma((U, u), \mathcal{F}) \longrightarrow \Gamma((U, u), \mathcal{F}).$$

Sheaves for which all these bijections are actually identities shall play an important role throughout. Let us introduce the following terminology, which comes from the special case $G = \mathbb{G}_{m,S}$:

Definition 1.3. A quasicoherent sheaf \mathcal{F} on \mathcal{G} is called *of weight zero* if the bijections in (1) are identities for all (U, u) and g as above.

The following characterization of sheaves of weight zero is well-known:

Lemma 1.4. *The functor $\mathcal{H} \mapsto F^*(\mathcal{H})$ is an equivalence between the category of quasicoherent sheaves on \mathcal{X} and the category of quasicoherent sheaves on \mathcal{G} of weight zero.*

Proof. Choose a smooth surjection $u : U \rightarrow \mathcal{G}$ from some scheme U . According to [13], Proposition 13.2.4, the category of quasicoherent sheaves on \mathcal{G} is equivalent to the category of quasicoherent sheaves on U endowed with a descent datum with respect to u . Let \mathcal{F} be a quasicoherent sheaf on \mathcal{G} of weight zero, with induced descent datum $\varphi : \text{pr}_1^*(\mathcal{F}_{U,u}) \rightarrow \text{pr}_2^*(\mathcal{F}_{U,u})$ on $U \times_{\mathcal{G}} U$. As discussed in the proof of Proposition 1.1, the morphism $U \times_{\mathcal{G}} U \rightarrow U \times_{\mathcal{X}} U$ is a $G_{U \times_{\mathcal{X}} U}$ -torsor. Since \mathcal{F} is of weight zero, φ is invariant under $G_{U \times_{\mathcal{X}} U}$, whence descends to $U \times_{\mathcal{X}} U$. In this way we obtain for the quasicoherent sheaf $\mathcal{F}_{U,u}$ on U a descent datum with respect to the smooth surjection $Fu : U \rightarrow \mathcal{X}$, which in turn defines a quasicoherent sheaf \mathcal{H} on \mathcal{G} . It is easy to see that there is a natural isomorphism $\mathcal{F} \simeq F^*(\mathcal{H})$, and that the functor $\mathcal{F} \mapsto \mathcal{H}$ is quasiinverse to $\mathcal{H} \mapsto F^*(\mathcal{H})$. \square

2. TAYLOR'S BIGGER BRAUER GROUP

In this section we recall and discuss Taylor's bigger Brauer group [22] in the general context of algebraic stacks. Taylor's idea is to attach to certain kinds of (not necessarily unital) associative algebras on \mathcal{X} a \mathbb{G}_m -gerbe, which in turn yields a cohomology class in $H^2(\mathcal{X}, \mathbb{G}_m)$. The collection of all such cohomology classes constitutes a subgroup, which is called the *bigger Brauer group* $\widetilde{\text{Br}}(\mathcal{X}) \subset H^2(\mathcal{X}, \mathbb{G}_m)$.

Let us now go into details. Suppose we have two quasicoherent sheaves \mathcal{M} and \mathcal{H} on \mathcal{X} , together with a pairing $\Phi : \mathcal{H} \otimes \mathcal{M} \rightarrow \mathcal{O}_{\mathcal{X}}$. This defines a quasicoherent associative $\mathcal{O}_{\mathcal{X}}$ -algebra $\mathcal{M} \otimes^{\Phi} \mathcal{H}$ as follows: The underlying quasicoherent sheaf is $\mathcal{M} \otimes \mathcal{H}$; the multiplication law is defined on local sections by

$$(m \otimes h) \cdot (m' \otimes h') = m \otimes \Phi(h, m')h'.$$

An important special case is that \mathcal{M} is locally free of finite rank, $\mathcal{H} = \mathcal{M}^{\vee}$ is the dual sheaf, and $\Phi(h, m) = h(m)$ is the evaluation pairing. Then $\mathcal{M} \otimes^{\Phi} \mathcal{H}$ is canonically isomorphic to the endomorphism algebra $\mathcal{E}nd(\mathcal{M})$, which contains a unit. Note, however, that in general $\mathcal{M} \otimes^{\Phi} \mathcal{H}$ does not contain a unit.

In the following we are interested in algebras that are locally of the form $\mathcal{M} \otimes^{\Phi} \mathcal{H}$, where one additionally demands that the pairing $\Phi : \mathcal{H} \otimes \mathcal{M} \rightarrow \mathcal{O}_{\mathcal{X}}$ is surjective. Given an $\mathcal{O}_{\mathcal{X}}$ -algebra \mathcal{A} , we use the following ad hoc terminology: A *local splitting* for \mathcal{A} is a sextuple $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi)$, where U is an algebraic space, $u : U \rightarrow \mathcal{X}$ is a morphism of S -stacks, \mathcal{M} and \mathcal{H} are quasicoherent \mathcal{O}_U -modules, $\Phi : \mathcal{H} \otimes \mathcal{M} \rightarrow \mathcal{O}_U$ is a surjective linear map, and $\psi : \mathcal{M} \otimes^{\Phi} \mathcal{H} \rightarrow \mathcal{A}_{U,u}$ is a bijection of algebras.

The local splittings form a category: A *morphism between two local splittings* $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi)$ and $(U', u', \mathcal{M}', \mathcal{H}', \Phi', \psi')$ is a quadruple (f, α, s, t) , where (f, α) is a morphism from $u : U \rightarrow \mathcal{X}$ to $u' : U' \rightarrow \mathcal{X}$, and $s : \mathcal{M}' \rightarrow f_*(\mathcal{M})$ and $t : \mathcal{H}' \rightarrow f_*(\mathcal{H})$ are linear maps of sheaves on U' ; we demand that the adjoint maps $f^*(\mathcal{M}') \rightarrow \mathcal{M}$ and $f^*(\mathcal{H}') \rightarrow \mathcal{H}$ are bijective and that the diagram

$$\begin{array}{ccc} \mathcal{M}' \otimes^{\Phi'} \mathcal{H}' & \xrightarrow{s \otimes t} & f_*(\mathcal{M} \otimes^{\Phi} \mathcal{H}) \\ \psi' \downarrow & & \downarrow \psi \\ \mathcal{A}_{U',u'} & \xrightarrow{\text{can}} & f_*(\mathcal{A}_{U,u}) \end{array}$$

is commutative. Composition is defined in the obvious way.

Let $\text{Split}(\mathcal{A})$ denote the category of all local splittings of \mathcal{A} with U affine. Then we have a forgetful functor

$$\text{Split}(\mathcal{A}) \longrightarrow (\text{Aff}/S), \quad (U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi) \longmapsto U,$$

which gives $\text{Split}(\mathcal{A})$ the structure of an S -stack. It comes along with a 1-morphism of S -stacks $\text{Split}(\mathcal{A}) \rightarrow \mathcal{X}$, sending a local splitting $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi)$ to the object in $\mathcal{X}_{U,u}$ induced by the morphism $u : U \rightarrow \mathcal{X}$. Moreover, $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi) \mapsto (U, u)$ makes $\text{Split}(\mathcal{A})$ into a stack over the site $\text{Big-et}(\mathcal{X})$.

Given a local section $s \in \Gamma((U, u), \mathbb{G}_{m,\mathcal{X}}) = \Gamma(U, \mathcal{O}_U^{\times})$ and a local splitting $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi) \in \text{Split}(\mathcal{A})_{U,u}$, we obtain an automorphism $(\text{id}_U, \text{id}_u, s, s^{-1})$ on this object. According to the result of Raeburn and Taylor ([17], Lemma 2.4) the resulting map of sheaves

$$\mathcal{O}_{\mathcal{X}}^{\times}|_{(\text{Aff}/U)} \longrightarrow \underline{\text{Aut}}_{\text{Split}(\mathcal{A})}(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi)$$

is bijective; moreover, all objects from $\text{Split}(\mathcal{A})_{U,u}$ are locally isomorphic. So if we demand that the algebra \mathcal{A} on \mathcal{X} admits a splitting over some $u : U \rightarrow \mathcal{X}$ that is smooth and surjective, the stack $\text{Split}(\mathcal{A}) \rightarrow \text{Big-et}(\mathcal{X})$ is a $\mathbb{G}_{m,\mathcal{X}}$ -gerbe, whence yields a cohomology class $[\mathcal{A}] \in H^2(\mathcal{X}, \mathbb{G}_m)$:

Definition 2.1. The algebra \mathcal{A} on \mathcal{X} is called a *central separable algebra* if it admits a local splitting $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi)$ with $u : U \rightarrow \mathcal{X}$ smooth surjective.

Note that this differs slightly from Taylor's approach in [22], Definition 2.1. By taking the existence of splittings as defining property, and not as a consequence, we avoid the technical problems discussed in [2].

We define the *bigger Brauer group* $\widetilde{\text{Br}}(\mathcal{X}) \subset H^2(\mathcal{X}, \mathbb{G}_m)$ as the subgroup generated by cohomology classes coming from central separable algebras as described above. Our task is to find conditions implying that the inclusion $\widetilde{\text{Br}}(\mathcal{X}) \subset H^2(\mathcal{X}, \mathbb{G}_m)$ is actually an equality. The following properties of quasicohherent sheaves will be useful:

Proposition 2.2. *Let \mathcal{F} be a quasicohherent sheaf on an algebraic S -stack \mathcal{G} . The following two conditions are equivalent:*

- (i) *There is a smooth surjection $u : U \rightarrow \mathcal{G}$ from an algebraic space U and a surjective linear map $\mathcal{F}_{U,u} \rightarrow \mathcal{O}_U$.*
- (ii) *There is a smooth surjection $v : V \rightarrow \mathcal{X}$ from an algebraic space V and a decomposition $\mathcal{F}_{V,v} \simeq \mathcal{K} \oplus \mathcal{O}_V$ for some quasicohherent sheaf \mathcal{K} on V .*

Proof. The implication (ii) \Rightarrow (i) is trivial. To see (i) \Rightarrow (ii), suppose we have a surjection $\mathcal{F}_{U,u} \rightarrow \mathcal{O}_U$. Choose an étale surjection $V \rightarrow U$, where $V = \bigcup V_\alpha$ is a disjoint union of affine schemes. Let $v : V \rightarrow \mathcal{G}$ be the induced morphism, and \mathcal{K} be the kernel of the induced surjection $\mathcal{F}_{V,v} \rightarrow \mathcal{O}_V$. This surjection must have a section, because quasicohherent sheaves on affine schemes have no higher cohomology. \square

Let us introduce a name for such sheaves:

Definition 2.3. Let \mathcal{F} be a quasicohherent sheaf on an algebraic S -stack \mathcal{G} . We say that \mathcal{F} *locally contains invertible summands* if the two equivalent conditions of Proposition 2.2 hold.

This notion was used in [19] to solve some problems on singularities in positive characteristics. For coherent sheaves on noetherian stacks, we have the following characterization involving the dual sheaf $\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathcal{G}})$:

Proposition 2.4. *Let \mathcal{F} be a coherent sheaf on a noetherian algebraic S -stack \mathcal{G} . Then the following are equivalent:*

- (i) *The sheaf \mathcal{F} locally contains invertible direct summands.*
- (ii) *The evaluation pairing $\mathcal{F} \otimes \mathcal{F}^\vee \rightarrow \mathcal{O}_{\mathcal{G}}$ is surjective.*
- (iii) *There is a smooth surjective morphism $u : U \rightarrow \mathcal{G}$ from some affine scheme $U = \text{Spec}(R)$, an R -module N , and a surjective linear mapping $\Gamma((U, u), \mathcal{F}) \otimes_R N \rightarrow R$.*

Proof. The implication (i) \Rightarrow (ii) is trivial: Choose a smooth surjection $u : U \rightarrow \mathcal{G}$ from some affine scheme U so that $\mathcal{F}_{U,u} \simeq \mathcal{K} \oplus \mathcal{O}_U$. Then the evaluation pairing $\mathcal{F}_{U,u} \otimes \mathcal{F}_{U,u}^\vee \rightarrow \mathcal{O}_U$ is obviously surjective, and so is $\mathcal{F} \otimes \mathcal{F}^\vee \rightarrow \mathcal{O}_{\mathcal{G}}$. The implication (ii) \Rightarrow (iii) is also trivial: Choose any smooth surjection $u : U \rightarrow \mathcal{G}$ from some affine scheme U and set $N = \Gamma((U, u), \mathcal{F}^\vee)$.

It remains to check (iii) \Rightarrow (i). Choose a smooth surjection $u : U \rightarrow \mathcal{G}$ from some affine scheme $U = \text{Spec}(R)$ admitting a surjection $\phi : \Gamma((U, u), \mathcal{F}) \otimes_R N \rightarrow R$. Then there are finitely many $f_1, \dots, f_r \in \Gamma((U, u), \mathcal{F})$ and $n_1, \dots, n_r \in N$ with $\phi(\sum f_i \otimes n_i) = 1$. Setting $s_i = \phi(f_i \otimes n_i)$, we obtain an affine open covering $U = V(s_1) \cup \dots \cup V(s_r)$. Replacing U by the disjoint union of the $V(s_i)$, we easily reduce to the case $r = 1$. This means that there is an $f \in \Gamma((U, u), \mathcal{F})$ and $n \in N$

with $\varphi(f \otimes n) = 1$. In other words, the map $f \mapsto \phi(f \otimes n)$ is surjective, which gives the desired surjection $\mathcal{F}_{U,u} \rightarrow \mathcal{O}_U$. \square

We finally examine the connection to central separable algebras. Suppose \mathcal{X} is an algebraic S -stack, and $\mathcal{G} \rightarrow \underline{\text{Big-et}}(\mathcal{X})$ is a $\mathbb{G}_{m,\mathcal{X}}$ -gerbe. Let $F : \mathcal{G} \rightarrow \mathcal{X}$ be the resulting morphism of algebraic S -stacks, as discussed in Section 1. Given a lisse-étale sheaf \mathcal{F} on \mathcal{G} and a smooth morphism $u : U \rightarrow \mathcal{G}$ from some algebraic space U , we denote by $\mathcal{F}_{U,u}$ the induced sheaf on U . For quasicohherent sheaves, the actions of $\mathbb{G}_{m,U}$ on $\mathcal{F}_{U,u}$ corresponds to a weight decomposition $\mathcal{F} = \bigoplus \mathcal{F}_n$, as explained in [10], Exposé I, Proposition 4.7.2. Here the direct sum runs through all $n \in \mathbb{Z}$, which is the character group of \mathbb{G}_m . A quasicohherent sheaf with $\mathcal{F} = \mathcal{F}_n$ is called of weight $w = n$.

Theorem 2.5. *Let \mathcal{G} be a \mathbb{G}_m -gerbe on a noetherian algebraic S -stack \mathcal{X} . Then the following are equivalent:*

- (i) *There is a central separable algebra \mathcal{A}' on \mathcal{X} whose \mathbb{G}_m -gerbe of splittings $\text{Split}(\mathcal{A}')$ is equivalent to \mathcal{G} .*
- (ii) *There is a coherent central separable algebra \mathcal{A} on \mathcal{X} whose \mathbb{G}_m -gerbe of splittings $\text{Split}(\mathcal{A})$ is equivalent to \mathcal{G} .*
- (iii) *There is a coherent sheaf \mathcal{F} on \mathcal{G} of weight $w = 1$ that locally contains invertible summands.*

Proof. The implication (ii) \Rightarrow (i) is trivial. To prove (i) \Rightarrow (iii), assume that $\mathcal{G} = \text{Split}(\mathcal{A}')$ for some central separable algebra \mathcal{A}' on \mathcal{X} . Let $\tilde{u} : U \rightarrow \mathcal{G}$ be a smooth morphism from an affine scheme U , and $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi) \in \mathcal{G}_{U,\tilde{u}}$ be the resulting object, as described in Section 2.1. We now use the sheaves \mathcal{M} on U to define a sheaf $\underline{\mathcal{M}}$ on \mathcal{G} by the tautological formula

$$\Gamma((U, \tilde{u}), \underline{\mathcal{M}}) = \Gamma(U, \mathcal{M}).$$

This obviously defines a presheaf on \mathcal{G} . It is easy to check that it satisfies the sheaf axiom, and that $\underline{\mathcal{M}}_{U,\tilde{u}} \simeq \mathcal{M}$, such that $\underline{\mathcal{M}}$ is quasicohherent. This quasicohherent sheaf is of weight $w = 1$: The sections $s \in \Gamma((U, \tilde{u}), \mathbb{G}_{m,\mathcal{X}}) = \Gamma(U, \mathcal{O}_U^\times)$ act via the automorphism $(\text{id}_U, \text{id}_u, s, s^{-1})$ on the object $(U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi) \in \mathcal{G}_{U,u}$, whence by multiplication-by- s on $\Gamma((U, \tilde{u}), \underline{\mathcal{M}})$.

To proceed, consider the ordered set $\mathcal{F}_\alpha \subset \underline{\mathcal{M}}$, $\alpha \in I$ of coherent subsheaves. The induced map $\varinjlim(\mathcal{F}_\alpha) \rightarrow \underline{\mathcal{M}}$ is bijective, by [13], Proposition 15.4. It remains to verify that some \mathcal{F}_α locally contains invertible summands. By construction, we have $\underline{\mathcal{M}}_{U,\tilde{u}} \simeq \mathcal{M}$, and a surjective pairing $\Phi : \mathcal{M} \otimes \mathcal{H} \rightarrow \mathcal{O}_U$. Setting $M_\alpha = \Gamma((U, \tilde{u}), \mathcal{F}_\alpha)$ and $N = \Gamma(U, \mathcal{H})$, we obtain a surjective pairing $\varinjlim(M_\alpha) \otimes N \rightarrow R$. Using that direct limits commute with tensor products, we infer that the map $M_\beta \otimes N \rightarrow R$ must already be surjective for some $\beta \in I$. According to Proposition 2.4, the sheaf $\mathcal{F} = \mathcal{F}_\beta$ locally contains invertible summands.

It remains to prove the implication (iii) \Rightarrow (ii). Let \mathcal{F} be a coherent sheaf on \mathcal{G} , of weight $w = 1$ and locally containing invertible summands. Then the evaluation pairing $\Phi : \mathcal{F}^\vee \otimes \mathcal{F} \rightarrow \mathcal{O}_{\mathcal{G}}$ is surjective, such that $\mathcal{F} \otimes^\Phi \mathcal{F}^\vee$ is a central separable algebra on \mathcal{G} , and the underlying coherent sheaf has weight zero. It follows from Lemma 1.4 that $\mathcal{F} \otimes^\Phi \mathcal{F}^\vee$ is isomorphic to the preimage of a nonunital associative algebra \mathcal{A} on \mathcal{X} . Moreover, given a smooth morphism $\tilde{u} : U \rightarrow \mathcal{G}$, we easily infer that we have a canonical isomorphism $\psi : \mathcal{A}_{U,\tilde{u}} \rightarrow \mathcal{F}_{U,\tilde{u}} \otimes^\Phi \mathcal{F}_{U,\tilde{u}}^\vee$, whence the algebra \mathcal{A} is central separable.

To finish the proof, we have to construct a functor of $\mathbb{G}_{m,\mathcal{X}}$ -gerbes $\mathcal{G} \rightarrow \text{Split}(\mathcal{A})$. Let $X \in \mathcal{G}_{U,u}$ be an object. Choose a morphism $\tilde{u} : U \rightarrow \mathcal{G}$ inducing this object, set $\mathcal{M} = \mathcal{F}_{U,\tilde{u}}$ and $\mathcal{H} = \mathcal{F}_{U,\tilde{u}}^\vee$, and let $\Phi : \mathcal{M} \otimes \mathcal{H} \rightarrow \mathcal{O}_U$ be the evaluation pairing. Together with the canonical isomorphism $\psi : \mathcal{A}_{U,u} \rightarrow \mathcal{F}_{U,\tilde{u}} \otimes^\Phi \mathcal{F}_{U,\tilde{u}}^\vee$ described above, we obtain the desired functor as

$$\mathcal{G} \longrightarrow \text{Split}(\mathcal{A}), \quad X \longmapsto (U, u, \mathcal{M}, \mathcal{H}, \Phi, \psi),$$

which is obviously compatible with the $\mathbb{G}_{m,\mathcal{X}}$ -action. \square

3. EXISTENCE OF CENTRAL SEPARABLE ALGEBRAS

We now come to our main result:

Theorem 3.1. *Let \mathcal{X} be a noetherian algebraic S -stack whose diagonal morphism $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ is quasiaffine. Then $\widetilde{\text{Br}}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{G}_m)$.*

Before we prove this, let us discuss two special cases. For schemes, the diagonal morphism is an embedding, whence automatically quasiaffine. Thus the preceding Theorem applies to schemes, which removes superfluous assumptions in results of Raeburn and Taylor [17] and the second author [18]. According to [13], Lemma 4.2, the diagonal morphism is quasiaffine even for Deligne–Mumford stacks. Thus:

Corollary 3.2. *Let \mathcal{X} be a noetherian scheme or a noetherian Deligne–Mumford S -stack. Then we have equality $\widetilde{\text{Br}}(\mathcal{X}) = H^2(\mathcal{X}, \mathbb{G}_m)$.*

Proof of Theorem 3.1: Fix a cohomology class $\alpha \in H^2(\mathcal{X}, \mathbb{G}_m)$ and choose a \mathbb{G}_m -gerbe $\mathcal{G} \rightarrow \text{Big-et}(\mathcal{X})$ representing α . Then there is an affine scheme U and a smooth surjective morphism $u : U \rightarrow \mathcal{X}$ so that $\mathcal{G}_{U,u}$ is nonempty. Note that $u : U \rightarrow \mathcal{X}$ is quasiaffine. To see this, let $v : V \rightarrow \mathcal{X}$ be a morphism from an affine scheme V . Then we have a commutative diagram with cartesian square

$$\begin{array}{ccc} U \times_{\mathcal{X}} V & \longrightarrow & U \times V & \xrightarrow{\text{pr}_2} & V \\ \downarrow & & \downarrow & & \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} & & \end{array}$$

The projection pr_2 is affine, because U is affine. The morphism $U \times_{\mathcal{X}} V \rightarrow U \times V$ is quasiaffine because Δ is quasiaffine. Whence the composition $U \times_{\mathcal{X}} V \rightarrow V$ is quasiaffine, which means that $u : U \rightarrow \mathcal{X}$ is quasiaffine.

By assumption, the induced gerbe $\mathcal{G} \times_{\mathcal{X}} U \rightarrow U$ is trivial. Hence there is coherent sheaf \mathcal{E} on $\mathcal{G} \times_{\mathcal{X}} U$ of weight $w = 1$ locally containing invertible summands. Choose a smooth surjection $v : V \rightarrow \mathcal{G} \times_{\mathcal{X}} U$ from some affine scheme V so that there is a surjection $\mathcal{E}_{V,v} \rightarrow \mathcal{O}_V$.

Now consider the other projection $F : \mathcal{G} \times_{\mathcal{X}} U \rightarrow \mathcal{G}$. This morphism is quasicompact and quasiseparated, so $F_*(\mathcal{E})$ is quasicoherent. The canonical map $F^*F_*(\mathcal{E}) \rightarrow \mathcal{E}$ is surjective by [7], Proposition 5.1.6, because F is quasiaffine. Hence the composition $F^*F_*(\mathcal{E})_{V,v} \rightarrow \mathcal{O}_V$ is surjective as well. Setting $v' = F \circ v : V \rightarrow \mathcal{G}$, we obtain a surjection $F_*(\mathcal{E})_{V,v'} \rightarrow \mathcal{O}_V$. Applying [13] Proposition 15.4, we write $F_*(\mathcal{E}) = \varinjlim \mathcal{F}_i$ as a direct limit of its coherent subsheaves. For some index i , the induced map $(\mathcal{F}_i)_{V,v'} \rightarrow \mathcal{O}_V$ must be surjective. Thus \mathcal{F}_i is a coherent sheaf on \mathcal{G} of weight $w = 1$ locally containing invertible summands. By Theorem 2.5, the cohomology class $\alpha \in H^2(\mathcal{X}, \mathbb{G}_m)$ lies in the bigger Brauer group. \square

The following example essentially due to Totaro ([21], Remark 1 in Introduction) shows that the assumption on the diagonal morphism $\Delta : \mathcal{X} \rightarrow \mathcal{X} \times \mathcal{X}$ in Theorem 3.1 is not superfluous. Let E be an elliptic curve over an algebraically closed ground field k , and \mathcal{L} be an invertible sheaf on E of degree zero, such that $\mathcal{L}^{\otimes t} \neq \mathcal{O}_E$ for $t \neq 0$. Consider the $\mathbb{G}_{m,E}$ -torsor $V = \text{Spec}(\bigoplus_{t \in \mathbb{Z}} \mathcal{L}^{\otimes t})$. According to [20], Chapter VII, §3.15, the torsor structure comes from a unique extension of k -group schemes $0 \rightarrow \mathbb{G}_m \rightarrow V \rightarrow E \rightarrow 0$. From this we obtain a morphism of algebraic k -stacks $BV \rightarrow BE$, which sends a V -torsor to its associated E -torsor. It follows that the morphism $BV \rightarrow BE$ is a $\mathbb{G}_{m,BE}$ -gerbe. Coherent sheaves on BV correspond to linear representations $V \rightarrow \text{GL}_k(n)$, $n \geq 0$. Using that the scheme $\text{GL}_k(n)$ is affine and $\Gamma(V, \mathcal{O}_V) = \bigoplus_{t \in \mathbb{Z}} \Gamma(E, \mathcal{L}^{\otimes t}) = k$, we infer that every coherent sheaf on BV is isomorphic to $\mathcal{O}_{BV}^{\oplus n}$. In particular, there are no nonzero coherent sheaves of weight $w = 1$. Summing up, the algebraic k -stack $\mathcal{X} = BE$ admits a $\mathbb{G}_{m,\mathcal{X}}$ -gerbe $\mathcal{G} = BV$ whose cohomology class does not lie in the bigger Brauer group.

4. APPENDIX: BIG-ÉTALE VS. LISSE-ÉTALE COHOMOLOGY

Let \mathcal{X} be an algebraic S -stack. Then we may view the lisse-étale site as a subcategory $\text{Lis-et}(\mathcal{X}) \subset \text{Big-et}(\mathcal{X})$ inside the big-étale site. This inclusion obviously sends coverings to coverings, whence the inclusion functor is continuous by [11], Exposé III, Proposition 1.6. Hence for all big-étale sheaves \mathcal{F} , the induced presheaf $\mathcal{F}_{\text{lis-et}} = \mathcal{F}|_{\text{Lis-et}(\mathcal{X})}$ on the lisse-étale site is a sheaf. Moreover, the induced restriction functor

$$\mathcal{X}_{\text{big-et}} \longrightarrow \mathcal{X}_{\text{lis-et}}, \quad \mathcal{F} \longmapsto \mathcal{F}_{\text{lis-et}}$$

commutes with all direct and inverse limits (by the formula for sheafification). In particular, the functor sends short exact sequences of abelian sheaves into short exact sequences. Whence $\mathcal{F} \mapsto H^i(\mathcal{X}_{\text{lis-et}}, \mathcal{F}_{\text{lis-et}})$ comprise a δ -functor on the category of big-étale abelian sheaves. By universality, the restriction map $\Gamma(\mathcal{X}_{\text{big-et}}, \mathcal{F}) \rightarrow \Gamma(\mathcal{X}_{\text{lis-et}}, \mathcal{F}_{\text{lis-et}})$ induces a natural transformation

$$H^i(\mathcal{X}_{\text{big-et}}, \mathcal{F}) \longrightarrow H^i(\mathcal{X}_{\text{lis-et}}, \mathcal{F}_{\text{lis-et}})$$

of δ -functors. It is not a priori clear that these canonical maps are bijections. The problem is that the left adjoint u^{-1} for the functor $u_*(\mathcal{F}) = \mathcal{F}_{\text{lis-et}}$ does not commute with finite inverse limits, such that our functor u_* does *not* yield a map of topoi $u = (u^{-1}, u_*)$ from $\mathcal{X}_{\text{big-et}}$ to $\mathcal{X}_{\text{lis-et}}$. However, a result of Gabber saves us from these troubles:

Proposition 4.1. *The canonical maps $H^i(\mathcal{X}_{\text{big-et}}, \mathcal{F}) \rightarrow H^i(\mathcal{X}_{\text{lis-et}}, \mathcal{F}_{\text{lis-et}})$ are bijective for all $i \geq 0$ and all big-étale abelian sheaves \mathcal{F} .*

Proof. First note that the topology on $\text{Lis-et}(\mathcal{X})$ is induced by the topology of $\text{Big-et}(\mathcal{X})$, in light of [11], Exposé III, Proposition 1.6. We now merely have to check that the inclusion of sites $\text{Lis-et}(\mathcal{X}) \subset \text{Big-et}(\mathcal{X})$ satisfies the assumptions A.1.1–A.1.4 of Gabber’s Theorem [16], Appendix A. Assumption A.1.1 requires that the final object $e \in \text{Big-et}(\mathcal{X})$ is covered by an object in $\text{Lis-et}(\mathcal{X})$; this holds because there is a smooth surjective morphism $u : U \rightarrow \mathcal{X}$ from an algebraic space U . Assumption A.1.2 demands that for every smooth $u : U \rightarrow \mathcal{X}$ from an algebraic space U and every covering family $U_i \rightarrow U$ in $\text{Big-et}(\mathcal{X})$, there is a refinement in $\text{Lis-et}(\mathcal{X})$; this trivially holds by choosing $U_i \rightarrow U$ itself. The remaining two assumptions have to do with existence of fiber products and hold obviously. \square

REFERENCES

- [1] K. Behrend: Derived l -adic categories for algebraic stacks. Mem. Amer. Math. Soc. 163 (2003).
- [2] S. Caenepeel, F. Grandjean: A note on Taylor's Brauer group. Pacific J. Math. 186 (1998), 13–27
- [3] A. de Jong: A result of Gabber. Preprint, <http://www.math.columbia.edu/~dejong/>
- [4] D. Edidin, B. Hassett, A. Kresch, A. Vistoli: Brauer groups and quotient stacks. Amer. J. Math. 123 (2001), 761–777.
- [5] J. Giraud: Cohomologie non abélienne. Grundlehren Math. Wiss. 179. Springer, Berlin, 1971.
- [6] A. Grothendieck: Sur quelques points d'algèbre homologique. Tohoku Math. J. 9 (1957), 119–221.
- [7] A. Grothendieck: Éléments de géométrie algébrique II: Étude globale élémentaire de quelques classes de morphismes. Publ. Math., Inst. Hautes Étud. Sci. 8 (1961).
- [8] A. Grothendieck: Le groupe de Brauer. In: J. Giraud (ed.) et al.: Dix exposés sur la cohomologie des schémas, pp. 46–189. North-Holland, Amsterdam, 1968.
- [9] A. Grothendieck et al.: Revêtements étales et groupe fondamental. Lect. Notes Math. 224, Springer, Berlin, 1971.
- [10] A. Grothendieck et al.: Schemas en groupes I. Lect. Notes Math. 151. Springer, Berlin, 1970.
- [11] A. Grothendieck et al.: Théorie des topos et cohomologie étale. Tome 1. Lect. Notes Math. 269. Springer, Berlin, 1972.
- [12] A. Grothendieck et al.: Théorie des topos et cohomologie étale. Tome 2. Lect. Notes Math. 270. Springer, Berlin, 1973.
- [13] G. Laumon, L. Moret-Bailly: Champs algébriques. Ergeb. Math. Grenzgebiete 39, Springer, Berlin, 2000.
- [14] M. Lieblich: Twisted sheaves and the period-index problem. Compos. Math. 144 (2008), 1–31.
- [15] M. Lieblich: Period and index in the Brauer group of an arithmetic surface. Preprint, math.NT/0702240.
- [16] M. Olsson: Sheaves on Artin stacks. J. Reine Angew. Math. 603 (2007), 55–112.
- [17] I. Raeburn, J. Taylor: The bigger Brauer group and étale cohomology. Pacific J. Math. 119 (1985), 445–463.
- [18] S. Schröer: The bigger Brauer group is really big. J. Algebra 262 (2003), 210–225.
- [19] S. Schröer: Singularities appearing on generic fibers of morphisms between smooth schemes. Michigan Math. 56 (2008), 55–76.
- [20] J.-P. Serre: Groupes algébriques et corps de classes. Actualités scientifiques et industrielles 1264. Hermann, Paris, 1975.
- [21] B. Totaro: The resolution property for schemes and stacks. J. Reine Angew. Math. 577 (2004), 1–22.
- [22] J. Taylor: A bigger Brauer group. Pacific J. Math. 103 (1982), 163–203.

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