

TOTAL SEPARABLE CLOSURE AND CONTRACTIONS

STEFAN SCHRÖER

29 January 2019

ABSTRACT. We show that on integral normal separated schemes whose function field is separably closed, for each pair of points the intersection of the resulting local schemes is local. This extends a result of Artin from rings to schemes. The argument relies on the existence of certain modifications in inverse limits. As an application, we show that Čech cohomology coincides with sheaf cohomology for the Nisnevich topology. Along the way, we generalize the characterization of contractible curves on surfaces by negative-definiteness of the intersection matrix to higher dimensions, using bigness of invertible sheaves on non-reduced schemes.

CONTENTS

Introduction	1
1. Contractions over Dedekind domains	3
2. Totally separably closed schemes	11
3. Application to Nisnevich cohomology	15
References	16

INTRODUCTION

This paper deals with the Zariski topology for a class of schemes that are in general highly non-noetherian, yet arise from noetherian schemes in a canonical way: We say that an integral scheme X is *totally separably closed* if it is normal and the function field $F = \mathcal{O}_{X,\eta} = \kappa(\eta)$ is separably closed. As abbreviation one also says that X is a *TSC scheme*. Each integral scheme X_0 has a total separable closure $X = \text{TSC}(X_0)$, defined as the integral closure with respect to a chosen separable closure $F = F_0^{\text{sep}}$ of the function field F_0 . Such schemes X are *everywhere strictly local*. In other words, all local rings $\mathcal{O}_{X,x}$, $x \in X$ are strictly local rings, that is, henselian with separably closed residue field. One may regard them as analogues of *Prüfer schemes*, where all local rings are valuation rings.

TSC schemes have some relevance with respect to the étale topology. Indeed, M. Artin [3] used them to prove that Čech cohomology equals sheaf cohomology for the étale topology over affine schemes $X = \text{Spec}(R)$. This result immediately extends to schemes with the *AF property*, which means that any finite subset admits an affine open neighborhood. Note, however, that by [4], Corollary 2, the AF property is

2010 *Mathematics Subject Classification*. 14E05, 14F20, 13B22.

Key words and phrases. Absolute integral closure, modifications, contractions, Nisnevich topology.

equivalent to quasiprojectivity, at least for normal schemes that are separated and of finite type over a ground field. Actually, Artin used algebraic closure rather than separable closure, but this makes no difference for the underlying topological spaces. See Huneke’s overview [16] for the role of absolute integral closure in commutative algebra.

One crucial step in Artin’s arguments is to show that affine integral TSC schemes X have the following very surprising property, which is of purely topological nature: For any pair of points $u, v \in X$ the intersection of local schemes

$$\mathrm{Spec}(\mathcal{O}_{X,u}) \cap \mathrm{Spec}(\mathcal{O}_{X,v}) \subset X$$

remains a local scheme. If we endow the underlying set X with the order relation $x \leq y \Leftrightarrow x \in \overline{\{y\}}$, the above property means that the supremum $\sup(u, v)$ exists for all pairs of points $u, v \in X$. This strange property almost never holds on noetherian schemes X_0 , and intuitively means that in inverse limits $X = \varprojlim X_\lambda$, common generalizations of $u_\lambda, v_\lambda \in X_\lambda$ are totally “ripped apart”. In some sense, this is a topological incarnation of the result of Schmidt that a field with two inequivalent henselian valuations is separably closed ([27], Satz 3. See [10], Theorem 4.4.1 for a modern account). The main goal of this paper is to establish Artin’s result in full generality:

Theorem. (See 2.1.) *For any separated integral TSC scheme X , the intersections $\mathrm{Spec}(\mathcal{O}_{X,u}) \cap \mathrm{Spec}(\mathcal{O}_{X,v}) \subset X$ are local for all points $u, v \in X$.*

In [28], I already obtained this for total separable closures of schemes X_0 that are separated and of finite type over a ground field k . The arguments rely on modifications and contractions in inverse limits $X = \varprojlim X_\lambda$, and do not apply in mixed characteristics. Here we modify our approach, and reduce the problem to proper schemes over excellent Dedekind domains. We then use different modifications X' and contractions \tilde{X} in inverse limits so that Artin’s result applies to the TSC scheme \tilde{X} , which is constructed to have the AF property. This is enough to conclude for the original TSC scheme X .

To carry this out, we have to analyze the existence of suitable *modifications* and *contractions*. On algebraic surfaces X , a curve $E = E_1 + \dots + E_r$ is contractible to points if and only if the intersection matrix $N = (E_i \cdot E_j)$ is negative-definite. This observation goes back to Mumford, Artin and Deligne, in various forms of generality. Note that in general the contractions $r : X \rightarrow Y$ yield algebraic spaces rather than schemes. The following generalization to higher dimensions seems to be of independent interest:

Theorem. (See 1.5.) *Let X be a normal scheme that is proper over an excellent Dedekind domain R , and $E = E_1 + \dots + E_r$ be a Weil divisor contained in a closed fiber for the structure morphism $X \rightarrow \mathrm{Spec}(R)$. If E is contractible to points, then for each effective Cartier divisor $D = \sum m_i E_i$, the invertible sheaf $\mathcal{O}_D(-D)$ on D is big.*

Here bigness for an invertible sheaf \mathcal{L} on some proper algebraic scheme Z , which is not necessarily reduced or irreducible, is defined in terms of the *Iitaka dimension*, which itself is given, up to a shift, by the Krull dimension of the ring $R(Z, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(Z, \mathcal{L}^{\otimes n})$. This generalization from integral to arbitrary

schemes was analyzed by Cutkosky [7], and his results on the multiplicity or volume $\text{mult}(\mathfrak{a}_\bullet) = \text{vol}(\mathfrak{a}_\bullet)$ for *graded families of ideals*, together with Huneke’s version [15] of the Briançon–Skoda Theorem, play a crucial role for the above.

As explained in [28], our main result on TSC schemes has immediate consequences for the *Nisnevich topology* of completely decomposed étale maps [24]. This is a variant of the étale topology, where the local rings are henselian local rings rather than strictly local rings. We get:

Theorem. (See 3.1.) *For each quasicompact separated scheme X and every abelian Nisnevich sheaf F , the edge maps*

$$\check{H}_{\text{Nis}}^p(X, F) \longrightarrow H_{\text{Nis}}^p(X, F)$$

from Čech cohomology to sheaf cohomology are bijective in every degree $p \geq 0$.

The paper is organized as follows: Section 1 contains our results on contractions for proper schemes over excellent Dedekind domains. The main result about TSC schemes is given in Section 2. The final Section 3 gives the application to Nisnevich cohomology.

Acknowledgement. This research was conducted in the framework of the research training group *GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology*, which is funded by the DFG.

1. CONTRACTIONS OVER DEDEKIND DOMAINS

Let $S = \text{Spec}(R)$ be the spectrum of a Dedekind domain R , and X be an proper S -scheme. We write $f : X \rightarrow S$ for the structure morphism. For simplicity, we assume that the scheme X is integral and that the ring R is excellent. Note that we do not assume flatness; in particular, the structure morphism may factor over some closed point $\sigma \in S$.

A closed subscheme $E \subset X$ is said to be *contractible to points* if there is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{r} & Y \\ & \searrow f & \swarrow g \\ & & S \end{array}$$

where Y is an algebraic space, the structure morphism $g : Y \rightarrow S$ is proper, and $r : X \rightarrow Y$ is a morphism with $\mathcal{O}_Y = r_*(\mathcal{O}_X)$ that is an open embedding on $X \setminus E$ such that the image $Z = r(E)$ consists of finitely many closed points. Their images in S are closed as well, and it follows that the connected components of E are contained in closed fibers for the structure morphism $f : X \rightarrow S$. The morphism $r : X \rightarrow Y$ is unique up to unique isomorphism, and depends only on the underlying closed set for $E \subset X$, which follows from [13], Lemma 8.11.1.

Algebraic spaces can be glued along open subsets in the same way as ringed spaces (a consequence from [25], Proposition 5.2.5). In particular, the closed subset $E \subset X$ is contractible to points if and only if each connected component is *contractible to a single point*. Using Stein factorization, one easily sees the following permanence property:

Proposition 1.1. *Assume X' is another proper S -scheme that is integral, and let $X' \rightarrow X$ be a morphism. Suppose that a closed subset $E \subset X$ is contractible to points, and that the morphism $X' \rightarrow X$ is finite over $X \setminus E$. Then the preimage $E' = E \times_X X'$ is contractible to points. Indeed, if $X \rightarrow Y$ is the contraction of $E \subset X$, then the Stein factorization Y' for the composition $X' \rightarrow Y$ is the contraction of $E' \subset X'$.*

For general closed subsets $E \subset X$ it is often difficult to verify contractibility. However, by applying the previous result to the blowing-up $X' \rightarrow X$ with center E one reduces to the case of effective Cartier divisors. Then more can be said:

Proposition 1.2. *Suppose $E \subset X$ is an effective Cartier divisor contained in some closed fiber $X_\sigma = f^{-1}(\sigma)$. Let $\mathcal{L} = \mathcal{O}_X(-E)$. If the restriction $\mathcal{L}|_E$ is ample then the closed subset $E \subset X$ is contractible to points.*

Proof. This immediately follows from Corollary 6.10 in Artin's work [2] on algebraic stacks: It suffices to treat the case that E is connected, and we need to check two conditions. The first condition is straightforward: for every coherent sheaf \mathcal{F} on E the cohomology group $H^1(E, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ vanishes for all $n \gg 0$, because $\mathcal{L}|_E$ is ample. The second condition is somewhat more intricate: Since E is proper, connected and contained in a closed fiber $X_\sigma = f^{-1}(\sigma)$, the rings $R_n = H^0(X, \mathcal{O}_{nE})$ are finite local artinian R -algebras. Write $k_n = R_n/\mathfrak{m}_{R_n}$ for their residue fields. The inclusions $E \subset 2E \subset \dots$ induce an inverse system $k_1 \supset k_2 \supset \dots$ of fields, all of which contain the residue field $\kappa = \kappa(\sigma)$ and have finite degree. Let $k = \bigcap k_n$ be their intersection, and choose an index n_0 so that the inclusions $k_{n+1} \subset k_n$ are equalities for all $n \geq n_0$. Consider the resulting morphism $E \rightarrow \text{Spec}(k)$ and, for each $n \geq 0$, the cartesian diagram

$$\begin{array}{ccc} R_n \times_{k_1} k & \longrightarrow & k \\ \downarrow & & \downarrow \\ R_n & \longrightarrow & k_1. \end{array}$$

Artin's second condition stipulates that the upper vertical arrows must be surjective. To see this, choose some index $m \geq \max(n_0, n)$. Then $k_m = k$ and the residue class map $R_m \rightarrow k_m = k$ factors over the fiber product $R_n \times_{k_1} k$, so the projection in question is surjective. \square

The converse does not hold: For example, if X is regular of dimension $d = 2$, and $E = E_1 + E_2$ is a curve with two irreducible components, having intersection matrix $N = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}$. The latter is negative-definite, so the curve E is contractible. The linear combination $D = 3E_1 + E_2$ is contractible as well, yet $\mathcal{O}_D(-D)$ is not ample, because $(D \cdot E_2) = -1$.

Nevertheless, it is natural to ask for some form of a converse. Indeed, we shall establish such an implication based on the notion of bigness rather than ampleness. Let us recall the relevant definitions: Suppose Z is a proper scheme over some ground field k . Given an invertible sheaf \mathcal{L} on Z we get a graded ring $R(Z, \mathcal{L}) = \bigoplus_{n \geq 0} H^0(Z, \mathcal{L}^{\otimes n})$, which is not necessarily of finite type or noetherian. Let $d = \dim(R)$ be its Krull dimension. The *Iitaka dimension* or *Kodaira–Iitaka dimension*

is defined as

$$\kappa(\mathcal{L}) = \begin{cases} d - 1 & \text{if } d \geq 1; \\ -\infty & \text{else.} \end{cases}$$

Note that it will be crucial to allow reducible and non-reduced schemes Z for what we have in mind. For integral normal schemes Z , the Iitaka dimension is a classical notion from birational geometry: If some $\mathcal{L}^{\otimes n_0}$ with $n_0 \geq 1$ has a non-zero global section, the number $\kappa(\mathcal{L})$ can also be seen as the maximal dimension of the images for the rational maps $X \dashrightarrow \mathbb{P}^m$ defined by $\mathcal{L}^{\otimes n}$, where $m = h^0(\mathcal{L}^{\otimes n}) - 1$ and $n \geq 1$ runs over the positive multiples of n_0 . We refer to the monograph of Lazarsfeld ([20], Section 2.2) for more details. Iitaka dimension was only recently extended to arbitrary proper schemes, by the work of Cutkosky on asymptotics of ideals and linear series. In fact, in [7], Section 7 he defined it in the more general context of *graded linear series*, which can be seen as graded subrings $L = \bigoplus_{n \geq 0} L_n$ inside $R(Z, \mathcal{L})$.

According to [7], Lemma 7.1 we have $\kappa(\mathcal{L}) \leq \dim(Z)$ for arbitrary proper schemes Z . In case of equality $\kappa(\mathcal{L}) = \dim(Z)$ one says that the invertible sheaf \mathcal{L} is *big*. We need the following observation on invertible sheaves that are not big:

Lemma 1.3. *Set $d = \dim(Z)$. If \mathcal{L} is not big, then for each coherent sheaf \mathcal{F} on Z there is a real constant $\beta \geq 0$ so that $h^0(\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \leq \beta n^{d-1}$ for all integers $n \geq 0$.*

Proof. This is a devissage argument similar to [14], Theorem 3.1.2. Let $\text{Coh}(Z)$ be the abelian category of all coherent sheaves \mathcal{F} on Z , and $\mathcal{C} \subset \text{Coh}(Z)$ be the subcategory of all sheaves for which the assertion holds. If $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is a short exact sequence, the resulting long exact sequence immediately gives the following implications:

$$(1) \quad \mathcal{F} \in \mathcal{C} \implies \mathcal{F}' \in \mathcal{C} \quad \text{and} \quad \mathcal{F}', \mathcal{F}'' \in \mathcal{C} \implies \mathcal{F} \in \mathcal{C}.$$

Furthermore, \mathcal{C} contains all coherent sheaves with $\dim(\mathcal{F}) \leq d - 1$, according to [7], Lemma 7.1. Let $Z_1, \dots, Z_r \subset Z$ be the irreducible components, endowed with the reduced scheme structure. By [7], Lemma 10.1 combined with Lemma 9.1 we have

$$\kappa(\mathcal{L}) = \max\{\kappa(\mathcal{L}|_{Z_1}), \dots, \kappa(\mathcal{L}|_{Z_r})\}.$$

In light of [7], Corollary 9.3 it follows that $\mathcal{O}_{Z_i} \in \mathcal{C}$, and hence $\mathcal{O}_{Z'} \in \mathcal{C}$ for every reduced closed subscheme $Z' \subset Z$. Note that the cited Corollary was formulated for projective rather than proper schemes, but the proof holds true without changes in the more general setting.

To proceed, we first suppose that \mathcal{F} is a torsion-free coherent \mathcal{O}_{Z_i} -module, say of rank $r \geq 0$. Let $\eta \in Z_i$ be the generic point, choose a bijection $\mathcal{F}_\eta \simeq \kappa(\eta)^{\oplus r}$, and let \mathcal{F}' be the resulting intersection $\mathcal{F} \cap \mathcal{O}_{Z_i}^{\oplus r}$ inside the sheaf $\mathcal{M}_{Z_i}^{\oplus r}$, where \mathcal{M}_{Z_i} denotes the quasicohherent sheaf of meromorphic functions. Then \mathcal{F}' is coherent and contained in both $\mathcal{O}_{Z_i}^{\oplus r}$ and \mathcal{F} . The quotient $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ has dimension $\leq d - 1$, thus $\mathcal{F}'' \in \mathcal{C}$. Using $\mathcal{F}' \subset \mathcal{O}_{Z_i}^{\oplus r}$ we infer with (1) that \mathcal{F}' and thus \mathcal{F} is contained in \mathcal{C} .

Next let \mathcal{F} be an $\mathcal{O}_{Z_{\text{red}}}$ -module, and write \mathcal{F}_i for the restriction $\mathcal{F}|_{Z_i}$ modulo its torsion subsheaf. Then we have a short exact sequence

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^r \mathcal{F}_i.$$

The term on the right lies in \mathcal{C} , by the preceding paragraph, and thus also the subsheaf $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$. The term on the left has $\dim(\mathcal{F}') \leq d - 1$, thus lies in \mathcal{C} , which again by (1) gives $\mathcal{F} \in \mathcal{C}$.

Finally, let \mathcal{F} be arbitrary, and $\mathcal{I} = \text{Nil}(\mathcal{O}_Z)$ be the nilradical, say with $\mathcal{I}^m = 0$. In the short exact sequences $0 \rightarrow \mathcal{I}^n \mathcal{F} / \mathcal{I}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathcal{I}^{n+1} \mathcal{F} \rightarrow \mathcal{F} / \mathcal{I}^n \mathcal{F} \rightarrow 0$, the term on the left is annihilated by \mathcal{I} , whence lies in \mathcal{C} . Using induction on $n \geq 0$, one sees that the $\mathcal{F} / \mathcal{I}^n \mathcal{F} \in \mathcal{C}$. The case $n = m$ yields $\mathcal{F} \in \mathcal{C}$. \square

It is easy to characterize bigness in dimension one:

Proposition 1.4. *Suppose the proper scheme Z is equidimensional, of dimension $d = 1$. Then the invertible sheaf \mathcal{L} is big if and only if $(\mathcal{L} \cdot Z') > 0$ for some irreducible component $Z' \subset Z$. In particular, this holds if $\deg(\mathcal{L}) > 0$.*

Proof. We have $h^0(\mathcal{L}^{\otimes n}) \geq \chi(\mathcal{L}^{\otimes n}) = \deg(\mathcal{L})n + \chi(\mathcal{O}_Z)$, where the degree is by definition $\deg(\mathcal{L}) = \chi(\mathcal{L}) - \chi(\mathcal{O}_Z)$. If \mathcal{L} is not big, Lemma 1.3 implies $\deg(\mathcal{L}) \leq 0$. Now suppose that $(\mathcal{L} \cdot Z') > 0$ for some irreducible component $Z' \subset Z$. By the above, the restriction $\mathcal{L}|_{Z'}$ is big. According to [7], Lemma 9.1 combined with Lemma 10.1, the invertible sheaf \mathcal{L} must be big. Conversely, assume \mathcal{L} is big. Then there is some irreducible component $Z' \subset Z$ such that $\mathcal{L}|_{Z'}$ is big. By [7], Lemma 7.1 we have $h^0(\mathcal{L}^{\otimes n}|_{Z'}) \geq \alpha n$ for some real constant $\alpha > 0$, and it follows $(\mathcal{L} \cdot Z') > 0$. \square

We now come to our converse for Proposition 1.2:

Theorem 1.5. *Suppose that X is normal, and let $E \subset X$ be an effective Weil divisor that is contractible to points, with irreducible components $E_1, \dots, E_r \subset E$. Suppose $D = \sum m_i E_i$ is a non-zero effective Cartier divisor supported on E , and let $\mathcal{L} = \mathcal{O}_X(-D)$. Then the restriction $\mathcal{L}|_D$ is big.*

Proof. It suffices to treat the case that E is connected. Let $r : X \rightarrow Y$ be the contraction, and $y = r(E)$ be the resulting closed point. Write $k = \kappa(y)$ for the residue field, choose a separable closure k^{sep} and consider the resulting geometric point $\bar{y} : \text{Spec}(k^{\text{sep}}) \rightarrow Y$ and the ensuing strictly local ring $\mathcal{O}_{Y, \bar{y}}$. We now replace the scheme Y by the spectrum of $\mathcal{O}_{Y, \bar{y}}$, and X by the fiber product $X \times_Y \text{Spec}(\mathcal{O}_{Y, \bar{y}})$. This brings us into the situation that the scheme Y is the spectrum of a strictly local excellent ring R , and $r : X \rightarrow Y$ is a proper morphism with $R = H^0(X, \mathcal{O}_X)$ that is an open embedding on $X \setminus E$ and maps E to the closed point $y \in Y$. Since the formal fibers of the excellent scheme $\text{Spec}(R)$ are geometrically regular, we may base-change to the formal completion and assume that the local noetherian ring R is complete.

Consider the short exact sequences $0 \rightarrow \mathcal{O}_D(-nD) \rightarrow \mathcal{O}_{(n+1)D} \rightarrow \mathcal{O}_{nD} \rightarrow 0$, for each integer $n \geq 0$. The term on the left is $\mathcal{L}^{\otimes n}|_D$, and we get a short exact sequence

$$(2) \quad 0 \longrightarrow H^0(D, \mathcal{L}^{\otimes n}|_D) \longrightarrow H^0(X, \mathcal{O}_{(n+1)D}) \longrightarrow H^0(X, \mathcal{O}_{nD}).$$

The schematic images for the morphisms $nD \rightarrow \text{Spec}(R)$ are of the form $\text{Spec}(R/\mathfrak{a}_n)$, for some inverse system of local Artin rings R/\mathfrak{a}_n . It yields a descending chain $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots$ of \mathfrak{m}_R -primary ideals given by

$$\mathfrak{a}_n = \{g \in R \mid g\mathcal{O}_X \subset \mathcal{O}_X(-nD)\}.$$

From this description we see that these ideals form a *graded family of ideals* in the sense of [7], that is, $\mathfrak{a}_m \cdot \mathfrak{a}_n \subset \mathfrak{a}_{m+n}$ for all $m, n \geq 0$. In other words, the subset $\bigoplus \mathfrak{a}_n T^n \subset R[T]$ is a subring, which one may call the *Rees ring* for the graded family of ideals. Since the complete local ring R is reduced, the limit

$$\alpha = \text{mult}(\mathfrak{a}_\bullet) = \text{vol}(\mathfrak{a}_\bullet) = \lim_{n \rightarrow \infty} \frac{\text{length}(R/\mathfrak{a}_n)}{n^d}$$

exists as a real number by [7], Theorem 4.7. Here $d = \dim(R)$, and the number α is called the *multiplicity* or *volume* of the graded family of ideals. One should think of it as a generalization of the classical *Hilbert–Samuel multiplicities* $e(\mathfrak{b}, R)$, which is defined in terms of the graded family of ideal powers $\mathfrak{b}_n = \mathfrak{b}^n$.

We now compute this number in two ways. For the first computation, we describe the ideals \mathfrak{a}_n in terms of valuations: Let $x_i \in E_i$ be the generic points. Since X is normal, the local rings \mathcal{O}_{X, x_i} are discrete valuation rings. Let $v_i : F^\times \rightarrow \mathbb{Z}$ be the corresponding normalized valuations on the field of fractions $F = \text{Frac}(R) = \kappa(\eta)$, where $\eta \in X$ is the generic point. Then

$$\mathfrak{a}_n = \{g \in R \mid v_1(g) \geq nm_1, \dots, v_r(g) \geq nm_r\}.$$

This reveals that the ideals \mathfrak{a}_n are integrally closed: Indeed, the codimension one points $x_1, \dots, x_r \in X$ admit a common affine neighborhood $U = \text{Spec}(A)$, according to [12], Theorem 1.5. Write $\mathfrak{p}_1, \dots, \mathfrak{p}_r \subset A$ for the corresponding prime ideals of height one. Then the localization $A' = S^{-1}A$ is a semilocal Dedekind domain, for the multiplicative system $S = A \setminus (\mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_r)$. We see

$$\mathfrak{a}_n = R \cap (\mathfrak{p}_1^{nm_1} A' \cap \dots \cap \mathfrak{p}_r^{nm_r} A'),$$

and this is integrally closed according to [29] Proposition 6.8.1 together with Remark 1.1.3 (8). Setting $\mathfrak{b} = \mathfrak{a}_1$, we moreover have $\mathfrak{b}^n \subset \mathfrak{a}_n$, and infer that the ideal \mathfrak{a}_n is the integral closure of the ideal \mathfrak{b}^n .

According to the Briançon–Skoda Theorem in Huneke’s form [15], Theorem 4.13, there is an integer $l \geq 0$ so that $\mathfrak{a}_n \subset \mathfrak{b}^{n-l}$ for all $n \geq l$. Note that this is already a consequence from Izumi’s Theorem as given by Hübl and Swanson [18], Theorem 1.2. It follows that $\text{length}(R/\mathfrak{a}_n) \geq \text{length}(R/\mathfrak{b}^{n-l})$. Passing to the limit, we obtain

$$\alpha = \lim_{n \rightarrow \infty} \frac{\text{length}(R/\mathfrak{a}_n)}{n^d} \geq \lim_{n \rightarrow \infty} \left(\frac{\text{length}(R/\mathfrak{b}^{n-l})}{(n-l)^d} \cdot \frac{(n-l)^d}{n^d} \right).$$

Indeed, both factors in the sequence on the right converge. The second factor converges to one, whereas the first factor tends to the Hilbert–Samuel multiplicity $e(\mathfrak{b}, R)$. But such Hilbert–Samuel multiplicities are always integers $e \geq 1$, according to [6], Chapter VIII, §4, No. 3. The upshot is that $\alpha \geq 1$.

Seeking a contradiction, we now assume that the restriction $\mathcal{L}|_D$ is not big, and compute the number α in another way. Recall that $\dim(X) = \dim(R) = d$, such that $\dim(E) = d - 1$. According to Lemma 1.5, we have $h^0(\mathcal{L}^{\otimes n}|_D) \leq \beta n^{d-2}$ for

some real constant $\beta > 0$. By definition of the ideals \mathfrak{a}_n , there are commutative diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{a}_{n+1} & \longrightarrow & R & \longrightarrow & H^0(X, \mathcal{O}_{(n+1)D}) \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{a}_n & \longrightarrow & R & \longrightarrow & H^0(X, \mathcal{O}_{nD}) \end{array}$$

with exact rows. Combining with the exact sequence (2), we see that the kernels for the surjection $R/\mathfrak{a}_{n+1} \rightarrow R/\mathfrak{a}_n$ are vector subspaces $V_n \subset H^0(D, \mathcal{L}^{\otimes n}|D)$. Inductively, we infer that

$$\text{length}(R/\mathfrak{a}_n) \leq \sum_{i=0}^{n-1} \dim(V_i) \leq \sum_{i=0}^{n-1} h^0(\mathcal{L}^{\otimes i}|E) \leq \beta \sum_{i=0}^{n-1} i^{d-2} \leq \gamma n^{d-1}$$

for some real constant $\gamma \geq 0$. This in turn gives

$$\alpha = \lim_{n \rightarrow \infty} \frac{\text{length}(R/\mathfrak{a}_n)}{n^d} \leq \lim_{n \rightarrow \infty} \frac{\gamma n^{d-1}}{n^d} = 0,$$

contradiction. \square

If X is a regular 2-dimensional scheme, with a curve $E \subset X$ that is contractible to a point, the intersection matrix $N = (E_i \cdot E_j)$ is negative-definite, according to Mumford [23] in the complex case, Artin [1] for algebraic surfaces, and Deligne [8], Exposé X, Corollary 1.8 in the arithmetic situation. So for every non-zero effective Cartier divisor $D = \sum m_i E_i$, we have $D^2 < 0$, and thus the restriction $\mathcal{L}|D$ of the invertible sheaf $\mathcal{L} = \mathcal{O}_X(-D)$ is big, according to Proposition 1.4. From this point of view, the preceding result can be seen as a generalization from dimension $d = 2$ to higher dimensions.

Now back to our general setting $f : X \rightarrow S = \text{Spec}(R)$. Let $E \subset X$ be a closed subset that is contractible to points. If the proper algebraic space Y resulting from the contraction $X \rightarrow Y$ admits an ample invertible sheaf, that is, comes from a projective scheme, we say that a closed subset $E \subset X$ is *projectively contractible to points*. This is a rather delicate condition that cannot be determined numerically in general.

The following is a variant of [28], Theorem 10.2. The new feature is that we have a *ground ring* rather than a ground field, and that the contraction is *projective*. These extensions will be essential for the application in the next section.

Proposition 1.6. *Let $E \subset X$ be an effective Cartier divisor that is contained in some closed fiber $X_\sigma = f^{-1}(\sigma)$. Furthermore, suppose that the structure morphism $f : X \rightarrow S$ is projective. Then there is an effective Cartier divisor $Z \subset E$ with the following property: On the blowing-up $g : X' \rightarrow X$ with center $Z \subset X$, the strict transform $E' \subset X'$ of $E \subset X$ becomes projectively contractible to points. Moreover, we could choose Z disjoint from any given finite subset $\{x_1, \dots, x_m\} \subset E$.*

Proof. Choose a very ample invertible sheaf \mathcal{L} on the projective scheme X so that there is a non-zero global section $s_0 \in H^0(X, \mathcal{L})$ that does not vanish at any of the finitely many points in $\text{Ass}(\mathcal{O}_E) \cup \{x_1, \dots, x_m\}$. Then the map $s_0 : \mathcal{O}_E \rightarrow \mathcal{L}|E$ is injective, and bijective at the points $x_1, \dots, x_m \in E$. The section s_0 defines an effective Cartier divisor $D \subset X$, and the intersection $Z = D \cap E$ remains Cartier in

E . Replacing \mathcal{L} and s_0 by suitable tensor powers, we may assume that $\mathcal{L}(-E)$ is very ample. Such a closed subscheme $Z \subset X$ is the desired center:

The exceptional divisor for the blowing-up $g : X' \rightarrow X$ is the effective Cartier divisor $g^{-1}(Z)$. Since X' is integral and $g : X' \rightarrow X$ is dominant, the preimages $g^{-1}(D)$ and $g^{-1}(E)$ are Cartier as well. Write $D', E' \subset X'$ for the strict transforms of D and E , respectively. Since the center Z is Cartier on E , the universal property of blowing-ups gives an X -morphism $E \rightarrow X'$ whose schematic image is the strict transform E' . In the same way, we have an X -morphism $D \rightarrow X'$ with image D' . Indeed, for each point $z \in Z$, let $f_1, f_2 \in \mathcal{O}_{X,z}$ be generators for the respective stalks of the ideal sheaves $\mathcal{O}_X(-E), \mathcal{O}_X(-D) \subset \mathcal{O}_X$. By assumption, they form a regular sequence. According to [22], Theorem 27 on page 98, they remain a regular sequence in the opposite order, which implies that the subscheme Z is indeed Cartier in both E and D .

According to [26], Lemma 4.4 the strict transforms $D', E' \subset X'$ are Cartier, with

$$g^{-1}(D) = D' + g^{-1}(Z) \quad \text{and} \quad g^{-1}(E) = E' + g^{-1}(Z)$$

as Cartier divisors on X' . In particular, we have $\mathcal{O}_{E'}(-E') = \mathcal{O}_E(Z - E)$ with respect to the identification $E' = E$. The latter sheaf is ample on E , because the sheaf $\mathcal{L}(-E) = \mathcal{O}_X(D - E)$ is ample on X . By Theorem 1.2, the Cartier divisor $E' \subset X'$ is contractible to points. Let $r : X' \rightarrow \tilde{X}$ be the resulting contraction, where \tilde{X} is a proper algebraic space.

It remains to construct an ample invertible sheaf on \tilde{X} . By the very definition of $X' = \text{Proj}(\bigoplus_{i \geq 0} \mathcal{I}^i)$ as a relative homogeneous spectrum, where $\mathcal{I} \subset \mathcal{O}_X$ is the ideal sheaf for the center $Z \subset X$, we have an invertible sheaf $\mathcal{O}_{X'}(1) = \mathcal{O}_{X'}(-g^{-1}(Z))$ that is relatively ample for the blowing-up $g : X' \rightarrow X$. Consider the invertible sheaf

$$\mathcal{L}' = g^*(\mathcal{L})(1) = \mathcal{O}_{X'}(g^{-1}(D) - g^{-1}(Z)) = \mathcal{O}_{X'}(D').$$

We claim that D' is disjoint from E' . This is well-known ([17], Chapter II, Exercise 7.12), but to fix ideas we provide an argument: Since $X' \setminus g^{-1}(Z) = X \setminus Z$, it suffices to check that $D' \cap g^{-1}(Z)$ and $E' \cap g^{-1}(Z)$ are disjoint. The inclusion $Z \subset D$ gives a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D \longrightarrow 0 \\ & & \downarrow & & \downarrow \text{id} & & \downarrow \\ 0 & \longrightarrow & \mathcal{I} & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Z \longrightarrow 0, \end{array}$$

and the Snake Lemma yields $0 \rightarrow \mathcal{O}_X(-D) \rightarrow \mathcal{I} \rightarrow \mathcal{O}_D(-Z) \rightarrow 0$. The term on the right is the ideal sheaf for the Cartier divisor $Z \subset D$, which coincides with $\mathcal{O}_D(-E)$. Restricting to Z results in the short exact sequence

$$0 \longrightarrow \mathcal{O}_Z(-D) \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \mathcal{O}_Z(-E) \longrightarrow 0.$$

Applying this reasoning to the inclusion $Z \subset E$, we infer that the above sequence splits, and obtain a direct sum decomposition $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_Z(-D) \oplus \mathcal{O}_Z(-E)$. Following Grothendieck's Convention, we regard sections $\Sigma = \sigma(Z)$ for the \mathbb{P}^1 -bundle

$$\mathbb{P}(\mathcal{I}/\mathcal{I}^2) = \text{Proj Sym}(\mathcal{I}/\mathcal{I}^2) = g^{-1}(Z) \longrightarrow Z$$

as invertible quotients $\varphi : \mathcal{I}/\mathcal{I}^2 \rightarrow \mathcal{N}$, via the identification $\Sigma = \text{Proj Sym}(\mathcal{N})$. In the direct sum decomposition $\mathcal{I}/\mathcal{I}^2 = \mathcal{O}_Z(-D) \oplus \mathcal{O}_Z(-E)$, the first projection

corresponds to the section $E' \cap g^{-1}(Z)$, whereas the second projection comes from $D' \cap g^{-1}(Z)$. It follows that the two sections are indeed disjoint. Hence \mathcal{L}' is trivial in some open neighborhood of E' , and consequently $\mathcal{L}' = r^*(\tilde{\mathcal{L}})$ for some invertible sheaf $\tilde{\mathcal{L}}$ on the algebraic space \tilde{X} .

Next, we verify that $\tilde{\mathcal{L}}$ is globally generated. Since the center $Z \subset X$ is locally of complete intersection, we may apply [5], Exposé VII, Lemma 3.5 together with the Projection Formula and obtain an identification $f_*(\mathcal{L}') = \mathcal{I}\mathcal{L}$. Since also $r_*(\mathcal{O}_{X'}) = \mathcal{O}_{\tilde{X}}$, we arrive at the identifications

$$(3) \quad H^0(\tilde{X}, \tilde{\mathcal{L}}) = H^0(X', \mathcal{L}') = \{\text{global sections } s \text{ of } \mathcal{L} \text{ with } s_Z = 0\}.$$

The first identification reveals that the base-locus for the invertible sheaf $\tilde{\mathcal{L}}$ must be contained in the image $r(D')$. The exact sequence $0 \rightarrow \mathcal{L}(-E) \rightarrow \mathcal{L} \rightarrow \mathcal{L}_E \rightarrow 0$ on the original scheme X yields a long exact sequence

$$(4) \quad 0 \longrightarrow H^0(X, \mathcal{L}(-E)) \xrightarrow{t_0} H^0(X, \mathcal{L}) \longrightarrow H^0(E, \mathcal{L}_E).$$

where $t_0 \in H^0(X, \mathcal{O}_X(E))$ is the canonical section defining the inclusion $E \subset X$. Now recall that the sheaf $\mathcal{L}(-E)$ is very ample and that the Cartier divisor $D \subset X$ is defined by a global section $s_0 \in H^0(X, \mathcal{L})$. For each point $x \in D \setminus Z = D \setminus E$ we may choose a global section $t \in H^0(X, \mathcal{L}(-E))$ with $t(x) \neq 0$. In turn, the global section $s_1 = tt_0$ of \mathcal{L} also has $s_1(x) = t(x)t_0(x) \neq 0$, thus it defines an effective Cartier divisor $D_1 \subset X$ with $x \notin D_1$ and $Z \subset D_1$. Under the identification (3), the resulting section s_1 of $\tilde{\mathcal{L}}$ does not vanish at $r(x') \in \tilde{X}$, where $g(x') = x$.

Now suppose we have a point $z \in Z$. The corresponding point $z' \in g^{-1}(Z) \cap D'$ on X' is an invertible quotient of $\mathcal{I}/\mathcal{I}^2 \otimes \kappa(z) = \mathcal{I} \otimes \kappa(z)$, whence defines a tangent vector at $z \in X$ not contained in Z , that is, a closed subscheme $T \subset \text{Spec}(\mathcal{O}_{X,z})$ of length two, satisfying $s_0|_T = 0$ and $t_0|_T \neq 0$. Since $\mathcal{L}(-E)$ is very ample, we may choose a global section t with $t(z) \neq 0$. As above, the global section $s_1 = tt_0$ of \mathcal{L} vanishes on Z but not on T , thus defines a Cartier divisor $D'_1 \subset X'$ that does not contain the point z' . Under the identification (3), the global section s_1 of $\tilde{\mathcal{L}}$ does not vanish at $r(z') \in \tilde{X}$. Summing up, we have shown that the sheaf $\tilde{\mathcal{L}}$ is globally generated.

The last step is to check that the globally generated invertible sheaf $\tilde{\mathcal{L}}$ is ample. Let $\tilde{C} \subset \tilde{X}$ be an integral curve contained in the fiber \tilde{X}_σ for the structure morphism $\tilde{X} \rightarrow S$. We merely have to show that $(\tilde{\mathcal{L}} \cdot \tilde{C}) > 0$. Let $C' \subset X'$ be the strict transform, such that $C' \not\subset E'$ and $(\tilde{\mathcal{L}} \cdot \tilde{C}) = (\mathcal{L}' \cdot C')$. If $g(C') \subset X$ is a point, that is, C' is a fiber of the \mathbb{P}^1 -bundle $g^{-1}(Z) = \mathbb{P}(\mathcal{I}/\mathcal{I}^2)$, we have $(\mathcal{L}' \cdot C') = (\mathcal{O}_{X'}(1) \cdot C') \geq 1$.

Now suppose that the image $C = g(C')$ is a curve rather than a point. If C is not contained in the center $Z \subset X$, fix a closed point $x \in C \setminus Z = C \setminus E$ and a global section t for the very ample sheaf $\mathcal{L}(-E)$ that vanishes at $x \in C$ but not at the generic point $\eta \in C$. The resulting global section $s_1 = tt_0$ of \mathcal{L} , via the exact sequence (4), vanishes along $\{x\} \cup Z$ but not at $\eta \in C$. From this we infer $(\tilde{\mathcal{L}} \cdot \tilde{C}) > 0$.

It remains to treat the case that $C \subset Z$. Let $\nu : B \rightarrow C'$ be the normalization map, and form the fiber product

$$P = B \times_X X' = B \times_C \mathbb{P}(\mathcal{I}/\mathcal{I}^2 \otimes \mathcal{O}_C) = \mathbb{P}(\mathcal{E}),$$

where the locally free sheaf $\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2$ on B is the sum of the two invertible sheafs $\mathcal{L}_1 = q^* \mathcal{O}_C(-D)$ and $\mathcal{L}_2 = q^* \mathcal{O}_C(-E)$. Here $q : B \rightarrow C$ is the composition of the normalization map $\nu : B \rightarrow C'$ and the induced map $g : C' \rightarrow C$. The two projections $\text{pr}_i : \mathcal{E} \rightarrow \mathcal{L}_i$ correspond to disjoint sections $\Sigma_i \subset \mathbb{P}(\mathcal{E})$, via $\Sigma_i = \text{Proj Sym}(\mathcal{L}_i)$. By functoriality of the construction, these Σ_1 and Σ_2 are the preimages of the sections $E' \cap g^{-1}(Z)$ and $D' \cap g^{-1}(Z)$ for $g^{-1}(Z) = \mathbb{P}(\mathcal{I}/\mathcal{I}^2) \rightarrow Z$ with respect to the canonical morphism $B \times_X X' \rightarrow X'$. In particular, $\mathcal{O}_P(\Sigma_2)$ is the pullback of \mathcal{L}' .

The scheme $P = \mathbb{P}(\mathcal{E})$ is a ruled surface over the proper regular curve B , so its Picard group modulo numerical equivalence takes the form $N(P) = \mathbb{Z}^2$. The closed cone of curves

$$\overline{\text{NE}}(P) \subset N(P) \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^2,$$

which for surfaces coincides with the pseudoeffective cone, must be generated by two extremal rays. Each fiber $F \subset P$ for the ruling has selfintersection number $F^2 = 0$. According to Proposition 1.1, the section $\Sigma_1 \subset P$ is contractible, so its selfintersection is $(\Sigma_1)^2 < 0$. In light of [19], Lemma 4.12, it follows that the numerical classes of F and Σ_1 are the two extremal rays for $\overline{\text{NE}}(P)$. This in turn implies $(\Sigma_2)^2 > 0$. In particular, $\mathcal{O}_P(\Sigma_2)$ is ample on Σ_2 . According to Fujita's result ([11], see also [9]), the invertible sheaf $\mathcal{O}_P(\Sigma_2)$ must be semiample. It follows that for some $n \geq 1$ the semiample sheaf $\mathcal{O}_P(n\Sigma_2)$ is the preimage of some ample sheaf on \tilde{P} , where $P \rightarrow \tilde{P}$ is the contraction of Σ_1 . Consequently $(\Sigma_2 \cdot \Sigma) > 0$ for every integral curve $\Sigma \neq \Sigma_1$. In particular, this holds for the section $\Sigma \subset P$ arising from the diagonal map $B \rightarrow B \times_X X' = P$. By construction, the projection $P \rightarrow X'$ induces a surjection $B = \Sigma \rightarrow C'$. Since $\mathcal{O}_P(\Sigma_2)$ is the preimage of $\mathcal{L}' = \mathcal{O}_{X'}(D')$, we must have $(\mathcal{L}' \cdot C') > 0$. \square

2. TOTALLY SEPARABLY CLOSED SCHEMES

Recall that an integral scheme X with generic point $\eta \in X$ is *totally separably closed* if it is normal and the function field $F = \mathcal{O}_{X,\eta} = \kappa(\eta)$ is separably closed. A space or a scheme is called *local* if it contains exactly one closed point. The main result of this paper is:

Theorem 2.1. *Let X be an integral separated scheme that is totally separably closed, and $u, v \in X$ be two points. Then the intersection $\text{Spec}(\mathcal{O}_{X,u}) \cap \text{Spec}(\mathcal{O}_{X,v})$ inside X is local.*

Proof. The intersection can be regarded as the underlying set of the schematic fiber product

$$P = \text{Spec}(\mathcal{O}_{X,u}) \times_X \text{Spec}(\mathcal{O}_{X,v}).$$

Its image contains the generic point $\eta \in X$, in particular P is non-empty. Furthermore, the scheme P is affine, because X is separated. Seeking a contradiction, we assume that the intersection is not local. Hence there are two closed points $\alpha \neq \beta$ inside P . Let $A, B \subset X$ be their closures in X . Both contain u and v . In fact, the

points $u, v \in A \cap B$ are generic points in the intersection. If the two points $u, v \in X$ admit a common affine open neighborhood, we immediately get a contradiction from [3], Corollary 1.8. The idea of this proof is to construct, starting from X , another integral separated scheme \tilde{X} that is totally separably closed and additionally enjoys the AF property, containing two points $\tilde{u} \neq \tilde{v}$ closely related to the original points $u \neq v$. Now the intersection $\tilde{P} = \text{Spec}(\mathcal{O}_{\tilde{X}, \tilde{u}}) \times_{\tilde{X}} \text{Spec}(\mathcal{O}_{\tilde{X}, \tilde{v}})$ is indeed local, and this will finally produce the desired contradiction.

Step 1: *We reduce to the case that X is the total separable closure of some proper \mathbb{Z} -scheme X_0 .* First of all, we may assume that our scheme X is quasicompact, simply by choosing affine open neighborhoods $U, V \subset X$ of $u, v \in X$ and replacing X by their union. Next, we write $X = \varprojlim X_\lambda$ as a filtered inverse limit of schemes X_λ , $\lambda \in L$ that are separated and of finite type over the ring $R = \mathbb{Z}$, with affine transition maps $X_\mu \rightarrow X_\lambda$, $\lambda \leq \mu$. This is possible according to [30], Appendix C, Proposition 7. Replacing X_λ by the schematic images of the projection $X \rightarrow X_\lambda$, we may assume that the X_λ are integral, and that the transition maps $X_\mu \rightarrow X_\lambda$ and the projections $X \rightarrow X_\lambda$ are dominant. Let $\eta_\lambda \in X_\lambda$ be the generic points, such that $\eta = (\eta_\lambda)_{\lambda \in L}$. The function fields $F_\lambda = \mathcal{O}_{X_\lambda, \eta_\lambda} = \kappa(\eta_\lambda)$ form a filtered direct system of subfields inside $F = \mathcal{O}_{X, \eta} = \kappa(\eta)$, with $F = \bigcup_{\lambda \in L} F_\lambda$. Let F_λ^{sep} be the relative separable algebraic closure of $F_\lambda \subset F$, and $\text{TSC}(X_\lambda)$ be the resulting integral closure of X_λ with respect to the field extension $F_\lambda \subset F_\lambda^{\text{sep}}$. Then the filtered inverse system X_λ induces a filtered inverse system $Y_\lambda = \text{TSC}(X_\lambda)$. The morphism $X \rightarrow X_\lambda$ induces compatible morphisms $X \rightarrow Y_\lambda$, giving an identification $X = \varprojlim Y_\lambda$.

Now suppose that the theorem is valid for all $Y_\lambda = \text{TSC}(X_\lambda)$. Let $u_\lambda, v_\lambda \in Y_\lambda$ be the images of $u, v \in X$. Then the schemes

$$P_\lambda = \text{Spec}(\mathcal{O}_{Y_\lambda, u_\lambda}) \times_{Y_\lambda} \text{Spec}(\mathcal{O}_{Y_\lambda, v_\lambda})$$

are local henselian. According to [3], Lemma 2.6 the inverse limit $P = \varprojlim P_\lambda$ is local henselian, contradiction.

This reduces us to the case that X is the total separable closure of some integral scheme X_0 that is separated and of finite type over the ring $R = \mathbb{Z}$. In light of Nagata's Compactification Theorem in the relative version obtained by Lütkebohmert [21], we may additionally assume that the structure morphism $X_0 \rightarrow \text{Spec}(\mathbb{Z})$ is proper. This concludes Step 1.

Step 2: *We may assume that the point $u \in X$ lies in a closed fiber $X \otimes \mathbb{F}_p$.* If both points $u, v \in X$ lie in the generic fiber $X \otimes \mathbb{Q}$, we could replace X_0 and X by their generic fibers. Now X_0 is proper over the field $k = \mathbb{Q}$, and we immediately get a contradiction to [28], Theorem 12.1. So we may assume without restriction that $u \in U$ lies in a closed fiber $X \otimes \mathbb{F}_p$, for some prime $p > 0$.

Step 3: *Here we write X as a filtered inverse system X_λ , $\lambda \in L$ of proper \mathbb{Z} -schemes so that the geometry of $A, B \subset X$ is captured by their images $A_\lambda, B_\lambda \subset X_\lambda$.* Let $F_0 \subset F$ be the inclusion of function fields coming from the canonical morphism $X \rightarrow X_0$. Changing the notation from Step 1, we now write $F_\lambda \subset F$, $\lambda \in L$ for the filtered direct system of subfields with $[F_\lambda : F_0] < \infty$, and let $X_\lambda \rightarrow X$ be the normalization of X_0 with respect to the field extension $F_0 \subset F_\lambda$. This gives a filtered inverse system X_λ of finite X_0 -schemes, with $X = \varprojlim X_\lambda$, where the

transition maps $X_\mu \rightarrow X_\lambda$, $\lambda \leq \mu$ are finite. Note that all structure morphisms $X_\lambda \rightarrow \text{Spec}(\mathbb{Z})$ are proper, that $X \rightarrow \text{Spec}(\mathbb{Z})$ is separated and universally closed, and that the projections $X \rightarrow X_\lambda$ are universally closed. Write

$$(5) \quad u_\lambda, v_\lambda, \alpha_\lambda, \beta_\lambda, \eta_\lambda \in X_\lambda$$

for the respective images of the points $u, v, \alpha, \beta, \eta \in X$. Let $A_\lambda, B_\lambda \subset X_\lambda$ be the closures of $\alpha_\lambda, \beta_\lambda \in X_\lambda$, which are also the images of $A, B \subset X$. In turn, we have $u_\lambda, v_\lambda \in A_\lambda \cap B_\lambda$. Replacing L by some cofinal subset, we may assume that the points in (5) are pairwise different.

Since inverse limits commute with inverse limits, we have $A \cap B = \varprojlim (A_\lambda \cap B_\lambda)$. For the local rings this means $\mathcal{O}_{A \cap B, u} = \varprojlim \mathcal{O}_{A_\lambda \cap B_\lambda, u_\lambda}$. Setting $C = \text{Spec}(\mathcal{O}_{A \cap B, u})$ and $C_\lambda = \text{Spec}(\mathcal{O}_{A_\lambda \cap B_\lambda, u_\lambda})$, we get $C \setminus \{u\} = \varprojlim (C_\lambda \setminus \{u_\lambda\})$. But the left-hand side is empty, because $u \in A \cap B$ is a generic point. Thus $1 = 0$ already holds as global sections on some C_λ . By symmetry, the same applies for the point $v \in A \cap B$. Replacing L by some cofinal subset, we thus may assume that $u_\lambda, v_\lambda \in A_\lambda \cap B_\lambda$ are generic points.

Step 4: *Reduction to the case that the connected components of $A_0 \cap B_0$ are irreducible.* The proper \mathbb{Z} -scheme $A_0 \cap B_0$ has only finitely many irreducible components. Fix an irreducible component $C_0 \subset A_0 \cap B_0$, let $C'_0 \subset A_0 \cap B_0$ be the union of the other irreducible components, and consider the normalized blowing-up $Y_0 \rightarrow X_0$ with center $Z_0 = C_0 \cap C'_0$. This morphism is an isomorphism over some open neighborhood of the set of generic points in $A_0 \cap B_0$, and the strict transforms of C_0 and C'_0 become disjoint on Y_0 . By induction on the number of irreducible components in $A_0 \cap B_0$, we find a normalized blowing-up $Y'_0 \rightarrow X_0$ that is an isomorphism over the set of generic points of $A_0 \cap B_0$ so that the strict transform A'_0, B'_0 have the property that the connected components of $A'_0 \cap B'_0$ are irreducible. Replacing X_0 by Y'_0 and X by $\text{TSC}(Y'_0)$ we may assume that the connected components of $A_0 \cap B_0$ are irreducible. In particular,

$$(6) \quad \overline{\{u_0\}} \cap \overline{\{v_0\}} = \emptyset.$$

Note that his property will later produce the desired contradiction.

Step 5: *Construction of two auxiliary filtered inverse systems X'_λ and X''_λ consisting of projective schemes.* Choose an affine open neighborhood $V_0 \subset X_0$ of v_0 not containing u_0 . By Chow's Lemma, there is a blowing-up $g_0 : X'_0 \rightarrow X_0$ with center Z_0 contained in $X_0 \setminus V_0$, such that X'_0 is projective. Set $Z'_0 = g_0^{-1}(\overline{\{u_0\}})$. Replacing X'_0 by some further normalized blowing-up of X'_0 with center Z'_0 , we may also assume that the closed set $g_0^{-1}(\overline{\{u_0\}})$ is the support of an effective Cartier divisor $E'_0 \subset X'_0$ contained in the closed fiber $X'_0 \otimes \mathbb{F}_p$.

Let $X'_\lambda \rightarrow X'_0$ be its normalization of X'_0 with respect to the finite field extension $F_0 \subset F_\lambda$. This gives a filtered direct system with finite transition maps, and we obtain a totally separably closed scheme $X' = \varprojlim X'_\lambda$. The projection $X' \rightarrow X'_0$ is affine and the scheme X'_0 satisfies the the AF' property, so the same holds for X' . The ensuing projective birational morphisms $X'_\lambda \rightarrow X_\lambda$ induces a birational morphism $X' \rightarrow X$ between integral schemes. The $X'_\lambda \rightarrow X_\lambda$ are isomorphisms over the open subsets $V_\lambda = V_0 \times_{X_0} X_\lambda$, which contains $v_\lambda, \alpha_\lambda, \beta_\lambda$, hence $X' \rightarrow X$ is an

isomorphism over $V = V_0 \times_{X_0} X$, which contains v, α, β . So we may regard the latter also as points $v', \alpha', \beta' \in X'$.

Let $A', B' \subset X'$ be the strict transforms of $A, B \subset X$, that is, the closures of $\alpha', \beta' \in X'$. According to [3], Corollary 1.8 together with [28], Theorem 7.6 the intersection $A' \cap B'$ is irreducible. Since $\text{Spec}(\mathcal{O}_{A' \cap B', v'}) = \text{Spec}(\mathcal{O}_{A \cap B, v})$, the point $v' \in A' \cap B'$ is generic, whence this must be the unique generic point. Consider the canonical morphism $\varphi : X' \rightarrow X_0$, which is a closed map. Suppose there would be a point u' in the intersection

$$(7) \quad A' \cap B' \cap \varphi^{-1}(u_0) = \overline{\{v'\}} \cap \varphi^{-1}(u_0).$$

Then $u_0 \in \varphi(\overline{\{v'\}}) = \overline{\{\varphi(v')\}} = \overline{\{v_0\}}$, contradicting (6). Thus the intersection (7) is empty. Passing to a cofinal subset of L , we may assume that already the fiber of $A'_0 \cap B'_0$ over $u_0 \in A_0 \cap B_0$ is empty. On the other hand, the morphisms $A'_0 \rightarrow A_0$ and $B'_0 \rightarrow B_0$ are proper and dominant, so there are points $r'_0 \in A'_0$ and $s'_0 \in B'_0$ lying in $\varphi^{-1}(u_0)$.

Now recall that the fiber $g_0^{-1}(\overline{\{u_0\}})$ is a Cartier divisor $E'_0 \subset X'_0$ lying inside the closed fiber $X' \otimes \mathbb{F}_p$. Write $n \geq 2$ for the common dimension of the schemes $X, X_\lambda, X', X'_\lambda$. Since $A_0, B_0 \subsetneq X_0$, the irreducible schemes A'_0, B'_0 have dimension $\leq n - 1$, and the intersections $A'_0 \cap E'_0$ and $B'_0 \cap E'_0$ have dimension $\leq n - 2$. On the other hand, each irreducible component of E'_0 has dimension $n - 1$, by Krull's Principal Theorem. According to Proposition 1.6, there is an effective Cartier divisor $Z'_0 \subset E'_0$ not containing $\{r'_0, s'_0\}$ so that on the blowing-up $X''_0 \rightarrow X'_0$ with center $Z'_0 \subset X'_0$ the strict transform E''_0 of E'_0 becomes projectively contractible to points. As above, we get a filtered inverse system of projective schemes X''_λ with finite transition maps, and the resulting $X'' = \varprojlim(X''_\lambda) = \text{TSC}(X''_0)$ comes with a birational morphism $X'' \rightarrow X'$ of integral schemes. Consider the composite morphism $h : X'' \rightarrow X$ and the corresponding $h_\lambda : X''_\lambda \rightarrow X_\lambda$. By construction, we have $h_0(E''_0) = \overline{\{u_0\}}$. This finishes Step 5.

Step 6: *Construction of the contractions $r_\lambda : X''_\lambda \rightarrow \tilde{X}_\lambda$ resulting in a contradiction.* Let $E''_\lambda \subset X''_\lambda$ be the preimages of $E''_0 \subset X''_0$. These subschemes are effective Cartier divisors, because X''_λ is integral and $X''_\lambda \rightarrow X''_0$ is dominant. According to Proposition 1.1, the E''_λ are projectively contractible to points. Moreover, the resulting contractions $r_\lambda : X''_\lambda \rightarrow \tilde{X}_\lambda$ coincide with the Stein factorization of $X''_\lambda \rightarrow \tilde{X}_0$, and yield yet another filtered inverse system \tilde{X}_λ of projective schemes with finite transition maps. We have a commutative diagram

$$\begin{array}{ccccc} \tilde{X}_\lambda & \xleftarrow{r_\lambda} & X''_\lambda & \xrightarrow{h_\lambda} & X_\lambda \\ \downarrow & & \downarrow & & \downarrow t_{0\lambda} \\ \tilde{X}_0 & \xleftarrow{r_0} & X''_0 & \xrightarrow{h_0} & X_0. \end{array}$$

Clearly, u_λ is contained in the finite set $t_{0\lambda}^{-1}(u_0)$, where $t_{0\lambda} : X_\lambda \rightarrow X_0$ denotes the transition map. Using the above commutative diagram, we infer that the $h_\lambda^{-1}(u_\lambda)$ is contained in the effective Cartier divisor E''_λ whose connected components are mapped to closed points in \tilde{X}_λ . But the fiber $h_\lambda^{-1}(u_\lambda)$ is connected, by Zariski's

Main Theorem. It follows that the set $h_\lambda^{-1}(u_\lambda)$ and in particular the elements $r_\lambda, s_\lambda \in h_\lambda^{-1}(u_\lambda)$ are mapped to the same point $\tilde{u}_\lambda \in \tilde{X}_\lambda$.

Now consider the resulting filtered inverse system $\tilde{X}_\lambda, \lambda \in L$ of projective schemes with finite transition maps. The inverse limit $\tilde{X} = \varprojlim \tilde{X}_\lambda$ is another totally separably closed scheme. Since the scheme \tilde{X}_0 is projective and the morphisms $\tilde{X} \rightarrow \tilde{X}_0$ is integral, \tilde{X} enjoys the AF property. The points $\tilde{u}_\lambda \in \tilde{X}_\lambda$ are compatible and yield a point $\tilde{u} \in \tilde{X}$. By construction, $X_\lambda \rightarrow \tilde{X}_\lambda$ are isomorphisms on an open neighborhood of $v_\lambda, \alpha_\lambda, \beta_\lambda$. So we may regard the latter as points on \tilde{X} , denoted by $\tilde{v}_\lambda, \tilde{\alpha}_\lambda, \tilde{\beta}_\lambda \in \tilde{X}_\lambda$. In turn, we get points $\tilde{v}, \tilde{\alpha}, \tilde{\beta}$ on the inverse limit \tilde{X} .

Since the points $\tilde{u}, \tilde{v} \in \tilde{X}$ admit a common affine open neighborhood, [3], Corollary 8.1 applies and we infer with [28], Theorem 7.6 that the intersection $\tilde{A} \cap \tilde{B}$ is irreducible. Arguing as above, we see that $\tilde{v} \in \tilde{A} \cap \tilde{B}$ must be the generic point, in particular $\tilde{u} \in \overline{\{\tilde{v}\}}$. Since the projection $\tilde{X} \rightarrow \tilde{X}_0$ is closed, we also have $\tilde{u}_0 \in \overline{\{\tilde{v}_0\}}$. The contraction $r_0 : X_0'' \rightarrow \tilde{X}_0$ is closed as well, hence

$$r_0(E_0'' \cap \overline{\{v_0''\}}) \supset r_0(r_0^{-1}(\tilde{u}_0) \cap \overline{\{v_0''\}}) = \{\tilde{u}_0\} \cap r_0(\overline{\{v_0''\}}) = \{\tilde{u}_0\} \cap \overline{\{\tilde{v}_0\}}.$$

We see that $E_0'' \cap \overline{\{v_0''\}} \neq \emptyset$. Finally, the blowing-up $h_0 : X_0'' \rightarrow X_0$ is a closed map with $h_0(E_0'') = \{u_0\}$ and $h_0(v_0'') = v_0$, hence the sets

$$h_0(E_0'' \cap \overline{\{v_0''\}}) \subset h_0(E_0'') \cap h_0(\overline{\{v_0''\}}) = \overline{\{u_0\}} \cap \overline{\{v_0\}}$$

are non-empty. But this contradicts (6). \square

Note that in Theorem 2.1 some assumption about separatedness is inevitable : For example, let k be an algebraically closed field, R_0 be the henselization of $k[x, y]$ at the maximal ideal $\mathfrak{m} = (x, y)$, and R be its total separable closure. Then R is a local integral domain of dimension two that is TSC. Let $U \subset \text{Spec}(R)$ be the complement of the closed point. Then U has dimension one and contains infinitely many closed points. Let R_1 and R_2 be two copies of R , and $X = \text{Spec}(R_1) \cup \text{Spec}(R_2)$ be the non-separated integral TSC scheme obtained by gluing along $U \subset \text{Spec}(R_i)$. For the two closed points $u, v \in X$ we have $\text{Spec}(\mathcal{O}_{X,u}) \cap \text{Spec}(\mathcal{O}_{X,v}) = U$, which is not local.

In this example, the diagonal $\Delta : X \rightarrow X \times X$ is not affine. It is conceivable that Theorem 2.1 holds true under the weaker assumption that the diagonal is merely affine rather than a closed embedding.

3. APPLICATION TO NISNEVICH COHOMOLOGY

Let X be a scheme, and write (Et/X) for the category of étale X -schemes. The *Nisnevich topology* on this category is the Grothendieck topology defined by the pretopology whose covering families $(U_i \rightarrow U)_{i \in I}$ are those where for each $x \in U$ there is some index $i \in I$ and some $x_i \in U_i$ mapping to x , such that the residue field extension $\kappa(x) \subset \kappa(x_i)$ is trivial [24]. We write X_{Nis} for the ensuing topos of presheaves on (Et/X) that satisfy the sheaf axiom for the Nisnevich topology. We refer to such sheaves as *Nisnevich sheaves*. Each point $x \in X$ yields a point $(P_*, P^*, \psi) : (\text{Set}) \rightarrow X_{\text{Nis}}$ in the sense of topos-theory, and the corresponding local ring of the structure sheaf with respect to the Nisnevich topology is the henselization

of the local ring $\mathcal{O}_{X,x}$ with respect to the Zariski topology. Every abelian Nisnevich sheaf F comes with a spectral sequence

$$E_2^{pq} = \check{H}^p(X_{\text{Nis}}, \underline{H}^q(F)) \implies H^{p+q}(X_{\text{Nis}}, F)$$

from Čech cohomology to sheaf cohomology (see for example [28], Appendix B).

Theorem 3.1. *If X is quasicompact and separated, then $\check{H}^p(X_{\text{Nis}}, \underline{H}^q(F)) = 0$ for all $p \geq 0$, $q \geq 1$ and all abelian Nisnevich sheaves F . In particular, the canonical maps*

$$\check{H}^p(X_{\text{Nis}}, F) \longrightarrow H^p(X_{\text{Nis}}, F)$$

from Čech cohomology to sheaf cohomology are bijective for all $p \geq 0$.

Proof. The result was already established in [28], Theorem 13.1 for schemes where the structure morphism $X \rightarrow \text{Spec}(\mathbb{Z})$ factors over the spectrum of a prime field. In other words, X is a k -scheme for some ground field k . This assumption entered only via [28], Theorem 12.1. But the latter holds true without the superfluous assumption of a ground field, by Theorem 2.1. \square

REFERENCES

- [1] M. Artin: Some numerical criteria for contractability of curves on algebraic surfaces. Am. J. Math. 84 (1962), 485–496.
- [2] M. Artin: Algebraization of formal moduli II: Existence of modifications. Ann. Math. 91 (1970), 88–135.
- [3] M. Artin: On the joins of Hensel rings. Advances in Math. 7 (1971), 282–296.
- [4] O. Benoist: Quasi-projectivity of normal varieties. Int. Math. Res. Not. IMRN 17 (2013), 3878–3885.
- [5] P. Berthelot, A. Grothendieck, L. Illusie (eds.): Théorie des intersections et théorème de Riemann–Roch (SGA 6). Springer, Berlin, 1971.
- [6] N. Bourbaki: Algèbre commutative. Chapitre 8–9. Masson, Paris, 1983.
- [7] S. Cutkosky: Asymptotic multiplicities of graded families of ideals and linear series. Adv. Math. 264 (2014), 55–113.
- [8] P. Deligne, N. Katz: Groupe de monodromie en géométrie algébrique (SGA 7 II). Springer, Berlin, 1973.
- [9] L. Ein: Linear systems with removable base loci. Comm. Algebra 28 (2000), 5931–5934.
- [10] A. Engler, A. Prestel: Valued fields. Springer, Berlin, 2005.
- [11] T. Fujita: Semipositive line bundles. J. Fac. Sci. Univ. Tokyo 30 (1983), 353–378.
- [12] P. Gross: The resolution property of algebraic surfaces. Compos. Math. 148 (2012), 209–226.
- [13] A. Grothendieck: Éléments de géométrie algébrique II: Étude globale élémentaire de quelques classes de morphismes. Publ. Math., Inst. Hautes Étud. Sci. 8 (1961).
- [14] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faisceaux cohérents. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961).
- [15] C. Huneke: Uniform bounds in Noetherian rings. Invent. Math. 107 (1992), 203–223.
- [16] C. Huneke: Absolute integral closure. In: A. Corso and C. Polini (eds.), Commutative algebra and its connections to geometry, pp. 119–135. Amer. Math. Soc., Providence, RI, 2011.
- [17] R. Hartshorne: Algebraic geometry. Springer, Berlin, 1977.
- [18] R. Hübl, I. Swanson: Discrete valuations centered on local domains. J. Pure Appl. Algebra 161 (2001), 145–166.
- [19] J. Kollár: Rational curves on algebraic varieties. Springer, Berlin, 1995.
- [20] R. Lazarsfeld: Positivity in algebraic geometry. I. Springer, Berlin, 2004.
- [21] W. Lütkebohmert: On compactification of schemes. Manuscr. Math. 80 (1993), 95–111.

- [22] H. Matsumura: Commutative algebra. Second edition. Benjamin/Cummings, Reading, Mass., 1980.
- [23] D. Mumford: The topology of normal singularities of an algebraic surface and a criterion for simplicity. Publ. Math., Inst. Hautes Étud. Sci. 9 (1961), 5–22.
- [24] Y. Nisnevich: The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory. In: J. Jardine, V. Snaith (eds.), Algebraic K-theory: connections with geometry and topology, pp. 241–342. Kluwer, Dordrecht, 1989.
- [25] M. Olsson: Algebraic spaces and stacks. American Mathematical Society, Providence, RI, 2016.
- [26] M. Perling, S. Schröer: Vector bundles on proper toric 3-folds and certain other schemes. Trans. Amer. Math. Soc. 369 (2017), 4787–4815.
- [27] F. Schmidt: Körper, über denen jede Gleichung durch Radikale auflösbar ist. Sitzungsber. Heidelberger Akad. Wiss. 2 (1933), 37–47.
- [28] S. Schröer: Geometry on totally separably closed schemes. Algebra Number Theory 11 (2017), 537–582.
- [29] I. Swanson, C. Hueke: Integral Closure of Ideals, Rings, and Modules. Cambridge University Press, Cambridge, 2006.
- [30] R. Thomason, T. Trobaugh: Higher algebraic K -theory of schemes and of derived categories. In: P. Cartier et al. (eds.), The Grothendieck Festschrift III, 247–435. Birkhäuser, Boston, MA, 1990.

MATHEMATISCHES INSTITUT, HEINRICH-HEINE-UNIVERSITÄT, 40204 DÜSSELDORF, GER-
MANY

E-mail address: `schroeer@math.uni-duesseldorf.de`