

THE BIGGER BRAUER GROUP IS REALLY BIG

STEFAN SCHRÖER

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ABSTRACT. I show that each étale n -cohomology class on noetherian schemes comes from a Čech cocycle, provided that any n -tuple of points admits an affine open neighborhood. Together with results of Raeburn and Taylor on the bigger Brauer group, this implies that for schemes such that each pair of points admits an affine open neighborhood, any étale \mathbb{G}_m -gerbe comes from a coherent central separable algebra. Such algebras are nonunital generalizations of Azumaya algebras. I also prove that, on normal noetherian schemes, each Zariski \mathbb{G}_m -gerbe comes from a central separable algebra.

INTRODUCTION

Grothendieck [16] asked whether each torsion class in $H_{\text{ét}}^2(X, \mathbb{G}_m)$ on a scheme X comes from an Azumaya algebra. This is a major open problem in the theory of Brauer groups. Gabber [8] proved it for affine schemes. But even for smooth projective threefolds the answer seems to be unknown. Edidin, Hassett, Kresch, and Vistoli [7] recently found counterexamples for nonseparated schemes.

To attack the problem, it is perhaps a good idea to modify it. Taylor [23] generalized the notion of Azumaya algebras to *central separable algebras*, which are not necessarily locally free or unital. Nevertheless, they come along with a \mathbb{G}_m -gerbe of splittings and therefore define a cohomology class in $H_{\text{ét}}^2(X, \mathbb{G}_m)$. Assuming that each finite subset in X admits an affine open neighborhood, Raeburn and Taylor [19] proved that each 2-cohomology class, torsion or not, comes from a coherent central separable algebra. Caenepeel and Grandjean [5] later fixed some problems in the original arguments.

Actually, the arguments of Raeburn and Taylor show that, on arbitrary noetherian schemes, each Čech 2-cohomology class comes from a coherent central separable algebra. Not every 2-cohomology class, however, comes from Čech cocycles. Rather, the obstruction is a 1-cocycle class with values in the presheaf $U \mapsto \text{Pic}(U)$.

Dealing with such obstruction, I prove a general convergence result for étale cohomology: The canonical map $\check{H}_{\text{ét}}^n(X, \mathcal{F}) \rightarrow H_{\text{ét}}^n(X, \mathcal{F})$ is bijective for any abelian sheaf \mathcal{F} provided each n -tuple of points $x_1, \dots, x_n \in X$ admits an affine open neighborhood. This generalizes a result of Artin [1], who assumed that each finite subsets lies in an affine neighborhood. For noetherian schemes such that each pair of points admits an affine open neighborhood, my result implies that $\widetilde{\text{Br}}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$. Here $\widetilde{\text{Br}}(X)$ is Taylor's *bigger Brauer group*, defined as the group of equivalence classes of central separable algebras.

Furthermore, we shall see that $H_{\text{zar}}^2(X, \mathbb{G}_m) \subset \widetilde{\text{Br}}(X)$ holds for any normal noetherian scheme. This applies to the nonseparated example constructed in [7], showing that there are central separable algebras neither equivalent to Azumaya algebras nor given by Čech cocycles.

The paper is organized as follows. The first section contains observation on tuples $x_1, \dots, x_n \in X$ admitting affine open neighborhoods. In Section 2, I prove the convergence result on étale cohomology. In the next section, I describe the obstruction map $H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$ in terms of gerbes and torsors. The result is purely formal and holds for any site. Section 4 contains the generalization of Raeburn's and Taylor's result on the bigger Brauer group. In Section 5, I show that each Zariski gerbe on a normal noetherian scheme lies in the bigger Brauer group. The last two sections contain examples: Section 6 deals with the nonseparated surface from [7], and Section 7 with the proper surfaces without ample line bundles from [20].

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1. TUPLES WITH AFFINE OPEN NEIGHBORHOODS

Given a scheme X and an integer $n \geq 2$, we may ask whether each n -tuple $x_1, \dots, x_n \in X$ admits an affine open neighborhood. Such conditions are related to the existence of ample line bundles (the generalized Chevalley Conjecture [17], page 327), embeddings into toric varieties [24], and étale cohomology [1]. In this section, I collect some elementary results concerning such conditions.

Proposition 1.1. *Let X be a scheme such that each pair $x_1, x_2 \in X$ admits an affine open neighborhood. Then X is separated.*

Proof. Let $U_\alpha \subset X$ be the family of all affine open subsets. Each point in $X \times X$ lies in some subset of the form $\text{Spec}(\kappa(x_1) \otimes \kappa(x_2))$ with $x_1, x_2 \in X$. Consequently, the $U_\alpha^2 \subset X^2$ form an affine open covering. Clearly, the diagonal $\Delta : X \rightarrow X^2$ is a closed embedding over each U_α^2 , hence a closed embedding. In other words, X is separated. \square

Given an integer $n \geq 1$ and an n -tuple $x_1, \dots, x_n \in X$, consider the subspace $S = \text{Spec}(\mathcal{O}_{X, x_1}) \cup \dots \cup \text{Spec}(\mathcal{O}_{X, x_n})$, which comprises all $x \in X$ specializing to one of the x_i . Setting $\mathcal{O}_S = i^{-1}(\mathcal{O}_X)$, where $i : S \rightarrow X$ is the canonical inclusion, we obtain a locally ringed space (S, \mathcal{O}_S) . It is covered by the schemes $\text{Spec}(\mathcal{O}_{X, x_i})$. This covering, however, is not necessarily an open covering, and (S, \mathcal{O}_S) is not necessarily a scheme.

Proposition 1.2. *With the preceding notation, the locally ringed space (S, \mathcal{O}_S) is an affine scheme if the tuple $x_1, \dots, x_n \in X$ admits an affine open neighborhood.*

Proof. To verify this we may assume that X is itself affine. Now the statement follows from [3], Chap. II, §3, No. 5, Proposition 17. \square

I suspect that the converse holds as well. This is indeed the case under some additional assumptions:

Proposition 1.3. *Suppose X is separated and of finite type over some noetherian ring R . Then (S, \mathcal{O}_S) is an affine scheme if and only if $x_1, \dots, x_n \in X$ admits an affine open neighborhood.*

Proof. We already saw that the condition is sufficient and have to verify necessity. Suppose (S, \mathcal{O}_S) is an affine scheme. To find the desired affine open neighborhood, we may assume that X is reduced by [10], Corollary 4.5.9. Adding the generic points $\eta \in X - S$ to the tuple $x_1, \dots, x_n \in X$, we may also assume that $S \subset X$ is dense.

Choose finitely many sections $g_1, \dots, g_m \in \Gamma(S, \mathcal{O}_S)$ so that the corresponding map $g : S \rightarrow \mathbb{A}_R^m$ is injective. We may view the g_i as rational functions on X whose domain of definition contains S . Therefore we can replace X by some suitable dense open subset and assume that the g_i extend to global sections $f_i \in \Gamma(X, \mathcal{O}_X)$. In turn, we have a morphism $f : X \rightarrow \mathbb{A}_R^m$.

Let $U \subset X$ be the subset of $x \in X$ that are isolated in their fiber $f^{-1}(f(x))$. This is an open subset by Chevalley's Semicontinuity Theorem ([13], Corollary 13.1.4). By construction, no $x \in S$ admits a generization in $f^{-1}(f(x))$, so $S \subset U$. Replacing X by U , we may assume that $f : X \rightarrow \mathbb{A}_R^m$ has discrete fibers. In other words, f is quasifinite. According to Zariski's Main Theorem ([13], Corollary 8.12.6), there is an open embedding of X into an affine scheme, hence \mathcal{O}_X is ample. By [11], Corollary 4.5.4, the tuple $x_1, \dots, x_n \in X$ admits an affine open neighborhood. \square

Here is another result in this direction. Recall that a scheme X is called *divisorial* if the open subset of the form $X_s \subset X$, where s is a global section of an invertible \mathcal{O}_X -module \mathcal{L} , generate the topology of X . This notion is due to Borelli [2].

Proposition 1.4. *Suppose X is a divisorial noetherian scheme. Then (S, \mathcal{O}_S) is an affine scheme if and only if $x_1, \dots, x_n \in X$ admits an affine open neighborhood.*

Proof. Suppose (S, \mathcal{O}_S) is an affine scheme. As in the previous proof, we may assume that X is reduced and that $S \subset X$ is dense. By quasicompactness, there is a finitely generated subgroup $P \subset \text{Pic}(X)$ such that the open subsets $X_s \subset X$, where s ranges over the global sections of the $\mathcal{L} \in P$, generate the topology. Choose generators $\mathcal{L}_1, \dots, \mathcal{L}_m \in P$. Then each $\mathcal{L}_i|_S$ is trivial because S is a semilocal affine scheme. Shrinking X if necessary, we may assume that each \mathcal{L}_i is trivial. Then \mathcal{O}_X is ample, and [11], Corollary 4.5.4 ensures that $x_1, \dots, x_n \in X$ admits an affine open neighborhood. \square

2. OBSTRUCTIONS AGAINST ČECH COCYCLES

Given a scheme X , let $X_{\text{ét}}$ be the site of étale X -schemes. Its Grothendieck topology is given by the quasicompact étale surjections. We call such morphism *refinements*, or *étale coverings*. For each abelian sheaf \mathcal{F} on $X_{\text{ét}}$, we have cohomology groups $H_{\text{ét}}^p(X, \mathcal{F})$. Sometimes we prefer to deal with the Čech cohomology groups $\check{H}_{\text{ét}}^p(X, \mathcal{F})$ instead. These groups are related by a natural transformation $\check{H}_{\text{ét}}^p(X, \mathcal{F}) \rightarrow H_{\text{ét}}^p(X, \mathcal{F})$ of ∂ -functors.

For $q \geq 0$, let $\mathcal{H}^q \mathcal{F}$ be the presheaf $U \mapsto H_{\text{ét}}^q(U, \mathcal{F})$. As explained in [18], Chapter III, Proposition 2.7, the composite functor $\Gamma(X, \mathcal{F}) = \check{H}^0(X, \mathcal{H}^0 \mathcal{F})$ gives a spectral sequence

$$\check{H}_{\text{ét}}^p(X, \mathcal{H}^q \mathcal{F}) \implies H_{\text{ét}}^{p+q}(X, \mathcal{F}).$$

We may view the Čech cohomology groups $\check{H}_{\text{ét}}^p(X, \mathcal{H}^q \mathcal{F})$ with $q > 0$ as obstructions against bijectivity of $\check{H}_{\text{ét}}^p(X, \mathcal{F}) \rightarrow H_{\text{ét}}^p(X, \mathcal{F})$. The goal of this section is to prove the following vanishing result:

Theorem 2.1. *Suppose X is a noetherian scheme. Let $n \geq 0$ be an integer such that each n -tuple $x_1, \dots, x_n \in X$ admits an affine open neighborhood. Then $\check{H}_{\text{ét}}^p(X, \mathcal{H}^q \mathcal{F}) = 0$ for all $p < n$, all $q > 0$, and any abelian sheaf \mathcal{F} on $X_{\text{ét}}$.*

In the case $n = 1$, this specializes to the well-known fact that $\check{H}_{\text{ét}}^0(X, \mathcal{H}^q \mathcal{F}) = 0$ for $q > 0$. The case $n = \infty$, that is, each finite subset lies in an affine open neighborhood, is Artin's result [1], Corollary 4.2. We may view Theorem 2.1 as a quantitative refinement of Artin's result. Here is an immediate application:

Corollary 2.2. *Suppose X is a noetherian scheme. Let $n \geq 0$ be such that each n -tuple $x_1, \dots, x_n \in X$ admits an affine open neighborhood. Then the canonical map $\check{H}_{\text{ét}}^p(X, \mathcal{F}) \rightarrow H_{\text{ét}}^p(X, \mathcal{F})$ is bijective for $p \leq n$, and injective for $p = n + 1$.*

Proof. The spectral sequence $\check{H}_{\text{ét}}^p(X, \mathcal{H}^q \mathcal{F}) \Rightarrow H_{\text{ét}}^{p+q}(X, \mathcal{F})$ has $E_r^{p,q} = 0$ for all $p < n$, all $q > 0$, and all $r > 0$ by Theorem 2.1. Hence the inclusion $E_{\infty}^{p,0} \subset \text{Gr } H_{\text{ét}}^p(X, \mathcal{F})$ is bijective for $p \leq n$. Furthermore $E_2^{p,0} = E_{\infty}^{p,0}$ for $p \leq n + 1$. In turn, the edge map $\check{H}_{\text{ét}}^p(X, \mathcal{F}) \rightarrow H_{\text{ét}}^p(X, \mathcal{F})$ is bijective for $p \leq n$, and injective for $p = n + 1$. \square

Let me also point out the following special case:

Corollary 2.3. *Let R be a noetherian ring, $Y = \bigcup_{\sigma \in \Delta} \text{Spec}(R[\sigma^{\vee} \cap M])$ a toric variety, and $X \subset Y$ a subscheme. Then the map $\check{H}_{\text{ét}}^2(X, \mathcal{F}) \rightarrow H_{\text{ét}}^2(X, \mathcal{F})$ is bijective.*

Proof. According to [24], page 709, each pair of points in a toric variety admits an affine open neighborhood. Now the statement follows from Corollary 2.2. \square

The proof of Theorem 2.1 requires a little preparation. Recall that a scheme is called *strictly local* if it is the spectrum of a henselian local ring with separably closed residue field.

Proposition 2.4. *Let X be a quasicompact scheme. The following are equivalent:*

- (i) *We have $H_{\text{ét}}^p(X, \mathcal{F}) = 0$ for all abelian sheaves \mathcal{F} and all $p > 0$.*
- (ii) *Each étale covering $U \rightarrow X$ admits a section.*
- (iii) *The scheme X is affine, and its connected components are strictly local.*

Proof. According to [1], Proposition 3.1, condition (ii) implies that X is affine. Now the equivalence (ii) \Leftrightarrow (iii) follows from [1], Proposition 3.2. To see the implication (ii) \Rightarrow (i), note that each \mathcal{F} -torsor is trivial on some étale covering $U \rightarrow X$, hence trivial, so the global section functor $H^0(X, \mathcal{F})$ is exact.

It remains to verify (i) \Rightarrow (ii). Seeking a contradiction, we assume that some étale covering $f : U \rightarrow X$ admits no section. Consider the sheaf $\mathcal{F} = f_!(\mathbb{Z}_U)$. This is the subsheaf $f_!(\mathbb{Z}_U) \subset f_*(\mathbb{Z}_U)$ defined via extension-by-zero. The Čech complex for the covering $U \rightarrow X$ is given by

$$H_{\text{ét}}^0(X, \mathcal{F}) \xrightarrow{d_0} H_{\text{ét}}^0(U, \mathcal{F}) \xrightarrow{d_1} H_{\text{ét}}^0(U^2, \mathcal{F}).$$

The constant section $1_U \in H_{\text{ét}}^0(U, f_*(\mathbb{Z}_U))$ clearly lies in the subgroup $H_{\text{ét}}^0(U, f_!(\mathbb{Z}_U))$. By construction, $1_U \in H_{\text{ét}}^0(U, \mathcal{F})$ lies in the kernel of d_1 , but not in the image of d_0 , and this holds true on all refinements of U . We conclude $\check{H}_{\text{ét}}^1(X, \mathcal{F}) \neq 0$. Since the

canonical map $\check{H}_{\text{ét}}^1(X, \mathcal{F}) \rightarrow H_{\text{ét}}^1(X, \mathcal{F})$ is injective, we also have $H_{\text{ét}}^1(X, \mathcal{F}) \neq 0$, contradiction. \square

Conforming with [1], Section 3, we call a scheme X *acyclic* if it satisfies the equivalent conditions in Proposition 2.4. For a point $x \in X$, let $\mathcal{O}_{X,x}^{\text{sh}}$ be the corresponding *strictly local ring*, that is, the strict henselization of $\mathcal{O}_{X,x}$. The following is a reformulation of Artin's fundamental result in [1]:

Proposition 2.5. *Let $x_1, \dots, x_n \in X$ be a tuple of points admitting an affine open neighborhood. Then the scheme $\text{Spec}(\mathcal{O}_{X,x_1}^{\text{sh}}) \times_X \dots \times_X \text{Spec}(\mathcal{O}_{X,x_n}^{\text{sh}})$ is acyclic.*

Proof. To check this, we may assume that X itself is affine. Now the assertion follows from [1], Theorem 3.4. \square

The following improvement will be the key step in proving Theorem 2.1:

Proposition 2.6. *Suppose X is a noetherian scheme such that every $(p+1)$ -tuple of points in X admits an affine open neighborhood. Let U be a quasicompact étale X -scheme, and $\beta \in H_{\text{ét}}^q(U^{p+1}, \mathcal{F})$, $q > 0$. Let V_0, \dots, V_k be quasicompact étale U -schemes, and $x_{k+1}, \dots, x_p \in U$ be points for some $0 \leq k \leq p$. Then there are refinements $V'_i \rightarrow V_i$ for $0 \leq i \leq k$, and affine étale neighborhoods $V'_i \rightarrow U$ of x_i for $k+1 \leq i \leq p$, such that $\beta|_{V'_0 \times \dots \times V'_p} = 0$.*

Proof. First, we prove by induction on k the following auxiliary statement: There are refinements $V'_i \rightarrow V_i$ for $i = 0, \dots, k$ such that $\beta|_{V'_0 \times \dots \times V'_k \times Z_{k+1} \times \dots \times Z_p} = 0$. Here we write $Z_i = \text{Spec}(\mathcal{O}_{U,x_i}^{\text{sh}})$ for the strictly local scheme corresponding to the points $x_i \in U$.

The induction starts with $k = -1$. Then there are no V_i , and the assertion boils down to Proposition 2.5. To see this, write each $Z_i = \varprojlim S_{i,\alpha_i}$ as inverse limits of affine étale X -schemes S_{i,α_i} . Then $Z_0 \times \dots \times Z_p = \varprojlim (S_{0,\alpha_0} \times \dots \times S_{p,\alpha_p})$, and [15], Exposé VII, Corollary 5.8 tells us that the canonical map

$$\varinjlim H_{\text{ét}}^q(S_{0,\alpha_0} \times \dots \times S_{p,\alpha_p}, \mathcal{F}) \longrightarrow H_{\text{ét}}^q(Z_0 \times \dots \times Z_p, \mathcal{F}_{\infty}) = 0$$

is bijective, where \mathcal{F}_{∞} is the inverse image of \mathcal{F} . We conclude that $\beta|_{V'_0 \times \dots \times V'_p} = 0$ for suitable $V'_i = S_{i,\alpha_i}$. Note that this is the only step in the proof where we need the assumption about affine neighborhoods of $(p+1)$ -tuples.

Now suppose the statement is already true for $k-1$. Fix a point $x_k \in V_k$, set $Z_k = \text{Spec}(\mathcal{O}_{V_k,x_k}^{\text{sh}})$, and choose refinements $V'_i \rightarrow V_i$ for $i = 0, \dots, k-1$ so that $\beta|_{V'_0 \times \dots \times V'_{k-1} \times Z_k \times \dots \times Z_p} = 0$. Write $Z_k = \varprojlim S_{\alpha}$ as the inverse limit of affine étale V_k -schemes S_{α} . According to [15], Exposé VII, Corollary 5.8, the canonical map

$$\begin{aligned} \varinjlim H_{\text{ét}}^q(V'_0 \times \dots \times V'_{k-1} \times S_{\alpha} \times Z_{k+1} \times \dots \times Z_p, \mathcal{F}_{\alpha}) \\ \longrightarrow H_{\text{ét}}^q(V'_0 \times \dots \times V'_{k-1} \times Z_k \times Z_{k+1} \times \dots \times Z_p, \mathcal{F}_{\infty}) \end{aligned}$$

is bijective, where \mathcal{F}_{α} and \mathcal{F}_{∞} are the inverse images of \mathcal{F} . We conclude that $\beta|_{V'_0 \times \dots \times V'_{k-1} \times S_{\alpha} \times Z_{k+1} \times \dots \times Z_p} = 0$ for some suitable index α . If $S_{\alpha} \rightarrow V_k$ is surjective, we are done by setting $V'_k = S_{\alpha}$. Otherwise, we finish the argument by applying noetherian induction to V_k . This proves the auxiliary statement.

It remains to construct the desired affine étale neighborhoods $V'_i \rightarrow U$ of the points $x_i \in U$ for $i = k+1, \dots, p$. For this, we write $Z_{k+1} = \varprojlim T_{\alpha}$ as the inverse limit of affine étale U -schemes T_{α} . Again by [15], Exposé VII, Corollary 5.8, the

canonical map

$$\begin{aligned} & \varinjlim H_{\text{ét}}^q(V'_0 \times \dots \times V'_k \times T_\alpha \times Z_{k+2} \times \dots \times Z_p, \mathcal{F}_\alpha) \\ & \longrightarrow H_{\text{ét}}^q(V'_0 \times \dots \times V'_k \times Z_{k+1} \times Z_{k+2} \times \dots \times Z_p, \mathcal{F}_\infty) \end{aligned}$$

is bijective, where \mathcal{F}_α and \mathcal{F}_∞ and the inverse images of \mathcal{F} . As above, we conclude that $\beta|_{V'_0 \times \dots \times V'_k \times T_\alpha \times Z_{k+2} \times \dots \times Z_p} = 0$ for some suitable index α . To finish the proof, set $V'_{k+1} = T_\alpha$ and apply induction on $p - k$. \square

Remark 2.7. If there are repetitions among the V_i or the x_i , say $V_i = V_j$ or $x_i = x_j$, then we may also assume $V'_i = V'_j$, by replacing both V'_i and V'_j by $V'_i \times_U V'_j$.

Proof of Theorem 2.1: Throughout, we regard X as base scheme and products of X -schemes as fibered products over X . Fix a Čech class $\gamma \in \check{H}^p(X, \mathcal{H}^q \mathcal{F})$ with $p < n$ and $q > 0$. Choose a refinement $U \rightarrow X$ and a cocycle $\beta \in H^q(U^{p+1}, \mathcal{F})$ representing γ .

It suffices to find a refinement $W \rightarrow U$ with $\beta|_{W^{p+1}} = 0$. For this, we shall construct by induction on m sequences of affine étale U -schemes $V_{m,1}, \dots, V_{m,m}$ such that $\beta|_{V_{m,i_0} \times \dots \times V_{m,i_p}} = 0$ for any set of indices $0 \leq i_0, \dots, i_p \leq m$. This clearly implies $\beta|_{W_m^{p+1}} = 0$, where $W_m = V_{m,1} \amalg \dots \amalg V_{m,m}$. In each stage of the induction, $V_{m+1,i}$ will be a refinement of $V_{m,i}$ for $i = 1, \dots, m$. The induction stops if $W_m \rightarrow U$ is surjective. We then set $W = W_m$ and have $\beta|_{W^{p+1}} = 0$.

Suppose we already have constructed $W_m = V_{m,1} \amalg \dots \amalg V_{m,m}$ as above, and that $W_m \rightarrow U$ is not yet surjective. Fix a point $x \in U$ not in the image and set $Z = \text{Spec}(\mathcal{O}_{U,x}^{\text{sh}})$. According to Proposition 2.6 and Remark 2.7, there is an affine étale neighborhood $V'_{m,m+1} \rightarrow U$ of the point x such that $\beta|_{V'^{p+1}_{m,m+1}} = 0$. Next, fix a tuple of indices $0 \leq i_0, \dots, i_p \leq m+1$. Applying Proposition 2.6 again, we may replace the $V'_{m,i}$ for $1 \leq i \leq m+1$ by further refinements so that $\beta|_{V'_{m,i_0} \times \dots \times V'_{m,i_p}} = 0$. Since there are only finitely many such tuples of indices, we may repeat this inductively until $\beta|_{V'_{m,i_0} \times \dots \times V'_{m,i_p}} = 0$ holds for all $0 \leq i_0, \dots, i_p \leq m+1$. Then we set $V_{m+1,i} = V'_{m,i}$ for $i = 1, \dots, m$, and $V_{m+1,m+1} = V'_{m,m+1}$, and $W_{m+1} = V_{m+1,1} \amalg \dots \amalg V_{m+1,m+1}$.

By construction, the image of $W_{m+1} \rightarrow U$ is strictly larger than the image of $W_m \rightarrow U$. Using noetherian induction, we conclude that the mapping $W_m \rightarrow U$ becomes surjective for some $m \geq 1$. Hence $W = W_n$ is the desired refinement with $\beta|_{W^{p+1}} = 0$. \square

3. GERBES AND 2-COHOMOLOGY

Theorem 2.2 implies that the injection $\check{H}_{\text{ét}}^2(X, \mathcal{F}) \rightarrow H_{\text{ét}}^2(X, \mathcal{F})$ is bijective for any scheme such that each pair $x_1, x_2 \in X$ admits an affine open neighborhood. There is no reason, however, that this holds in general. In this section we shall describe the obstruction in geometric terms.

We shall work in an abstract setting: Fix an arbitrary site with terminal object X and an abelian sheaf \mathcal{F} . Then we have cohomology groups $H^p(X, \mathcal{F})$. The spectral sequence $\check{H}^p(X, \mathcal{H}^q \mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$ gives an exact sequence

$$0 \longrightarrow \check{H}^2(X, \mathcal{H}^0 \mathcal{F}) \longrightarrow H^2(X, \mathcal{F}) \longrightarrow \check{H}^1(X, \mathcal{H}^1 \mathcal{F}) \xrightarrow{d} \check{H}^3(X, \mathcal{H}^0 \mathcal{F}).$$

The *obstruction map* $H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$ is the obstruction for a cohomology class to come from a Čech cocycle. The task now is to describe an obstruction map in terms of gerbes and torsors.

To do so, let me recall the following geometric interpretation of the universal ∂ -functor $H^p(X, \mathcal{F})$ for $p = 0, 1, 2$: We may define $H^1(X, \mathcal{F})$ as the group of isomorphism classes of \mathcal{F} -torsors, and $H^2(X, \mathcal{F})$ as the group of equivalence classes of \mathcal{F} -gerbes. Recall that a *gerbe* is a stack in groupoids $\mathcal{G} \rightarrow X_{\text{ét}}$ satisfying the following properties: The objects in \mathcal{G} are locally isomorphic, and for each $V \rightarrow X$ there is a refinement $U \rightarrow V$ with \mathcal{G}_U nonempty. An \mathcal{F} -gerbe is a gerbe \mathcal{G} , together with isomorphisms $\rho_T : \mathcal{F}_U \rightarrow \text{Aut}_{T/U}$ for each object $T \in \mathcal{G}_U$, such that the ρ_T are compatible with restrictions, and that the diagram

$$\begin{array}{ccc} \mathcal{F}_U & \xrightarrow{\rho_T} & \text{Aut}_{T/U} \\ \text{id} \downarrow & & \downarrow f \mapsto gfg^{-1} \\ \mathcal{F}_U & \xrightarrow{\rho_{T'}} & \text{Aut}_{T'/U} \end{array}$$

is commutative for each U -isomorphism $g : T \rightarrow T'$ (see [9], Chapter IV, Definition 2.2.1). Two \mathcal{F} -gerbes $\mathcal{G}, \mathcal{G}'$ are *equivalent* if there is a functor of stacks $\mathcal{G} \rightarrow \mathcal{G}'$ compatible with the \mathcal{F} -action on automorphism groups. Such functors are automatically equivalences by [9], Chapter IV, Corollary 2.2.7.

The $H^p(X, \mathcal{F})$, $p = 0, 1, 2$ form a ∂ -functor as follows: Given a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

and an \mathcal{F}'' -torsor \mathcal{T}'' , its liftings $(\mathcal{T}, \mathcal{T} \rightarrow \mathcal{T}'')$ to an \mathcal{F} -torsor \mathcal{T} form an \mathcal{F}' -gerbe representing the coboundary $\partial(\mathcal{T}'')$. According to [9], Chapter III, Proposition 3.5.1, and Chapter IV, Lemma 3.4.3, the group $H^p(X, \mathcal{F})$ vanishes on injective sheaves for $p = 1, 2$, hence is a universal ∂ -functor, which justifies the notation.

It is easy to express the obstruction map $H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$ in terms of gerbes and torsors: Let \mathcal{G} be an \mathcal{F} -gerbe. Choose a covering $U \rightarrow X$ admitting an object $T \in \mathcal{G}_U$. Then the sheaf $\text{Isom}(p_0^*T, p_1^*T)$ is an \mathcal{F}_{U^2} -torsor on U^2 , where $p_i : U^2 \rightarrow U$ are the projections omitting the i -th factor. Its isomorphism class is a Čech 1-cochain in $C^1(U, \mathcal{H}^1\mathcal{F})$.

Lemma 3.1. *The $\mathcal{H}^1\mathcal{F}$ -valued 1-cochain $\text{Isom}(p_0^*T, p_1^*T)$ is a 1-cocycle.*

Proof. Set $\mathcal{T} = \text{Isom}(p_0^*T, p_1^*T)$, and let $p_i : U^3 \rightarrow U^2$ be the projections omitting the i -th factor. We have to see that $p_1^*\mathcal{T}$ is isomorphic to the *contracted product* $p_2^*\mathcal{T} \wedge^{\mathcal{F}} p_0^*\mathcal{T}$. The latter is the quotient of $p_2^*\mathcal{T} \times p_0^*\mathcal{T}$ by the \mathcal{F}_{U^2} -action $(h_0, h_2) \cdot f = (h_0 \circ f, f^{-1} \circ h_2)$. Using the semisimplicial identities $p_i \circ p_j = p_{j-1} \circ p_i$, $i < j$, we obtain

$$\begin{aligned} p_0^*\mathcal{T} &\simeq \text{Isom}((p_0p_0)^*T, (p_1p_0)^*T), & p_2^*\mathcal{T} &\simeq \text{Isom}((p_1p_0)^*T, (p_1p_1)^*T), \\ p_1^*\mathcal{T} &\simeq \text{Isom}((p_0p_0)^*T, (p_1p_1)^*T). \end{aligned}$$

Composition gives a map $p_2^*\mathcal{T} \times p_0^*\mathcal{T} \rightarrow p_1^*\mathcal{T}$, which induces the desired bijection $p_2^*\mathcal{T} \wedge^{\mathcal{F}} p_0^*\mathcal{T} \simeq p_1^*\mathcal{T}$. Note that this bijection is canonical. \square

Lemma 3.2. *There is a well-defined linear map $H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$ given by $\mathcal{G} \mapsto \text{Isom}(p_0^*T, p_1^*T)$.*

Proof. You easily check that the cohomology class of $\text{Isom}(p_0^*T, p_1^*T)$ neither depends on the choice of the refinement $U \rightarrow X$ nor on the choice of the object $T \in \mathcal{G}_U$. If $\mathcal{G}, \mathcal{G}'$ are two \mathcal{F} -gerbes representing the same cohomology class, then there is a functor $\mathcal{G} \rightarrow \mathcal{G}'$ compatible with the \mathcal{F} -action on automorphism groups. It follows that the isomorphism class of $\text{Isom}(p_0^*T, p_1^*T)$ depends only on the equivalence class of \mathcal{G} .

It remains to check that the map $H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$ is linear. To see this, choose an injective resolution $\mathcal{F} \rightarrow \mathcal{I}^\bullet$. Given a section $s \in H^0(X, \mathcal{I}^2)$ contained in the image of $f : \mathcal{I}^1 \rightarrow \mathcal{I}^2$, let $f^{-1}(s) \subset \mathcal{I}^1$ be the induced $\mathcal{I}^0/\mathcal{F}$ -torsor, and \mathcal{G}' the corresponding \mathcal{F} -gerbe of \mathcal{I}^0 -liftings of $f^{-1}(s)$. Let $\mathcal{G} \subset \mathcal{G}'$ be the subcategory of liftings $\mathcal{I}_U^0 \rightarrow f^{-1}(s)_U$ to the trivial torsor. Since \mathcal{I}^0 is injective, any \mathcal{I}_U^0 -torsor is trivial. Therefore, the inclusion $\mathcal{G} \subset \mathcal{G}'$ is actually a substack hence an equivalence of \mathcal{F} -gerbes. Note that any cohomology class is representable by such an \mathcal{F} -gerbe \mathcal{G} , because $\mathcal{F} \rightarrow \mathcal{I}^\bullet$ is an injective resolution.

Now choose lifting $\tilde{s} \in H^0(U, \mathcal{I}^1)$ of s over some refinement $U \rightarrow X$. This defines the lifting $\mathcal{I}_U^0 \rightarrow f^{-1}(s)_U$, $0 \mapsto \tilde{s}_U$, that is, an object $T \in \mathcal{G}_U$. Now a morphism $p_0^*T \rightarrow p_1^*T$ is precisely a lifting of $p_1^*(\tilde{s}) - p_0^*(\tilde{s}) \in H^0(U^2, \mathcal{I}^0/\mathcal{F})$ to \mathcal{I}^0 . Consequently, the torsor $\text{Isom}(p_0^*T, p_1^*T)$ is nothing but the image of $p_1^*(\tilde{s}) - p_0^*(\tilde{s}) \in H^0(U^2, \mathcal{I}^0/\mathcal{F})$ under the coboundary $H^0(U^2, \mathcal{I}^0/\mathcal{F}) \rightarrow H^1(U^2, \mathcal{F})$ induced by the exact sequence $0 \rightarrow \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^0/\mathcal{F} \rightarrow 0$. Using this description, we immediately infer that $\mathcal{G} \mapsto \text{Isom}(p_0^*T, p_1^*T)$ is linear. \square

Proposition 3.3. *An \mathcal{F} -gerbe \mathcal{G} lies in the image of $\check{H}^2(X, \mathcal{F}) \rightarrow H^2(X, \mathcal{F})$ if and only if the class of $\text{Isom}(p_0^*T, p_1^*T)$ vanishes in $\check{H}^1(X, \mathcal{H}^1\mathcal{F})$. In other words, we have an exact sequence $0 \rightarrow \check{H}^2(X, \mathcal{F}) \rightarrow H^2(X, \mathcal{F}) \rightarrow \check{H}^1(X, \mathcal{H}^1\mathcal{F})$.*

Proof. According to [9], Chapter IV, Corollary 2.5.3, an \mathcal{F} -gerbe \mathcal{G} comes from $\check{H}^2(X, \mathcal{F})$ if and only if it admits an object $T \in \mathcal{G}_U$ over some refinement $U \rightarrow X$ with $p_0^*(T) \simeq p_1^*(T)$, hence $\text{Isom}(p_0^*T, p_1^*T)$ is trivial.

Now suppose $\mathcal{T} = \text{Isom}(p_0^*T, p_1^*T)$ has trivial cohomology class. Replacing U by a refinement, we find an \mathcal{F} -torsor \mathcal{P} on U with $\text{Isom}(p_1^*\mathcal{P}, p_0^*\mathcal{P}) \simeq \mathcal{T}$. According to [9], Chapter III, Proposition 2.3.2 there is a twisted object $T' \in \mathcal{G}_U$ satisfying $\mathcal{P} = \text{Isom}(T, T')$. Then $\text{Isom}(p_0^*T', p_1^*T')$, being isomorphic to

$$\text{Isom}(p_0^*T', p_0^*T) \wedge \text{Isom}(p_0^*T, p_1^*T) \wedge \text{Isom}(p_1^*T, p_1^*T') = p_0^*(\mathcal{P}^{-1}) \wedge \mathcal{T} \wedge p_1^*(\mathcal{P}),$$

is trivial, and we conclude that the class of \mathcal{G} lies in $\check{H}^2(X, \mathcal{F})$. \square

4. CENTRAL SEPARABLE ALGEBRAS

In this section I apply Theorem 2.1 to the bigger Brauer group. Throughout, X denotes a noetherian scheme. Let me recall some notions from Raeburn and Taylor [19]. Given two coherent \mathcal{O}_X -modules \mathcal{E}, \mathcal{F} and a pairing $\lambda : \mathcal{F} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$, we obtain a coherent \mathcal{O}_X -algebra $\mathcal{E} \otimes^\lambda \mathcal{F}$ as follows: The underlying \mathcal{O}_X -module is $\mathcal{E} \otimes \mathcal{F}$, and the multiplication law is

$$(e \otimes f) \cdot (e' \otimes f') = e\lambda(f, e') \otimes f' = e \otimes \lambda(f, e')f'.$$

Usually, $\mathcal{E} \otimes^\lambda \mathcal{F}$ is neither commutative nor unital. We are mainly interested in the case that λ is surjective; this ensures that \mathcal{E}, \mathcal{F} , and $\mathcal{E} \otimes^\lambda \mathcal{F}$ are faithful \mathcal{O}_X -modules.

Now let \mathcal{A} be a coherent \mathcal{O}_X -algebra. A *splitting* for \mathcal{A} is a quadruple $(\mathcal{E}, \mathcal{F}, \lambda, s)$, where \mathcal{E}, \mathcal{F} are coherent \mathcal{O}_X -modules, $\lambda : \mathcal{F} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$ is a surjective pairing, and $s : \mathcal{A} \rightarrow \mathcal{E} \otimes^\lambda \mathcal{F}$ is an \mathcal{O}_X -algebra bijection. We say that \mathcal{A} is *elementary* if it admits

a splitting. If there is an étale covering $U \rightarrow X$ so that \mathcal{A}_U admits a splitting, we say that \mathcal{A} is a *central separable algebra*.

Suppose \mathcal{A} is a central separable algebra. For each étale map $U \rightarrow X$, let \mathcal{S}_U be the groupoid of splittings for \mathcal{A}_U ; a morphism $(\mathcal{E}, \mathcal{F}, \lambda, s) \rightarrow (\mathcal{E}', \mathcal{F}', \lambda', s')$ of splittings is a pair of bijections $e : \mathcal{E} \rightarrow \mathcal{E}'$ and $f : \mathcal{F} \rightarrow \mathcal{F}'$ such that the diagrams

$$\begin{array}{ccc} \mathcal{F} \otimes \mathcal{E} & \xrightarrow{\lambda} & \mathcal{O}_X & & \mathcal{A} & \xrightarrow{s} & \mathcal{E} \otimes \mathcal{F} \\ f \otimes e \downarrow & & \downarrow \text{id} & \text{and} & \text{id} \downarrow & & \downarrow e \otimes f \\ \mathcal{F}' \otimes \mathcal{E}' & \xrightarrow{\lambda'} & \mathcal{O}_X & & \mathcal{A} & \xrightarrow{s'} & \mathcal{E}' \otimes \mathcal{F}' \end{array}$$

commute. Clearly, the fibered category $\mathcal{S} \rightarrow X_{\text{ét}}$ is a stack in Giraud's sense ([9], Chapter II, Definition 1.2.1). According to [19], Lemma 2.3, the splittings for \mathcal{A} are locally isomorphic. Furthermore, each splitting $(\mathcal{E}, \mathcal{F}, \lambda, s)$ comes along with a sheaf homomorphism

$$\mathbb{G}_m \longrightarrow \text{Aut}_{(\mathcal{E}, \mathcal{F}, \lambda, s)}, \quad \xi \longmapsto (\xi, 1/\xi),$$

which is bijective by [19], Lemma 2.4. In other words, \mathcal{S} is a \mathbb{G}_m -gerbe. So each central separable algebra \mathcal{A} defines via the gerbe \mathcal{S} a cohomology class in $H_{\text{ét}}^2(X, \mathbb{G}_m)$.

Next, let us recall Taylor's definition of the bigger Brauer group. You easily check that central separable algebras are closed under taking opposite algebras and tensor products. Two central separable algebras $\mathcal{A}, \mathcal{A}'$ are called *equivalent* if there are elementary algebras $\mathcal{B}, \mathcal{B}'$ with $\mathcal{A} \otimes \mathcal{B} \simeq \mathcal{A}' \otimes \mathcal{B}'$. The set of equivalence classes $\widetilde{\text{Br}}(X)$ is called the *bigger Brauer group*. Addition is given by tensor product, and inverses are given by opposite algebras.

The map $\mathcal{A} \mapsto \mathcal{S}$ induces an inclusion $\widetilde{\text{Br}}(X) \subset H_{\text{ét}}^2(X, \mathbb{G}_m)$ of abelian groups. Raeburn and Taylor [19] showed that this inclusion is a bijection provided that each finite subset of X admits a common affine neighborhood. We may relax this assumptions:

Theorem 4.1. *Let X be a noetherian scheme with the property that each pair $x, y \in X$ admits an affine open neighborhood. Then $\widetilde{\text{Br}}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$.*

Proof. The proof of Raeburn and Taylor actually shows that, on an arbitrary noetherian scheme, each Čech 2-cohomology class comes from a coherent central separable \mathcal{O}_X -algebra ([19], Theorem 3.6). According to Theorem 2.1, we have $\check{H}_{\text{ét}}^2(X, \mathbb{G}_m) = H_{\text{ét}}^2(X, \mathbb{G}_m)$, and in turn $\widetilde{\text{Br}}(X) = H_{\text{ét}}^2(X, \mathbb{G}_m)$. \square

5. NORMAL NOETHERIAN SCHEMES

Hilbert's Theorem 90 implies that the map $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \rightarrow H_{\text{ét}}^2(X, \mathbb{G}_m)$ is injective. The goal of this section is to construct central separable algebras representing classes from this subgroup. Throughout, we shall assume that X is a normal noetherian scheme.

Let Div_X and \mathcal{Z}_X^1 be the sheaves of Cartier divisors and Weil divisors with respect to the Zariski topology, and $\mathcal{P}_X = \mathcal{Z}_X^1 / \text{Div}_X$ the corresponding quotient sheaf. Similarly, let $\text{Div}(X)$ and $Z^1(X)$ be the groups of Cartier divisors and Weil divisors, and $\text{Cl}(X) = Z^1(X) / \text{Div}(X)$. Setting $P(X) = \Gamma(X, \mathcal{P}_X)$, we obtain an inclusion $\text{Cl}(X) \subset P(X)$.

Proposition 5.1. *Let X be a normal noetherian scheme. Then there is a canonical identification $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) = P(X) / \text{Cl}(X)$.*

Proof. Let \mathcal{M}_X^\times be the sheaf of invertible rational functions. The exact sequence $1 \rightarrow \mathcal{O}_X^\times \rightarrow \mathcal{M}_X^\times \rightarrow \mathcal{D}iv_X \rightarrow 0$ gives an exact sequence

$$H_{\text{zar}}^1(X, \mathcal{M}_X^\times) \longrightarrow H_{\text{zar}}^1(X, \mathcal{D}iv_X) \xrightarrow{\partial} H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \longrightarrow H_{\text{zar}}^2(X, \mathcal{M}_X^\times).$$

The outer groups $H_{\text{zar}}^n(X, \mathcal{M}_X^\times)$ vanish; to check this, use the spectral sequence $H_{\text{zar}}^p(X, R^q i_* \mathcal{O}_{X^{(0)}}^\times) \Rightarrow H_{\text{zar}}^{p+q}(X^{(0)}, \mathcal{O}_{X^{(0)}}^\times)$, where $i : X^{(0)} \rightarrow X$ is the inclusion of the generic points. Now the exact sequence $0 \rightarrow \mathcal{D}iv_X \rightarrow \mathcal{Z}_X^1 \rightarrow \mathcal{P}_X \rightarrow 0$ gives an exact sequence

$$\text{Div}(X) \longrightarrow Z^1(X) \longrightarrow P(X) \xrightarrow{\partial} H_{\text{zar}}^1(X, \mathcal{D}iv_X) \longrightarrow H_{\text{zar}}^1(X, \mathcal{Z}_X^1).$$

The term on the right vanishes, because \mathcal{Z}_X^1 is flabby, and the result follows. \square

Weil divisors give rise to central separable algebras in the following way: Given finitely many $C_1, \dots, C_n \in Z^1(X)$, consider the coherent reflexive sheaves

$$\mathcal{E} = \bigoplus_{\nu=1}^n \mathcal{O}_X(C_\nu) \quad \text{and} \quad \mathcal{F} = \bigoplus_{\nu=1}^n \mathcal{O}_X(-C_\nu).$$

Let $\lambda_{\nu\mu} : \mathcal{O}_X(C_\nu) \otimes \mathcal{O}_X(-C_\mu) \rightarrow \mathcal{O}_X$ be the pairing defined as

$$f \otimes g \mapsto \begin{cases} f(g) & \text{if } \nu = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

The $(n \times n)$ -matrix of pairings $\lambda = (\lambda_{\nu\mu})$ defines a pairing $\lambda : \mathcal{F} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$. As described in Section 4, this yields a coherent \mathcal{O}_X -algebra $\mathcal{A} = \mathcal{E} \otimes^\lambda \mathcal{F}$.

Clearly, the pairing $\lambda : \mathcal{F} \otimes \mathcal{E} \rightarrow \mathcal{O}_X$ is surjective if at each point $x \in X$ at least one Weil divisor C_i is Cartier. Under this assumption, \mathcal{A} is a central separable \mathcal{O}_X -algebra endowed with a splitting. We shall use such algebras for the following result:

Theorem 5.2. *Suppose X is a normal noetherian scheme. Then we have inclusions $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \subset \widetilde{\text{Br}}(X)$ of subgroups in $H_{\text{ét}}^2(X, \mathbb{G}_m)$.*

Proof. Fix a class $\alpha \in H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$, and choose a representant $s \in P(X)$ with respect to the canonical surjection $P(X) \rightarrow H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$ from Proposition 5.1. Then $s \in P(X)$ is given by a collection of Weil divisors $D_i \in Z^1(U_i)$ on some open covering $X = U_1 \cup \dots \cup U_n$, such that $D_i - D_j$ are Cartier on the overlaps $U_{ij} = U_i \cap U_j$. We may extend each D_i from U_i to X and denote the resulting Weil divisor $D_i \in Z^1(X)$ by the same letter. For each $U_i \subset X$, set

$$\mathcal{E}_i = \bigoplus_{\nu=1}^n \mathcal{O}_{U_i}(D_i - D_\nu) \quad \text{and} \quad \mathcal{F}_i = \bigoplus_{\nu=1}^n \mathcal{O}_{U_i}(D_\nu - D_i).$$

As above, this yields a coherent \mathcal{O}_{U_i} -algebra $\mathcal{A}_i = \mathcal{E}_i \otimes^{\lambda_i} \mathcal{F}_i$. They are central separable because $D_i - D_\nu$ is Cartier on U_i for $\nu = i$.

These \mathcal{O}_{U_i} -algebras glue together as follows: For each overlap $U_{ij} = U_i \cap U_j$, consider the invertible $\mathcal{O}_{U_{ij}}$ -module $\mathcal{L}_{ij} = \mathcal{O}_{U_{ij}}(D_i - D_j)$. We have canonical isomorphisms

$$\mathcal{E}_i|_{U_{ij}} \otimes \mathcal{L}_{ji} \longrightarrow \mathcal{E}_j|_{U_{ij}} \quad \text{and} \quad \mathcal{L}_{ij} \otimes \mathcal{F}_i|_{U_{ij}} \longrightarrow \mathcal{F}_j|_{U_{ij}}.$$

The canonical bijections $\mathcal{L}_{ji} \otimes \mathcal{L}_{ij} \rightarrow \mathcal{O}_{U_{ij}}$ yield isomorphisms $\lambda_{ij} : \mathcal{A}_j|_{U_{ij}} \rightarrow \mathcal{A}_i|_{U_{ij}}$. These isomorphisms obviously satisfy the cocycle condition $\lambda_{ij} \circ \lambda_{jk} = \lambda_{ik}$ on triple

overlaps. We deduce that there is a coherent central separable \mathcal{O}_X -algebra \mathcal{A} with $\mathcal{A}|_{U_i} = \mathcal{A}_i$.

It remains to check that the \mathcal{O}_X^\times -gerbe \mathcal{S} of splittings for \mathcal{A} has cohomology class $\alpha \in H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$. Let $f : \mathcal{Z}_X^1 \rightarrow \mathcal{P}_X$ be the canonical surjection, and \mathcal{G}' be the \mathcal{O}_X^\times -gerbe of \mathcal{M}_X^\times -liftings for the Div_X -torsor $f^{-1}(s) \subset \mathcal{Z}_X^1$. Then \mathcal{G}' has class $\alpha \in H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$ because $s \mapsto \alpha$. Note that $H^1(U, \mathcal{M}_X^\times) = 0$ for any open subset $U \subset X$. Therefore, the fibered subcategory $\mathcal{G} \subset \mathcal{G}'$ of liftings of $f^{-1}(s)$ to the trivial \mathcal{M}_X^\times -torsor \mathcal{M}_X^\times is an \mathcal{O}_X^\times -subgerbe.

To finish the proof, we construct a functor $\mathcal{G} \rightarrow \mathcal{S}$ compatible with \mathcal{O}_X^\times -actions. Suppose we have an object in \mathcal{G} over an open subset $V \subset X$, that is, an equivariant map $\mathcal{M}_V^\times \rightarrow f^{-1}(s)|_V$. Let $D \in \Gamma(V, f^{-1}(s))$ be the image of the unit section $1 \in \Gamma(V, \mathcal{M}_X^\times)$. Then $D - D_i$ are Cartier on $V_i = V \cap U_i$. Consider the coherent reflexive \mathcal{O}_{V_i} -modules $\mathcal{E}'_i = \mathcal{E}_i \otimes \mathcal{O}_{V_i}(D - D_i)$ and $\mathcal{F}'_i = \mathcal{F}_i \otimes \mathcal{O}_{V_i}(D_i - D)$. We have splittings $\mathcal{A}|_{V_i} = \mathcal{E}'_i \otimes^{\lambda_i} \mathcal{F}'_i$. Note that

$$\mathcal{E}'_i = \bigoplus_{\nu=1}^n \mathcal{O}_{V_i}(D - D_\nu) \quad \text{and} \quad \mathcal{F}'_i = \bigoplus_{\nu=1}^n \mathcal{O}_{V_i}(D_\nu - D).$$

Obviously, the sheaves \mathcal{E}'_i glue together and give a coherent \mathcal{O}_V -module \mathcal{E}' . Similarly, the \mathcal{F}'_i glue and give a coherent \mathcal{O}_V -module \mathcal{F}' . In turn, we obtain a splitting $\mathcal{A}|_V = \mathcal{E}' \otimes^{\lambda'} \mathcal{F}'$.

Summing up, we have defined for each object in \mathcal{G} an object in \mathcal{S} . It is easy to see that this construction is functorial and respects the \mathcal{O}_X^\times -action on automorphism groups. Therefore, the central separable \mathcal{O}_X -algebra \mathcal{A} has class $\alpha \in H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$. \square

Next, we describe the obstruction against cocycles. Fix a cohomology class $\alpha \in H_{\text{zar}}^2(X, \mathcal{O}_X^\times)$ and choose $s \in P(X)$ mapping to α . Then there is an open covering $X = U_1 \cup \dots \cup U_n$ and Weil divisors $D_i \in Z^1(U_i)$ representing $s|_{U_i}$, such that $D_i - D_j$ are Cartier on the overlaps U_{ij} .

Proposition 5.3. *The cocycle $U_{ij} \mapsto \mathcal{O}_{U_{ij}}(D_i - D_j)$ represents the image of α under the obstruction map $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \rightarrow \check{H}_{\text{zar}}^1(X, \mathcal{H}^1(\mathcal{O}_X^\times))$.*

Proof. Consider the exact sequence $1 \rightarrow \mathcal{O}_X^\times \rightarrow \mathcal{M}_X^\times \rightarrow \mathcal{Z}^1 \rightarrow \mathcal{P}_X \rightarrow 0$. Since $H_{\text{zar}}^1(U, \mathcal{M}_X^\times) = 0$ for any open subset $U \subset X$, we may argue as in the proof of Proposition 3.2 and infer that $U_{ij} \mapsto \mathcal{O}_{U_{ij}}(D_i - D_j)$ represents the image of α under the obstruction map $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \rightarrow \check{H}_{\text{zar}}^1(X, \mathcal{H}^1(\mathcal{O}_X^\times))$. \square

6. NONSEPARATED SURFACES

Recall that the Brauer group $\text{Br}(X) \subset H_{\text{ét}}^2(X, \mathbb{G}_m)$ is the subgroup generated by Azumaya algebras, and that the *cohomological Brauer group* $\text{Br}'(X) \subset H_{\text{ét}}^2(X, \mathbb{G}_m)$ is the torsion subgroup. In this section we discuss the example of Edidin, Hassett, Kresch, and Vistoli [7] of a scheme with $\text{Br}(X) \neq \text{Br}'(X)$.

Let A be a strictly local normal noetherian ring of dimension two that is nonfactorial. In other words, A is neither regular nor an E_8 -singularity ([4], Proposition 3.3). Set $Y = \text{Spec}(A)$ and let $W \subset Y$ be the complement of the closed point. Define $X = U_1 \cup U_2$ as the union of two copies of Y glued along W . Then X is a normal nonseparated surface with two closed points $x_1 \in U_1, x_2 \in U_2$.

The theory of quotient stacks was used in [7] to prove $\mathrm{Br}(X) \neq \mathrm{Br}'(X)$. Let us present a different argument. The covering $X = U_1 \cup U_2$ gives an exact sequence

$$\bigoplus_{i=1}^2 H^1(U_i, \mathbb{G}_m) \longrightarrow H^1(U_1 \cap U_2, \mathbb{G}_m) \longrightarrow H^2(X, \mathbb{G}_m) \longrightarrow \bigoplus_{i=1}^2 H^2(U_i, \mathbb{G}_m)$$

for both Zariski and étale cohomology. The outer terms vanish because the U_i are strictly local. Together with Hilbert's Theorem 90, this implies $H_{\mathrm{ét}}^2(X, \mathbb{G}_m) = H_{\mathrm{zar}}^2(X, \mathbb{G}_m)$. Hence *every cohomology class comes from a central separable \mathcal{O}_X -algebra* by Theorem 5.2.

Using Proposition 5.1 and the canonical bijections $\mathrm{Cl}(X) = \mathrm{Pic}(W) = \mathrm{Cl}(Y)$, we conclude

$$H_{\mathrm{ét}}^2(X, \mathbb{G}_m) = \mathrm{Cl}(U_1) \oplus \mathrm{Cl}(U_2) / \mathrm{Cl}(Y) \simeq \mathrm{Cl}(Y) \neq 0.$$

This implies $\check{H}_{\mathrm{zar}}^2(X, \mathcal{O}_X^\times) = 0$. Indeed, suppose some class in $H_{\mathrm{zar}}^2(X, \mathbb{G}_m)$ represented by a pair of Weil divisors $(D_1, D_2) \in \mathrm{Cl}(U_1) \oplus \mathrm{Cl}(U_2)$ vanishes in the obstruction group $\check{H}_{\mathrm{zar}}^1(X, \mathcal{H}^1(\mathcal{O}_X^\times))$. By Proposition 5.3, the invertible sheaf $\mathcal{O}_W(D_1 - D_2)$ is of the form $\mathcal{L}_1|_W \otimes \mathcal{L}_2|_W$ with $\mathcal{L}_i \in \mathrm{Pic}(U_i) = 0$. It follows that our pair (D_1, D_2) is zero in $H_{\mathrm{ét}}^2(X, \mathbb{G}_m)$. Summing up, *only the trivial cohomology class comes from a cocycle*.

As explained in [21], Proposition 1.5, each Azumaya \mathcal{O}_X -algebra \mathcal{A} is of the form $\mathrm{End}(\mathcal{E})$ for some reflexive \mathcal{O}_X -module \mathcal{E} , say of rank $r > 0$, with $\mathcal{E}_{U_i} = \mathcal{O}_{U_i}(D_i)^{\oplus r}$ for some Weil divisors $D_i \in Z^1(U_i)$. Furthermore, the class of \mathcal{A} is the image of $-(D_1, D_2)$ in $H_{\mathrm{zar}}^2(X, \mathbb{G}_m)$. Since $\Gamma(U_1, \mathcal{A}) = \Gamma(W, \mathcal{A}) = \Gamma(U_2, \mathcal{A})$, we have $D_1 \sim D_2$ and conclude $\mathrm{Br}(X) = 0$. In other words, *only the trivial cohomology class comes from an Azumaya algebra*.

7. NONPROJECTIVE PROPER SURFACES

In this section I discuss the cohomology groups $H_{\mathrm{ét}}^2(X, \mathbb{G}_m)$ for some nonprojective proper surfaces constructed in [20]. Let me recall the construction: Fix an algebraically closed ground field k , let E be an elliptic curve, and choose two closed points $e_1, e_2 \in E$. Let $Y \rightarrow \mathbb{P}^1 \times E$ be the blowing-up of the points $(0, e_1), (\infty, e_2)$, and $g: Y \rightarrow X$ the contraction of the strict transforms $E_1, E_2 \subset Y$ of $0 \times E, \infty \times E$. Then X is a proper normal algebraic surface containing two singularities $x_1, x_2 \in X$ of genus g . As explained in [20], it has no ample line bundles if the divisor classes $e_1, e_2 \in \mathrm{Pic}(E) \otimes \mathbb{Q}$ are linearly independent.

Proposition 7.1. *We have $H_{\mathrm{zar}}^2(X, \mathcal{O}_X^\times) \simeq \mathrm{Pic}(E) / \mathbb{Z}e_1 + \mathbb{Z}e_2$.*

Proof. The sheaf $\mathcal{P}_X = \mathcal{Z}_X^1 / \mathrm{Div}_X$ is a skyscraper sheaf supported by the singular locus $\{x_1, x_2\}$, with stalks $\mathcal{P}_{x_i} = \mathrm{Cl}(\mathcal{O}_{X, x_i})$. According to Proposition 5.1, we have

$$H_{\mathrm{zar}}^2(X, \mathcal{O}_X^\times) = \mathrm{Cl}(\mathcal{O}_{X, x_1}) \oplus \mathrm{Cl}(\mathcal{O}_{X, x_2}) / \mathrm{Cl}(X).$$

The terms on the right are $\mathrm{Cl}(\mathcal{O}_{X, x_i}) = \mathrm{Pic}(Y \otimes \mathcal{O}_{X, x_i}) / \mathbb{Z}E_i$, where $Y \otimes \mathcal{O}_{X, x_i}$ denotes the fiber product $Y \times_X \mathrm{Spec}(\mathcal{O}_{X, x_i})$. Moreover, the canonical mapping $\mathrm{Pic}(Y \otimes \mathcal{O}_{X, x_i}) \rightarrow \mathrm{Pic}(Y \otimes \mathcal{O}_{X, x_i}^\wedge)$ is injective. Grothendieck's Existence Theorem gives $\mathrm{Pic}(Y \otimes \mathcal{O}_{X, x_i}^\wedge) = \mathrm{Pic}(nE_i)$ for some $n > 0$, and we have $\mathrm{Pic}(nE_i) = \mathrm{Pic}(E_i)$ because E_i is elliptic. Consequently $\mathrm{Cl}(\mathcal{O}_{X, x_i}) = \mathrm{Pic}(E_i) / \mathbb{Z}e_i$.

The group $\mathrm{Cl}(X)$ is generated by the images of $\mathrm{Pic}(E)$, $\mathrm{Pic}(\mathbb{P}^1)$, and the exceptional divisors for the contraction $Y \rightarrow \mathbb{P}^1 \times E$. The latter two types restrict to zero in $\mathrm{Cl}(\mathcal{O}_{X, x_i})$. The result now follows from the snake lemma. \square

Proposition 7.2. *The inclusion $H_{\text{zar}}^2(X, \mathcal{O}_X^\times) \subset H_{\text{ét}}^2(X, \mathbb{G}_m)$ is bijective.*

Proof. We have $H_{\text{ét}}^2(E, \mathbb{G}_m) = 0$ because the ground field is algebraically closed ([16], Corollary 1.2). In turn $H_{\text{ét}}^2(\mathbb{P}^1 \times E, \mathbb{G}_m)$ vanishes ([8], page 193, Theorem 2). By birational invariance, $H_{\text{ét}}^2(Y, \mathbb{G}_m)$ vanishes as well ([16], Corollary 7.2). Now the commutative diagram

$$\begin{array}{ccccccc} \text{Pic}(Y_{\text{zar}}) & \longrightarrow & H_{\text{zar}}^0(X, R^1 g_* \mathcal{O}_Y^\times) & \longrightarrow & H_{\text{zar}}^2(X, \mathcal{O}_X^\times) & \longrightarrow & H_{\text{zar}}^2(Y, \mathcal{O}_Y^\times) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Pic}(Y_{\text{ét}}) & \longrightarrow & H_{\text{ét}}^0(X, R^1 g_* \mathbb{G}_m) & \longrightarrow & H_{\text{ét}}^2(X, \mathbb{G}_m) & \longrightarrow & H_{\text{ét}}^2(Y, \mathbb{G}_m). \end{array}$$

The map on the left is bijective by Hilbert's Theorem 90. The map next to the left is nothing but the sum of the maps $\text{Pic}(Y \otimes \mathcal{O}_{X, x_i}) \rightarrow \text{Pic}(Y \otimes \mathcal{O}_{X, x_i}^{\text{sh}})$. But both $\text{Pic}(Y \otimes \mathcal{O}_{X, x_i})$ and $\text{Pic}(Y \otimes \mathcal{O}_{X, x_i}^{\text{sh}})$ are equal to $\text{Pic}(E_i)$ as shown in the proof for Proposition 7.1. We infer $H_{\text{zar}}^2(X, \mathbb{G}_m) = H_{\text{ét}}^2(X, \mathbb{G}_m)$ using the 5-Lemma. \square

Proposition 7.3. *We have $\check{H}_{\text{zar}}^2(X, \mathcal{O}_X^\times) = 0$.*

Proof. We have to check that the map $H_{\text{zar}}^2(X, \mathbb{G}_m) \rightarrow \check{H}_{\text{zar}}^1(X, \mathcal{H}^1 \mathbb{G}_m)$ is injective. Pick some $s \in P(X)$. Choose an open covering $U_i \subset X$ so that s lifts to Weil divisors $D_i \in Z^1(U_i)$. The image of s in $\check{H}_{\text{zar}}^1(X, \mathcal{H}^1 \mathbb{G}_m)$ is represented by the 1-cocycle $U_{i,j} \mapsto \mathcal{O}_{U_{i,j}}(D_i - D_j)$. Suppose this class is zero. After refining the covering, there are Cartier divisors $C_i \in \text{Div}(U_i)$ with $D_i - D_j = C_i - C_j$. After re-indexing, we may assume $x_1 \in U_1$ and $x_2 \in U_2$. Since D_1 is principal on $\text{Spec}(\mathcal{O}_{X, x_1})$, and D_2 is principal on $\text{Spec}(\mathcal{O}_{X, x_2})$, we infer that $C_1 - C_2$ is a principal divisor on the Dedekind scheme $S = \text{Spec}(\mathcal{O}_{X, x_1}) \times_X \text{Spec}(\mathcal{O}_{X, x_2})$, which comprises all points $x \in X$ with $\{x_1, x_2\} \subset \overline{\{x\}}$. But this implies that s is the restriction of a global reflexive rank one sheaf, such that s maps to zero in $H_{\text{zar}}^2(X, \mathbb{G}_m)$. \square

Question 7.4. Is the inclusion $\check{H}_{\text{ét}}^2(X, \mathbb{G}_m) \subset H_{\text{ét}}^2(X, \mathbb{G}_m)$ bijective? Does the obstruction group $\check{H}_{\text{ét}}^1(X, \mathcal{H}^1 \mathbb{G}_m)$ vanish?

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MATHEMATISCHE FAKULTÄT, RUHR-UNIVERSITÄT, 44780 BOCHUM, GERMANY
E-mail address: s.schroer@ruhr-uni-bochum.de