MORET-BAILLY FAMILIES AND NON-LIFTABLE SCHEMES

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Abstract. Generalizing the Moret-Bailly pencil of supersingular abelian surfaces to higher dimensions, we construct for each field of characteristic $p > 0$ a smooth projective variety with trivial dualizing sheaf that does not formally lift to characteristic zero. Our approach heavily relies on local unipotent group schemes, the Beauville–Bogomolov Decomposition for Kähler manifolds with $c_1 = 0$, and equivariant deformation theory in mixed characteristics.

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Introduction

Every compact Kähler manifold V with Chern class $c_1 = 0$ has unobstructed deformations, although the obstruction group $H^2(V, \Theta_V)$ is usually non-zero. This foundational fact relies on the T^1*-Lifting Theorem* (confer [8], [51], [50], [32], [40]). It holds, in particular, for complex tori, hyperkähler manifolds, and Calabi–Yau manifolds. In fact, the *Beauville–Bogomolov Decomposition Theorem* asserts that a compact Kähler manifold V with $c_1 = 0$ admits a finite étale covering $V' \to V$ that splits into a product $V' = V_1 \times \ldots \times V_r$ where the factors belong to these three classes ([7], [5]).

Much less is known for smooth proper scheme Y in characteristic $p > 0$ that have $c_1 = 0$, in the sense that the dualizing sheaf ω_Y is numerically trivial. Under strong additional assumptions, analogues of the T^1-Lifting Theorem ([18], [43]) and the Decomposition Theorem [39] hold true. In light of the liftability of abelian varieties ([31], [35]) and K3-surfaces [12], it is natural to wonder whether any such Y admit a

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lifting to characteristic zero. This, however, turns out to be false already for Calabi–Yau threefolds. The first example was given by Hirokado [27] in characteristic $p = 3$. The second author [44] found further examples in characteristic $p = 2, 3$ using quotients of the *Moret-Bailly pencil of supersingular abelian surfaces* [34]. Further examples in dimension three at certain bounded sets of primes where constructed by Schoen [42], Hirokado, Ito and Saito ([28] and [29]), Cynk and van Straten [10], and Cynk and Schütt [11]. Finally, Achinger and Zdanowicz ([2], [53]) produced for each prime $p \geq 5$ a non-liftable Calabi–Yau manifold of dimension $2p$, based on the failure of Kodaira Vanishing as observed by Totaro [52].

The goal of this paper is to generalize the Moret-Bailly pencil to higher dimensions: Let $A = E_1 \times \ldots \times E_g$ be a product of supersingular elliptic curves. Roughly speaking, the embeddings of the local unipotent group scheme $\alpha_p = \mathbb{G}_a[F]$ into the abelian variety A are parameterized by the projectivization $\mathbb{P}^n = \mathbb{P}(\mathfrak{a})$ of the Lie algebra $\mathfrak{a} = \mathrm{Lie}(A)$, where $n + 1 = g$. In fact, any inclusion $\mathscr{O}_{\mathbb{P}^n}(-d) \subset \mathscr{O}_{\mathbb{P}^n}^{\oplus(n+1)}$ that is locally a direct summand corresponds to a family $H \subset A \times \mathbb{P}^n$ of such finite local group schemes Setting $X = A \times \mathbb{P}^n$, we then form the family of quotients $Y = X/H$, which comes with a radical surjection $\epsilon : X \to Y$ and an induced fibration $\varphi : Y \to \mathbb{P}^n$.

We call the fibrations $\varphi : Y \to \mathbb{P}^n$ and also the smooth proper schemes Y *Moret-Bailly families*, since they are higher-dimensional analogs of the famous construction of a non-isotrivial family of abelian surfaces over the projective line [34]; such families were already mentioned by Grothendieck in [25], Remark 4.6.

The dimension is $\dim(Y) = 2n + 1 = 2p - 3$. It is easy to compute the Betti numbers $b_i(Y)$, but the cohomological invariants $h^i(\mathscr{O}_Y)$ remain mysterious. Using a result of Achinger [1] on the splitting type of the Frobenius push-forward on toric varieties, one may express $h^i(\mathscr{O}_Y)$ via lattice point counts. It turns out that the canonical projection $Y \to A^{(p)} = A/A[F]$ is the Albanese map, and the dualizing sheaf ω_Y is the pullback of $\mathscr{O}_{\mathbb{P}^n}(m)$, for the integer $m = d(p-1) - (n-1)$. In turn, we have $c_1 = 0$ if and only if $d(p - 1) = g$, and in this case $\omega_Y = \mathscr{O}_Y$. The main result of this paper is:

Theorem. (See Thm. 8.2.) *Suppose $d = 1$ and $g = p - 1$ and $p \geq 3$. Then the Moret-Bailly family Y does not formally lift to characteristic zero.*

Apparently, this are the first examples of non-liftable manifolds with $c_1 = 0$ that do not belong the class of abelian varieties, hyperkähler manifolds, Calabi–Yau manifolds, or products thereof.

To show that such Y does not lift formally to characteristic zero, we first establish that it does not lift projectively, and afterwards prove that each formal lifting carries an ample invertible sheaf. For the first step, we assume that a projective lifting $\mathfrak{Y} \to \mathrm{Spec}(R)$ exists, and use the existence of relative Hilbert schemes $\mathrm{Hilb}_{\mathfrak{Y}/R}$ to show that properties of the Albanese map $V \to \mathrm{Alb}_{V/\mathbb{C}}$ for the resulting complex fiber $V = \mathfrak{Y} \otimes_R \mathbb{C}$ would contradict the Beauville–Bogomolov Decomposition Theorem for Kähler manifolds with $c_1 = 0$. This step actually holds whenever $d(p - 1) = g$.

In the second step, we use Rim's equivariant deformation theory [41] and its generalization to mixed characteristics [45] to show that each formal lifting $\mathfrak{Y} \to \mathrm{Spf}(R)$ admits an ample sheaf, which together with Grothendieck's Existence Theorem gives the desired contradiction. This relies on the computation of *weights* in the groups $H^2(Y,\mathscr{O}_Y)$ and $H^1(Y,\Theta_Y)$ for the action of $G=\{\pm 1\}$ coming from the sign involution on A. Although we have no closed formula for the dimension of the cohomology groups, this gives enough information to conclude that the sign involution on Y and an equivariant ample invertible sheaf $\mathscr{L}$ extend to all infinitesimal deformations, which leads to the desired ample sheaf.

The paper is organized as follows: In Section 1 we collect some facts on families of algebraic group schemes in characteristic $p>0$ and discuss the four-term complex that describes certain infinitesimal quotients. In Section 2 we apply this to families of abelian varieties. The Moret-Bailly families $\varphi: Y\to\mathbb{P}^n$ are introduced in Section 3, where we compute the dualizing sheaf and Betti numbers. Section 4 contains a description of the Picard scheme and the Albanese map. Fibers of the Albanese map play a crucial role in Section 5, where we prove that Moret-Bailly families with $c_1=0$ do not projectively lift to characteristic zero. Here the main ingredient are relative Hilbert schemes, and the Beauville–Bogomolov decomposition over the complex numbers. In Section 6 we express the cohomology groups $H^i(Y,\mathscr{O}_Y)$ in terms of cohomology on $\mathbb{P}^n$ for coefficient sheaves that involve Frobenius pullbacks and exterior powers. This is used in Section 7 to compute weights in $H^2(Y,\mathscr{O}_Y)$ and $H^1(Y,\Theta_Y)$, which are the crucial obstruction groups for infinitesimal deformations. The final Section 8 contains the proof that Moret-Bailly families Y with $c_1=0$ do not formally lift to characteristic zero, by using equivariant deformation theory in mixed characteristics.

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1. Families of algebraic group schemes

Let S be a base scheme. A *family of algebraic group schemes* is a scheme X, together with a morphism $X\to S$ that is flat and of finite presentation, endowed with the structure of a relative group scheme. Here we could allow algebraic spaces as well. However, for the sake of exposition we stay in the realm of schemes, which is sufficient for our applications.

Let us assume that the sheaf of Kähler differentials $\Omega^1_{X/S}$ is locally free. Then the tangent sheaf $\Theta_{X/S}=\underline{\operatorname{Hom}}(\Omega^1_{X/S},\mathscr{O}_X)$ is locally free as well. The *sheaf of Lie algebras* $\operatorname{Lie}_{X/S}$ is the pullback of $\Theta_{X/S}$ along the neutral section $e:S\to X$. This is a locally free sheaf, endowed with a Lie bracket, such that the fibers $\mathfrak{g}=\operatorname{Lie}_{X/S}\otimes\kappa(a)$, $a\in S$ become Lie algebras over the residue fields $\kappa(a)$.

Now suppose that S has characteristic $p>0$. Then the sheaf of Lie algebras $\operatorname{Lie}_{X/S}$ acquires the *p-map* as additional structure, such that the $\mathfrak{g}=\operatorname{Lie}_{X/S}\otimes\kappa(a)$ become *restricted Lie algebras* over $\kappa(a)$. Recall that the map

$$\operatorname{Hom}(X,Y)\longrightarrow\operatorname{Hom}(\operatorname{Lie}_{X/S},\operatorname{Lie}_{Y/S}) \tag{1}$$

is bijective provided that X has height at most one. Note that the hom set on the right comprises $\mathscr{O}_S$-linear maps compatible with Lie bracket and p-map, and that *height at most one* means that the relative Frobenius $F : X \to X^{(p)}$ is trivial. Here $X^{(p)}$ is the pullback of X along the absolute Frobenius map $F_S : S \to S$, and the morphism F comes from the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ \downarrow & & \downarrow \\ S & \xrightarrow[F_S]{} & S \end{array}$$

of absolute Frobenius maps. Moreover, each sheaf $\mathscr{H}$ of restricted Lie algebras that is locally free of finite rank arises from some family of algebraic group scheme H of height at most one. In fact, H is the relative spectrum of the sheaf of algebras

$$\mathscr{A} = \mathscr{H}om(U^{[p]}(\mathscr{H}), \mathscr{O}_S), \tag{2}$$

where $U^{[p]}(\mathscr{H})$ is the quotient of the sheaf $U(\mathscr{H})$ of universal enveloping algebras by some sheaf of ideals defined via the p-map, as explained in [13], Chapter II, §7, No. 4, compare also [15], Exposé VII, Theorem 7.2.

Let $\mathscr{H} \subset \mathrm{Lie}_{X/S}$ be a subsheaf that is locally a direct summand, and assume that $\mathscr{H}$ is stable under both Lie bracket and p-map. Let $H \to S$ be the corresponding family of group schemes of height at most one, with $\mathrm{Lie}_{H/S} = \mathscr{H}$. We now consider the inclusion $H \subset X$ and form the resulting quotient $Y = X/H$. Such a quotient exists as an algebraic space. It is actually a scheme, because the projection $X \to Y$ is a finite universal homeomorphism ([37], Theorem 6.2.2). If H is normal, Y inherits the structure of a family of algebraic groups.

Proposition 1.1. *The structure morphism $Y \to S$ is flat and of finite presentation. Moreover, it is smooth provided that $X \to S$ is smooth.*

Proof. The projection $\epsilon : X \to Y$ is faithfully flat and of finite presentation, because it is a torsor with respect to $H \times Y$. The assertion now follows from fppf descent. □

In the special case $\mathscr{H} = \mathrm{Lie}_{X/S}$ the group scheme H coincides with the kernel $X[F]$ of the relative Frobenius map. In the general case, we thus have an S-morphism $X/H \to X^{(p)}$.

We now assume that X and is smooth. Then the same holds for Y, and the homomorphism $X \to X^{(p)}$ is an epimorphism, such that $X/H = X^{(p)}$. In turn we obtain an exact sequence $0 \to \mathscr{H} \to \mathrm{Lie}_{X/S} \to \mathrm{Lie}_{Y/S}$ of families of restricted Lie algebras. By assumption, the inclusion on the left is locally a direct summand, so the cokernel $\mathscr{K} = \mathrm{Lie}_{X/S} / \mathscr{H}$ is locally free. Since forming the quotient $Y = X/H$ commutes with base-change, the inclusion $\mathscr{K} \subset \mathrm{Lie}_{Y/S}$ is locally a direct summand. Let $K \subset Y$ be the corresponding family of group schemes of height at most one. The isomorphism theorem ensures $Y/K = X/X[F] = X^{(p)}$. This gives a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{H} & \longrightarrow & \mathrm{Lie}_{X/S} & \longrightarrow & \mathrm{Lie}_{Y/S} & & \\
 & & & & & \searrow & \downarrow & \searrow & \\
 & & 0 & \longrightarrow & \mathscr{H}^{(p)} & \longrightarrow & \mathrm{Lie}_{X^{(p)}/S} & \longrightarrow & \mathrm{Lie}_{Y^{(p)}/S}
\end{array}$$

The two diagonal maps vanish, hence the vertical map factors over $\mathscr{H}^{(p)}$ and we obtain the *four-term complex*

$$0 \longrightarrow \mathscr{H} \longrightarrow \mathrm{Lie}_{X/S} \longrightarrow \mathrm{Lie}_{Y/S} \longrightarrow \mathscr{H}^{(p)} \longrightarrow 0. \tag{3}$$

Sequences like this already appear in Ekedahl's work on foliations of smooth algebraic schemes ([17], Corollary 3.4).

Theorem 1.2. *The above complex of restricted Lie algebras is exact.*

Proof. By construction, the complex is exact at all terms, with the possible exception of $\mathscr{H}^{(p)}$. Our task is thus to verify that $\mathrm{Lie}_{Y/S} \to \mathscr{H}^{(p)}$ is surjective. By the Nakayama Lemma, it suffices to do so after tensoring with $\kappa(a)$, for $a \in S$. Since the formation of quotients commutes with base-change, so does the formation of the complex. It thus suffices to treat the case that S itself is the spectrum of a field. Now the terms become finite-dimensional vector spaces. The outer terms have the same dimension, and the same holds for the inner terms. In turn, the rank of $\mathrm{Lie}_{Y/S} \to \mathscr{H}^{(p)}$ coincides with the dimension of $\mathscr{H}^{(p)}$, so the map in question must be surjective. □

2. Families of abelian varieties

We keep the assumption of the preceding section, and assume now that $\psi : X \to S$ is a family of abelian varieties of relative dimension $g \geq 0$. According to [3], Theorem 7.3 the relative Picard functor is representable, and $P = \mathrm{Pic}^0_{X/S}$ is called the *family of dual abelian varieties*. The sheaf of Lie algebras is given by $\mathrm{Lie}_{P/S} = R^1\psi_*(\mathscr{O}_X)$, with trivial bracket.

There is also a very useful identification $P = \underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)$ explained in [26], Exposé VII, which we briefly recall: Let $Y \to S$ be another family of abelian varieties. From the Leray–Serre spectral sequence for $X \to S$, one sees that a homomorphism $Y \to P$ corresponds to an invertible sheaf $\mathscr{L}$ on $X \times Y$ together with compatible trivializations over $0 \times Y$ and $X \times 0$, up to isomorphism. But the category of such *birigidified invertible sheaves* is equivalent to the category of *biextensions* of X and Y by $\mathbb{G}_m$ (loc. cit. Example 2.9.5). Using $\mathrm{Biext}^1(X, Y; \mathbb{G}_m) = \mathrm{Ext}^1(X \otimes^L Y, \mathbb{G}_m)$ (loc. cit. Corollary 3.6.5), the Cartan identification with $\mathrm{Ext}^1(Y, \underline{\mathrm{RHom}}(X, \mathbb{G}_m))$, the composite functor spectral sequence and the vanishing of $\underline{\mathrm{Hom}}(X, \mathbb{G}_m)$, one obtains the desired identification

$$\mathrm{Hom}(Y, P) = \mathrm{Biext}^1(X, Y; \mathbb{G}_m) \xrightarrow{\simeq} \mathrm{Hom}(Y, \underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)).$$

This is functorial in Y, hence $\underline{\mathrm{Ext}}^1(X, \mathbb{G}_m)$ is representable by $P = \mathrm{Pic}^0_{X/S}$.

Now suppose that the base scheme S has characteristic $p > 0$. Let $\mathscr{H} \subset \mathrm{Lie}_{X/S}$ be a subsheaf that is locally a direct summand, and stable under the p-map, and $H \subset X$ the corresponding family of group schemes of height at most one. Then the quotient

$Y = X/H$ is again a family of g-dimensional abelian varieties. Clearly, $\underline{\mathrm{Hom}}(X, \mathbb{G}_m)$ vanishes, and the induced map $\mathrm{Pic}^0_{Y/S} \to \mathrm{Pic}^0_{X/S}$ is faithfully flat. In turn, the long exact Ext sequence for the short exact sequence $0 \to H \to X \to Y \to 0$ yields the short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(H, \mathbb{G}_m) \longrightarrow \mathrm{Pic}^0_{Y/S} \longrightarrow \mathrm{Pic}^0_{X/S} \longrightarrow 0, \tag{4}$$

Here the term on the left is called the *Cartier dual* of H, which is also a family of group schemes of height at most one. Likewise, $0 \to H' \to Y \to X^{(p)} \to 0$ with $H' = X[F]/H$ gives a short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(H', \mathbb{G}_m) \longrightarrow \mathrm{Pic}^0_{X^{(p)}/S} \longrightarrow \mathrm{Pic}^0_{Y/S} \longrightarrow 0 \tag{5}$$

Write $\varphi : Y \to S$ and $\psi : X \to S$ for the structure morphisms.

Proposition 2.1. *Suppose the p-map $\mathrm{Lie}_{X/S} \to \mathrm{Lie}_{X/S}$ factors over the subsheaf $\mathscr{H}$. Then we have a four-term exact sequence*

$$0 \longrightarrow \mathscr{K} \longrightarrow R^1\psi_*(\mathscr{O}_X)^{(p)} \longrightarrow R^1\varphi_*(\mathscr{O}_Y) \longrightarrow \mathscr{K}^{(p)} \longrightarrow 0,$$

with the sheaf $\mathscr{K} = \mathscr{H}om(\mathrm{Lie}_{X/S}/\mathscr{H}, \mathscr{O}_S)$. Moreover, the sequence is natural with respect to the inclusion $H \subset X$.

Proof. Consider the family $H' = X[F]/H$ of group schemes of height at most one. Its Cartier dual $K = \underline{\mathrm{Hom}}(H', \mathbb{G}_m)$ is a family of finite group schemes. The latter have height at most one, by our assumption on the p-map on $\mathrm{Lie}_{X/S}$. By Lemma 2.2 below, the sheaf of Lie algebras $\mathscr{K} = \mathrm{Lie}_{K/S}$ coincides with the linear dual of $\mathrm{Lie}_{H'/S} = \mathrm{Lie}_{X/S}/\mathscr{H}$. Our assertion in now is a consequence of Theorem 1.2, applied to the short exact sequence (5). The four-term sequence is natural with respect to $H \subset X$, because the same holds for the two short exact sequences (4) and (5). □

The preceding proof relies on the following observations: Let (LocLib/S) be the category of locally free sheaves of finite rank, and (Grp/S) be the category of families of algebraic group schemes. Consider the functors

$$\mathscr{E} \longmapsto V \quad \text{and} \quad \mathscr{E} \longmapsto V^*,$$

where $V = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathscr{E}^\vee))$ and $V^* = \mathrm{Spec}(\mathrm{Sym}^\bullet(\mathscr{E}))$ are the vector bundles with $\mathrm{Lie}_{V/S} = \mathscr{E}$ and $\mathrm{Lie}_{V^*/S} = \mathscr{E}^\vee$, with trivial Lie brackets and p-maps. Note that we follow Grothendieck's convention from [21], Section 9.6. The Frobenius kernels $G = V[F]$ and $G^* = V^*[F]$ are families of finite local group schemes.

Lemma 2.2. *The contravariant functors $\mathscr{E} \mapsto G^*$ and $\mathscr{E} \mapsto \underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ are naturally isomorphic. In particular, there is an identification*

$$\mathrm{Lie}_{\underline{\mathrm{Hom}}(G,\mathbb{G}_m)/S} = \mathscr{H}om(\mathrm{Lie}_{G/S}, \mathscr{O}_S)$$

of locally free sheaves that is natural in G.

Proof. The natural identification arises as follows: Let $T = \mathrm{Spec}(A)$ be an affine S-scheme, and consider the resulting A-module $E = \Gamma(T, \mathscr{E}_T)$, which is finitely generated and projective. According to (1), the group of A-valued points in the Cartier dual $\underline{\mathrm{Hom}}(G, \mathbb{G}_m)$ is the set of linear maps

$$E = \mathrm{Lie}_{G/S} \otimes A \longrightarrow \mathrm{Lie}_{\mathbb{G}_m/S} \otimes A = A$$

that are compatible with p-maps. On the left-hand side, the p-map vanishes, whereas on the right hand side it is nothing but $\lambda \mapsto \lambda^p$. So these linear maps can be seen as vectors in the dual $\mathrm{Hom}_A(E, A)$ annihilated by the relative Frobenius map. The latter coincide with the A-valued points of the Frobenius kernel $G^* = V^*[F]$. □

The following property of families of abelian varieties $\varphi : Y \to S$ will be crucial for later computations:

Proposition 2.3. *For all $s \geq 0$, the higher direct image sheaves $R^s\varphi_*(\mathscr{O}_Y)$ are locally free, their formation commutes with base-change, and the canonical maps $\Lambda^s R^1\varphi_*(\mathscr{O}_Y) \to R^s\varphi_*(\mathscr{O}_Y)$ are bijective. Moreover, the Leray–Serre spectral sequence*

$$E_2^{r,s} = H^r(S, R^s\varphi_*(\mathscr{O}_Y)) \Longrightarrow H^{r+s}(Y, \mathscr{O}_Y)$$

has trivial differentials on the E_i-page provided $p-1$ does not divide $i-1$.

Proof. The first assertion is [6], Proposition 2.5.2. For the second assertion, note that the differentials on the i-th page are certain additive maps $d_i : E_i^{r,s} \to E_i^{r+i,s-i+1}$, which are natural with respect to the family of abelian varieties Y. In particular, for each integer n the multiplication on Y induces an endomorphism n^* on $\Lambda^s R^1\varphi_*(\mathscr{O}_Y)$. The latter is multiplication by n^s. To check this it suffices to treat the case where S is the spectrum of a field k and $s = 1$. Then the proper commutative group schemes form an abelian category, and $Y \mapsto H^1(Y, \mathscr{O}_Y)$ is a contravariant functor into the abelian category of all k-vector spaces, with the property $H^1(\mathscr{O}_{Y_1\times Y_2}) = H^1(\mathscr{O}_{Y_1}) \oplus H^1(\mathscr{O}_{Y_2})$. Now recall that any functor $F : \mathcal{C} \to \mathcal{C}'$ between additive categories that respects products also respects the $\mathbb{Z}$-module structure on hom sets ([30], Proposition 8.2.15), and thus $n^* = n$.

Now suppose that there is an element $a \in E_i^{r,s}$ whose image $b \in E_i^{r+i,s-i+1}$ is non-zero. The latter can be seen as a basis vector inside an $\mathbb{F}_p$-vector space. Recall that the multiplicative group $\mathbb{F}_p^\times$ is cyclic of order $p-1$. Choose an integer n that generates $\mathbb{F}_p^\times$. Since $p-1 \nmid i-1$ we have $n^{i-1} \not\equiv 1$ modulo p. One gets

$$n^s b = n^s d_i(a) = d_i(n^s a) = n^{s-i+1} d_i(a) = n^{s-i+1} b$$

from the naturality of the Leray–Serre spectral sequence. Comparing coefficients gives $n^{1-i} \equiv 1$ modulo p, contradiction. □

3. Moret-Bailly families

Let k be a ground field of characteristic $p > 0$. Recall that an abelian variety A of dimension $g \geq 1$ is called *superspecial* if the Lie algebra $\mathfrak{g} = \mathrm{Lie}(A)$ has trivial p-map. For $g = 1$ this means that $A = E$ is a supersingular elliptic curve. Moreover, the products $A = E_1 \times \ldots \times E_g$ of supersingular elliptic curves are superspecial. Note that if k is algebraically closed, the converse holds ([38], Theorem 2). If moreover $g \geq 2$, the isomorphism class of A does not depend on the factors ([47], Theorem 3.5). We need the following well-known existence result:

Lemma 3.1. *In each dimension $g \geq 1$ there is a superspecial abelian variety A.*

Proof. Using the three Weierstraß equations

$$y^2 + xy = x^3 + \frac{36}{1728-j}x + \frac{1}{1728-j} \quad \text{and} \quad y^2 + y = x^3 \quad \text{and} \quad y^2 = x^3 + x,$$

one sees that each invariant $j \in k$ is attained by some elliptic curve (compare [49], Example on page 36). So it suffices to show that there are supersingular j-values in the prime field $k = \mathbb{F}_p$. For $p = 2$ this is $j = 0$. Suppose now $p \geq 3$. Recall that an elliptic curve in Legendre form $E : y^2 = x(x-1)(x-\lambda)$ is supersingular if and only if λ is a root of the Hasse polynomial $P(T) = \sum_{i=0}^m \binom{m}{i}^2 T^i$, where $m = (p-1)/2$. One may view the spectrum of $k[\lambda]$ as the coarse moduli space for elliptic curves E with level structure $(\mathbb{Z}/2\mathbb{Z})_k^2 \subset E$. The group $G = \mathrm{GL}_2(\mathbb{F}_2)$ acts freely via the level structures, and the ring of invariants $k[\lambda]^G = k[j]$ is the coarse moduli space for the Deligne–Mumford stack $\mathscr{M}_{1,1}$. According to [9], Theorem 1 the Hasse polynomial has at least one root over k if and only if $p \not\equiv 1$ modulo 4; then a supersingular E over $\mathbb{F}_2$ already exists with level structure, and can be put in Legendre form. Suppose now $p \equiv 1$ modulo 4, and consider the subgroup $H \subset G$ generated interchanging the 2-division points with coordinate $x = 0, 1$. This is given by the change of coordinates $x = -x' + 1$, and induces $\lambda \mapsto 1 - \lambda$ on the coarse moduli space. By loc. cit. the Hasse polynomial viewed as element in the ring of invariants $k[\lambda]^H = k[\lambda - \lambda^2]$ acquires a root, which gives the desired supersingular j-value. ∎

Fix some integers $n, d \geq 1$ and choose a superspecial abelian variety A of dimension $g = n + 1$. In turn, every non-zero vector in $\mathfrak{g} = \mathrm{Lie}(A)$ gives an inclusion $\alpha_p \subset A$. Choose an identification $\mathfrak{g} = k^{n+1}$. Set $X = A \times \mathbb{P}^n$, and view this as the constant family of abelian varieties over $\mathbb{P}^n = \operatorname{Proj} k[T_0, \ldots, T_n] = \mathbb{P}(\mathfrak{g}^\vee)$. Choose some homogeneous polynomials $Q_0, \ldots, Q_n$ of degree $d \geq 1$ without common zero on the projective space, and consider the resulting inclusion

$$\mathscr{O}_S(-d) \subset \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} = \mathfrak{g} \otimes_k \mathscr{O}_{\mathbb{P}^n} = \mathrm{Lie}_{X/\mathbb{P}^n}\,.$$

Let $H \subset X$ be the family of height-one group schemes with $\mathrm{Lie}_{H/S} = \mathscr{O}_S(-d)$, and form the resulting quotient $Y = X/H$. This is smooth and proper, of dimension $\dim(Y) = 2n + 1$, and with $h^0(\mathscr{O}_Y) = 1$. Write

$$\varphi : Y = X/H = (A \times \mathbb{P}^n)/H \longrightarrow \mathbb{P}^n$$

for the structure morphism, which is a family of supersingular abelian varieties of dimension $g = n + 1$, and $\epsilon : X \to Y$ for the quotient map.

We call Y a *Moret-Bailly family*, because the above generalizes the pencils in [34] to arbitrary dimensions. Note that the construction $Y = Y_{A,q}$ depends on the superspecial abelian variety A and the finite flat morphism $q : \mathbb{P}^n \to \mathbb{P}^n$ defined by the homogeneous polynomials Q_i, but we usually neglect this in notation.

The cokernel $\mathscr{E}_d$ for the inclusion $\mathscr{O}_{\mathbb{P}^n}(-d) \subset \mathfrak{g} \otimes \mathscr{O}_{\mathbb{P}^n}$ is locally free and sits in the short exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-d) \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \mathscr{E}_d \longrightarrow 0, \tag{6}$$

thus has $\det(\mathscr{E}_d) = \mathscr{O}_{\mathbb{P}^n}(d)$. Note that for $Q_i = T_i$ this becomes the *Euler sequence* (compare [36], page 6), and $\mathscr{E}_1 = \Theta_{\mathbb{P}^n/k}(-1)$. In general, we have $\mathscr{E}_d = q^*(\Theta_{\mathbb{P}^n/k}(-1))$, where $q : \mathbb{P}^n \to \mathbb{P}^n$ is the morphism of degree $\deg(q) = d^n$ defined by the homogeneous polynomials Q_i.

Note that the Frobenius pullback $\mathscr{E}_{pd} = \mathscr{E}_d^{(p)}$ is obtained by taking the p-powers $Q_0^p, \ldots, Q_n^p$. To simplify notation, we write $\mathscr{E}_{-d} = \underline{\mathrm{Hom}}(\mathscr{E}_d, \mathscr{O}_{\mathbb{P}^n})$ for the dual sheaves, and also set $\mathscr{E}_{-pd} = \mathscr{E}_{-d}^{(p)}$.

Proposition 3.2. *The sheaf of Lie algebras and the first direct image are given by*

$$\mathrm{Lie}_{Y/\mathbb{P}^n} = \mathscr{E}_d \oplus \mathscr{O}_{\mathbb{P}^n}(-dp) \quad \textit{and} \quad R^1\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n}(d) \oplus \mathscr{E}_{-pd}.$$

Moreover, the p-map in $\mathrm{Lie}_{Y/\mathbb{P}^n}$ *is trivial on the first summand, and sends the second summand to the first, for all such splittings.*

Proof. By Theorem 1.2, the sheaf of Lie algebras is an extension of $\mathscr{O}_{\mathbb{P}^n}(-pd)$ by $\mathscr{E}_d$. All such extensions split: We have $\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^n}(-pd), \mathscr{E}_d) = H^1(\mathbb{P}^n, \mathscr{E}_d(pd))$, and tensoring (6) with $\mathscr{O}_{\mathbb{P}^n}(pd)$ yields an exact sequence

$$H^1(\mathbb{P}^n, \mathfrak{g} \otimes_k \mathscr{O}_{\mathbb{P}^n}(pd)) \longrightarrow H^1(\mathbb{P}^n, \mathscr{E}_d(pd)) \longrightarrow H^2(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(pd - d)).$$

The outer terms vanish, because the invertible sheaves $\mathscr{O}_{\mathbb{P}^n}(pd)$ and $\mathscr{O}_{\mathbb{P}^n}(pd-d)$ are ample. Thus $\mathrm{Lie}_{Y/\mathbb{P}^n}$ splits.

Next we verify the assertion on the p-map. By construction, $\mathscr{E}_d$ is a quotient for $\mathrm{Lie}_{X/S}$, whereas $\mathscr{O}_{\mathbb{P}^n}(-pd)$ is a subsheaf of $\mathrm{Lie}_{X^{(p)}/S}$. Since $X = A \times \mathbb{P}^n$ comes from a superspecial abelian variety A, the p-maps vanish on these sheaves of Lie algebras, and the assertion follows.

Finally, we analyze the first direct image. The dual $\mathscr{E}_{-d}$ for the quotient $\mathscr{E}_d = \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}/\mathscr{O}_{\mathbb{P}^n}(-d)$ comes with an inclusion into $\mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} = \mathrm{Lie}(A^{(p)}) \otimes_k \mathscr{O}_{\mathbb{P}^n}$ that is locally a direct summand. In turn, the cokernel is isomorphic to $\mathscr{O}_{\mathbb{P}^n}(d)$. According to Theorem 2.1, the direct image $R^1\varphi_*(\mathscr{O}_Y)$ is an extension of $\mathscr{E}_{-pd}$ by this cokernel $\mathscr{O}_{\mathbb{P}^n}(d)$. As in the preceding paragraph, one verifies that all such extensions split. $\square$

This computation has the following consequence:

Corollary 3.3. *For each rational point $a \in \mathbb{P}^n$, there are only finitely many other rational points $b \in \mathbb{P}^1$ such that $\varphi^{-1}(a) \simeq \varphi^{-1}(b)$. Moreover, any field of definition F for the generic fiber $\varphi^{-1}(\eta)$ has* $\mathrm{trdeg}(F) = n$.

Proof. We may assume that k is algebraically closed. Choose some odd prime $l \neq p$, and some symplectic level structure $(\mathbb{Z}/l\mathbb{Z})^{2g} \to A$. This descents to a family of symplectic level structures for Y. Let $\mathscr{A}_{g,l}$ be the Artin stack of g-dimensional abelian varieties endowed with such a structure. This is actually an algebraic space ([19], Chapter IV, Corollary 2 on page 131) that is separated and of finite type. Our Moret-Bailly family corresponds to a morphism $h : \mathbb{P}^n \to \mathscr{A}_{g,l}$. It suffices to check that h is quasi-finite, because every abelian variety has only finitely many such level structures. Suppose it is not quasi-finite. Then there is a curve $C \subset \mathbb{P}^n$ that maps to a closed point. Using Chow's Lemma, one easily sees that any morphism from a proper scheme to an algebraic space that is separated and locally of finite type and which contracts a curve also contracts numerically equivalent curves. It follows that the whole image $h(\mathbb{P}^n)$ is a closed point, and thus $Y = B \times \mathbb{P}^n$ for some abelian variety B. In turn, $\mathrm{Lie}_{Y/\mathbb{P}^n} = \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}$, in contradiction to the proposition. The assertion on the field of definition is proven in an analogous way. $\square$

To simplify notation, we now write $\mathscr{O}_Y(m)$ for the preimages under the structure morphism $\varphi : Y \to \mathbb{P}^n$ of the invertible sheaves $\mathscr{O}_{\mathbb{P}^n}(m)$.

Corollary 3.4. *The dualizing sheaf takes the form* $\omega_Y = \mathscr{O}_Y(m)$ *for the integer* $m = d(p-1) - (n+1)$. *In particular,* $c_1 = 0$ *holds if and only if* $d(p-1) = n+1$, *and in this case we actually have* $\omega_Y = \mathscr{O}_Y$.

Proof. Since Y is smooth, the dualizing sheaf is $\omega_Y = \det(\Omega^1_{Y/k})$. We have a short exact sequence $0 \to \varphi^*(\Omega^1_{\mathbb{P}^n/k}) \to \Omega^1_{Y/k} \to \Omega^1_{Y/\mathbb{P}^n} \to 0$. The cokernel is isomorphic to the preimage of the dual for $\mathrm{Lie}_{Y/\mathbb{P}^n}$, because $\varphi : Y \to \mathbb{P}^n$ is a family of smooth algebraic group schemes. The sheaf of Lie algebras equals $\mathscr{E}_d \oplus \mathscr{O}_{\mathbb{P}^n}(-pd)$, and we have $\det(\mathscr{E}_d) = \mathscr{O}_{\mathbb{P}^n}(d)$. Furthermore $\det(\Omega^1_{\mathbb{P}^n}) = \mathscr{O}_{\mathbb{P}^n}(-n-1)$. Combining all this we obtain $\omega_Y = \mathscr{O}_Y(m)$ for the integer $m = -n - 1 - d + pd$. $\square$

It follows that the Kodaira dimension takes the values $\mathrm{Kod}(Y) \in \{\infty, 0, n\}$ depending on the sign of the integer $m = d(p-1) - (n+1)$. Moreover, in the canonical model $Z = \mathrm{Proj} \bigoplus_{t \geq 0} H^0(Y, \omega^{\otimes t})$ is given by the respective schemes $Z = \varnothing, \mathbb{P}^0, \mathbb{P}^n$. Analogous statements hold for the anticanonical models.

It is easy to determine the Betti numbers $b_i \geq 0$, defined as the ranks of the l-adic cohomology groups $H^i(\bar{Y}, \mathbb{Z}_l(i)) = \varprojlim_{\nu \geq 0} H^i(\bar{Y}, \mu_{l^\nu}^{\otimes i})$, where $\bar{Y} = Y \otimes k^{\mathrm{alg}}$ is the base-change to some algebraic closure, and $l > 0$ is a prime different from p.

Proposition 3.5. *The l-adic cohomology groups* $H^i(\bar{Y}, \mathbb{Z}_l(i))$ *are free of rank* $b_i = \sum_j \binom{2n+2}{i-j}$, *where the sum runs over all even j. In particular, we have* $b_1 = 2n+2$ *and* $b_2 = 2n^2 + 3n + 2$ *and* $b_{2n+1} = 2^{2n+1}$.

Proof. We may assume that k is algebraically closed. The quotient map $\epsilon : X \to Y$ is a finite universal homeomorphism, so the l-adic cohomology groups for $Y = X/H$ and $X = A \times \mathbb{P}^n$ coincide. Taking cohomology with coefficients in $R = \mathbb{Z}/l^\nu\mathbb{Z}$, we have $H^\bullet(\mathbb{P}^n) = R[h]/(h^{n+1})$ and $H^\bullet(A) = \Lambda^\bullet H^1(A)$. These R-modules are free, where the generator h has degree two, and $H^1(A)$ is of rank $2n+2$. In turn,

$$H^i(X) = \bigoplus_j H^j(\mathbb{P}^n) \otimes_R H^{i-j}(A)$$

by the Künneth Formula ([4], Exposé XVII, Theorem 5.4.3) and the assertion on $H^i(Y, \mathbb{Z}_l(i))$ is a direct consequence. The values b_1 and b_2 follow immediately. In middle degree, we get $b_{2n+1} = \sum_s \binom{2n+2}{s}$, where the sum runs over all odd s. This sum is half of $(1+1)^{2n+2} - (1-1)^{2n+2} = 2^{2n+2}$. $\square$

Note that since $b_{2n+1} \neq 0$, the method introduced by Hirokado [27] to establish non-liftability apparently does not apply.

4. The Picard scheme and the Albanese map

Keep the notation from the previous section, such that $Y = (A \times \mathbb{P}^n)/H$ is a Moret-Bailly family formed with some superspecial abelian variety A of dimension $g = n+1$ and some homogeneous polynomials $Q_0, \ldots, Q_n$ of degree $d \geq 1$, without common zero on $\mathbb{P}^n$. Let $\varphi : Y \to \mathbb{P}^n$ be the structure morphism. We now examine the Picard scheme for Y. Recall that its Lie algebra is the cohomology group $H^1(Y, \mathscr{O}_Y)$.

Proposition 4.1. *The Picard scheme* $\mathrm{Pic}_{Y/k}$ *has dimension* $n+1$, *and furthermore* $h^1(\mathscr{O}_Y) = \binom{n+d}{d}$. *In particular, the Picard scheme is smooth if and only if* $d = 1$.

Proof. We may assume that k is algebraically closed. To compute the dimension $d \geq 0$ of the Picard scheme, choose a prime $l \neq p$ that does not divide the order of the torsion part in $\mathrm{NS}(Y)$. The Kummer sequence $0 \to \mu_l \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ implies that $2d = b_1$. According to Proposition 3.5, we have $b_1 = 2n+2$.

The Leray–Serre spectral sequence for $\varphi : Y \to \mathbb{P}^n$ gives an exact sequence

$$0 \longrightarrow H^1(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}) \to H^1(Y, \mathscr{O}_Y) \longrightarrow H^0(\mathbb{P}^n, R^1\varphi_*(\mathscr{O}_Y)) \longrightarrow H^2(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}).$$

The outer terms vanish, and we merely have to compute the global sections of $R^1\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n}(d) \oplus \mathscr{E}_{-pd}$. The first summand contributes $h^0(\mathscr{O}_{\mathbb{P}^n}(d)) = \binom{n+d}{d}$. It remains to check that the sheaf $\mathscr{E}_{-pd}$ has no non-zero global sections. Dualizing the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^n}(-pd) \to \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{E}_{pd} \to 0$, we get an exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^n, \mathscr{E}_{-pd}) \longrightarrow H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}) \longrightarrow H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(pd)).$$

The map on the right is given by the homogeneous polynomials $Q_0^p, \ldots, Q_n^p$. These are linearly independent, so the map must be injective. It follows that $H^0(\mathbb{P}^n, \mathscr{E}_{-pd}) = 0$. This shows $h^1(\mathscr{O}_Y) = \binom{n+d}{d}$. $\square$

Set $X = A \times \mathbb{P}^n$. The family $H \subset X$ of finite group schemes of height one sits inside the constant family $X[F] = A[F] \times \mathbb{P}^n$. In turn, we get an induced homomorphism $Y \to A^{(p)} \times \mathbb{P}^n$ between families of abelian varieties, and write $\psi : Y \to A^{(p)}$ for the composition with the projection.

Proposition 4.2. *The morphism $\psi : Y \to A^{(p)}$ is flat, and every geometric fiber is non-reduced, with reduction isomorphic to the projective n-space. Moreover, the canonical map $\mathscr{O}_{A^{(p)}} \to \psi_*(\mathscr{O}_Y)$ is bijective.*

Proof. The quotient map $\epsilon : X \to Y$ if faithfully flat, and so is the composition $\psi \circ \epsilon : X = A \times \mathbb{P}^n \to A^{(p)}$. By descent, $\psi : Y \to A^{(p)}$ must be flat.

The fiber $Z = \psi^{-1}(0)$ over the origin is the family $(A[F] \times \mathbb{P}^n)/H$ of height-one group schemes, with $\mathrm{Lie}_{Z/\mathbb{P}^n} = \mathscr{E}_d$. We claim that $h^0(\mathscr{O}_Z) = 1$. The universal enveloping algebra is $U(\mathscr{E}_d) = \mathrm{Sym}^\bullet(\mathscr{E}_d)$, and the restricted quotient becomes $U^{[p]}(\mathscr{E}_d) = \mathrm{Sym}^\bullet(\mathscr{E}_d)/\mathscr{E}_d^{(p)}\,\mathrm{Sym}^\bullet(\mathscr{E}_d)$. According to (2), Z is the relative spectrum of the corresponding sheaf of Hopf algebras

$$\mathscr{A} = \mathscr{H}om(U^{[p]}(\mathscr{E}_d), \mathscr{O}_{\mathbb{P}^n}) \subset \mathscr{H}om(\mathrm{Sym}^\bullet(\mathscr{E}_d), \mathscr{O}_{\mathbb{P}^n}).$$

The term on the right is the product of the coherent sheaves $\mathscr{H}om(\mathrm{Sym}^i(\mathscr{E}_d), \mathscr{O}_{\mathbb{P}^n})$, and the term on the left is already contained in the corresponding sum. The summands are *divided powers* $\Gamma^i(\mathscr{F}) = \mathrm{Sym}^i(\mathscr{F}^\vee)^\vee$, for the dual sheaf $\mathscr{F} = \mathscr{E}_d^\vee$. It thus suffices to verify that the divided powers have no non-zero global sections for $i \geq 1$. We proceed by induction. The case $i = 1$ was already treated in the proof for Proposition 4.1. Now suppose $i \geq 2$, and that the assertion is true for $i-1$. The surjection $\mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{E}_d$ induces a canonical surjection $\mathrm{Sym}^{i-1}(\mathscr{E}_d) \otimes \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathrm{Sym}^i(\mathscr{E}_d)$. Dualizing the latter gives an inclusion $\Gamma^i(\mathscr{F}) \subset \Gamma^{i-1}(\mathscr{F}) \otimes \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}$, which completes the induction. Note that the inclusion is a piece from the Eagon–Northcott complex [16]. Summing up, this establishes $h^0(\mathscr{O}_Z) = 1$.

Now consider the fiber over a geometric point $\bar{b} : \mathrm{Spec}(\Omega) \to A^{(p)}$, with image $b \in A^{(p)}$. Making a base-change, it suffices to treat the case that $k = \Omega$ is algebraically closed and that $b \in A^{(p)}$ is rational. Then $b = a^{(p)}$ for some rational point $a \in A$, and $\psi^{-1}(b) = ((a + A[F]) \times \mathbb{P}^n)/H$. This is isomorphic to $\psi^{-1}(0) = (A[F] \times \mathbb{P}^n)/H$ via translation by a, so the fiber is non-reduced, with reduction isomorphic to $\mathbb{P}^n$.

It remains to compute $\psi_*(\mathscr{O}_Y)$. We just saw that the function $b \mapsto h^0(\mathscr{O}_{Y_b}) = 1$ is constant on the reduced scheme $A^{(p)}$. It follows that the direct image sheaf is locally free of rank one, hence the canonical map $\mathscr{O}_{A^{(p)}} \to \psi_*(\mathscr{O}_Y)$ is bijective. □

As explained by Serre [46], there is a morphism $Y \to V$ to some abelian variety V such that every other such morphism $Y \to V'$ arises via composition with some unique $V \to V'$. Note that the latter usually does not respect the origin. This $V = \mathrm{Alb}_{Y/k}$ is called the *Albanese variety*, and $Y \to \mathrm{Alb}_{Y/k}$ is the *Albanese map*.

Proposition 4.3. *The morphism $\psi : Y \to A^{(p)}$ is the Albanese map.*

Proof. Let $f : Y \to B$ be a morphism into another abelian variety. We have to show that this map factors uniquely over $\psi : Y \to A^{(p)}$. The composition $f \circ \epsilon : X \to B$ factors over the projection $\mathrm{pr}_1 : X = A \times \mathbb{P}^n \to A$. This gives a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{pr}_1} & A \\ {\scriptstyle\epsilon}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ Y & \xrightarrow[f]{} & B. \end{array}$$

For each rational point $a \in \mathbb{P}^n$, the fiber $H_a \subset X_a = A$ is a copy of α_p whose schematic image in Y and hence also in B is a rational point $b \in B$. The inclusion $\mathscr{O}_{\mathbb{P}^n}(-d) \subset \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}$ defining the family of subgroup schemes $H \subset X$ is locally a direct summand, hence the canonical map $\bigcup_a H_a \to A$ from the disjoint union has schematic image $A[F]$. In the commutative diagram

$$\begin{array}{ccccc} \bigcup_a H_a & \xrightarrow{\mathrm{can}} & H & \xrightarrow{\mathrm{pr}_1} & A \\ & \searrow & {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle h} \\ & & Y & \xrightarrow[f]{} & B \end{array}$$

the composition $h \circ \mathrm{pr}_1 \circ \mathrm{can}$ factors over the origin $0 \in B$, and the schematic image of $\mathrm{pr}_1 \circ \mathrm{can}$ is $A[F]$. Thus $h : A \to B$ factors over $A/A[F] = A^{(p)}$. The factorization is unique, because the composition $X \to A \to A/A[F]$ is faithfully flat, hence an epimorphism. □

5. Non-existence of projective liftings

We keep the assumptions of the preceding section, and furthermore assume that $n \geq 2$ and $d(n+1) = p - 1$, such that our Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$ of dimension $\dim(Y) = 2n + 1 \geq 5$ has dualizing sheaf $\omega_Y = \mathscr{O}_Y$. Recall that one says that Y *projectively lifts to characteristic zero* if there is a local noetherian ring R with residue field $k = R/\mathfrak{m}_R$ such that the canonical map $\mathbb{Z} \to R$ is injective, together with a projective flat morphism $\mathfrak{Y} \to \mathrm{Spec}(R)$ whose closed fiber is isomorphic to Y. Note that one may assume that R is also complete and one-dimensional. The goal of this section is to establish the following:

Theorem 5.1. *The scheme Y does not projectively lift to characteristic zero.*

The proof requires some preparation and is given at the end of this section. It suffices to treat the case that k is algebraically closed. Seeking a contradiction, we assume that Y projectively lifts. Then there is a complete discrete valuation ring R with residue field $R/\mathfrak{m}_R = k$ whose field of fractions $F = \mathrm{Frac}(R)$ has characteristic zero, together with a proper flat morphism $\mathfrak{Y} \to \mathrm{Spec}(R)$ with closed fiber $Y = \mathfrak{Y} \otimes_R k$. Write $V = \mathfrak{Y} \otimes_R F$ for the generic fiber, which is a smooth proper scheme with $h^0(\mathscr{O}_V) = 1$.

We start by examining the Picard scheme of V. The component of the origin $P = \mathrm{Pic}^0_{V/F}$ is an abelian variety. Consider the dual abelian variety $\mathrm{Pic}^0_{P/F}$. After passing to a finite extension of R, we may assume that the structure morphism $\mathfrak{Y} \to \mathrm{Spec}(R)$ admits a section. In particular, the generic fiber V contains a rational point. Then there is a Poincaré sheaf $\mathscr{P}$ on $V \times P$, and we may assume that it is numerically trivial on the fibers of the first projection. As explained in [24], Theorem 3.3, the resulting

$$\Psi : V \longrightarrow \mathrm{Pic}^0_{P/F}, \quad v \longmapsto [\mathscr{P}|\{v\} \times P]$$

is the Albanese map, and we write $\mathrm{Alb}_{V/F} = \mathrm{Pic}^0_{P/F}$.

Proposition 5.2. *For every rational point $\lambda \in \mathrm{Alb}_{V/F}$, the fiber $V_\lambda = \Psi^{-1}(\lambda)$ of the Albanese map is smooth, with $c_1 = 0$ and $\dim(V_\lambda) = n$.*

Proof. Applying the Specialization Theorem ([4], Exposé XVI, Corollary 2.2) to the smooth proper morphism $\mathfrak{Y} \to \mathrm{Spec}(R)$, we conclude that the Betti numbers for the closed and generic fiber coincide. From Proposition 3.5 we see that $b_1(V) = 2(n+1)$. In turn, the abelian variety $\mathrm{Pic}^0_{V/F}$ has dimension $n+1$, and the same holds for the Albanese variety $\mathrm{Alb}_{V/F}$.

The invertible sheaf $\omega_{\mathfrak{Y}/R} = \det(\Omega^1_{\mathfrak{Y}/R})$ is trivial on the closed fiber, hence at least numerically trivial on the generic fiber. In other words, the scheme V has $c_1 = 0$. By the Beauville–Bogomolov Decomposition Theorem ([7], [5]), there is a finite étale covering $f : V' \to V$ having a decomposition $V' = A' \times W$ where the first factor is an abelian variety and the second factor has $h^1(\mathscr{O}_W) = 0$. Strictly speaking, this exists only over F^{alg}, but by achieve this over F by enlarging our discrete valuation ring R. Obviously, the projection $\mathrm{pr} : V' \to A'$ is the Albanese map. In turn, we get a commutative diagram

$$\begin{array}{ccc} V' & \xrightarrow{\mathrm{pr}_1} & A' \\ {\scriptstyle f}\downarrow & & \downarrow \\ V & \longrightarrow & \mathrm{Alb}_{V/F}\,. \end{array}$$

Since the image of V generates $\mathrm{Alb}_{V/F}$, the same holds for the image of A', hence $A' \to \mathrm{Alb}_{V/F}$ is surjective, and thus smooth. In turn, the composition $V' \to \mathrm{Alb}_{V/F}$ is smooth, and by descent the same holds for $V \to \mathrm{Alb}_{V/F}$.

For each rational point $\lambda \in \mathrm{Alb}_{V/F}$, the inclusion of the fiber $V_\lambda \subset V$ is lci, with conormal sheaf $\mathscr{I}/\mathscr{I}^2 = \mathscr{O}_{V_\lambda}^{\oplus r}$, where $r = n+1$ is the dimension of $\mathrm{Alb}_{V/F}$. The Adjunction Formula then ensures that V_λ has $c_1 = 0$. Furthermore, $\dim(V_\lambda) = \dim(V) - \dim(\mathrm{Alb}_{V/F}) = (2n+1) - (n+1) = n$. □

According to Proposition 4.3, the composition $A \times \mathbb{P}^n \to A^{(p)}$ of the quotient map with the projection induces the Albanese map $\psi : Y \to A^{(p)}$, and the reduced preimage of the origin

$$Z = \psi^{-1}(0)_{\mathrm{red}} = \mathbb{P}^n_k.$$

is a copy of the projective n-space. We now exploit the existence of the relative Hilbert scheme $\mathrm{Hilb}_{\mathfrak{Y}/R}$, which parameterizes flat families of closed subschemes [23], and regard the closed subscheme $Z \subset Y$ as a k-valued point $\xi = [Z]$ in the relative Hilbert scheme.

Proposition 5.3. *The structure morphism* $\mathrm{Hilb}_{\mathfrak{Y}/R} \to \mathrm{Spec}(R)$ *is smooth near* ξ.

Proof. Let $\mathscr{I} \subset \mathscr{O}_Y$ be the sheaf of ideal for the closed subscheme $Z \subset Y$. According to [23], Corollary 5.4 it suffices to check that the obstruction group $\mathrm{Ext}^1(\mathscr{I}/\mathscr{I}^2, \mathscr{O}_Z)$ vanishes. Since the inclusion $Z \subset Y$ is lci, the conormal sheaf $\mathscr{N} = \mathscr{I}/\mathscr{I}^2$ is locally free, and the obstruction group becomes $H^1(Z, \mathscr{N}^\vee)$.

To compute the sheaf $\mathscr{N}$, we consider the commutative diagram

$$
\begin{array}{ccccc}
Z & \longrightarrow & Y & \longrightarrow & \mathrm{Spec}(k) \\
& \searrow^{\simeq} & \downarrow{\scriptstyle\varphi} & & \\
& & \mathbb{P}^n & & \\
& & \downarrow & & \\
& & \mathrm{Spec}(k) & &
\end{array} \tag{7}
$$

where the arrows are either smooth or lci. The vertical part yields a short exact sequence $0 \to \varphi^*\Omega^1_{\mathbb{P}^n/k} \to \Omega^1_{Y/k} \to \Omega^1_{Y/\mathbb{P}^n} \to 0$. The kernel is locally a direct summand, so the restriction

$$0 \longrightarrow \varphi^*\Omega^1_{\mathbb{P}^n/k}|Z \longrightarrow \Omega^1_{Y/k}|Z \longrightarrow \Omega^1_{Y/\mathbb{P}^n}|Z \longrightarrow 0,$$

remains exact. The horizontal part yields another short exact sequence

$$0 \longrightarrow \mathscr{I}/\mathscr{I}^2 \longrightarrow \Omega^1_{Y/k}|Z \longrightarrow \Omega^1_{Z/k} \longrightarrow 0.$$

Using the commutativity of (7), we see that the inclusion of $\varphi^*\Omega^1_{\mathbb{P}^n/k}|Z$ splits the above extension, so the projection $\mathscr{I}/\mathscr{I}^2 \to \Omega^1_{Y/\mathbb{P}^n}|Z$ is bijective. With respect to the identification $Z = \mathbb{P}^n$ this becomes $\mathscr{N}^\vee = \mathscr{E}_d \oplus \mathscr{O}_{\mathbb{P}^n}(-pd)$.

One now easily checks that $H^1(Z, \mathscr{N}^\vee) = 0$. Indeed, the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^n}(-d) \to \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{E}_d \to 0$ yields an exact sequence

$$H^1(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1}) \longrightarrow H^1(\mathbb{P}^n, \mathscr{E}_d) \to H^2(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-d))$$

The term on the left vanishes. This also holds for the term on the right for $n \neq 2$. Since $n = d(p-1) \geq d$ this also remains true for $n = 2$. Furthermore, we have $H^1(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(-pd)) = 0$, because $n \geq 2$. $\square$

By Hensel's Lemma, the relative Hilbert scheme admits an R-valued point passing through $\xi \in \mathrm{Hilb}_{\mathfrak{Y}/R}$. Let $\mathfrak{Z} \subset \mathfrak{Y}$ be the corresponding flat family of closed subschemes, with closed fiber $\mathfrak{Z} \otimes_R k = Z = \mathbb{P}^n_k$.

Proposition 5.4. *The generic fiber* $\mathfrak{Z} \otimes_R F \subset \mathfrak{Y} \otimes_R F = V$ *is isomorphic to* $\mathbb{P}^n_F$, *and must be contained in some fiber of the Albanese map* $\Psi : V \to \mathrm{Alb}_{V/F}$.

Proof. Using $H^1(\mathbb{P}^n, \Theta_{\mathbb{P}^n}) = 0$ we inductively construct compatible isomorphisms $\mathbb{P}^n \otimes R/\mathfrak{m}_R^{n+1} \to \mathfrak{Z} \otimes_R R/\mathfrak{m}_R^{n+1}$. Grothendieck's Existence Theorem gives $\mathbb{P}^n_R \simeq \mathfrak{Z}$. Since every morphism from the projective line to an abelian variety is constant, the scheme $\mathfrak{Z}_F$ must be contained in some fiber of $\Psi : V \to \mathrm{Alb}_{V/F}$. □

Proof of Theorem 5.1: Let $\lambda \in \mathrm{Alb}_{V/F}$ be the rational point such that the fiber $V_\lambda = \Psi^{-1}(\lambda)$ of the Albanese map contains the generic fiber of the family of subschemes $\mathfrak{Z} \subset \mathfrak{Y}$. The inclusion $\mathbb{P}^n_F \subset V_\lambda$ must be a connected component, because $\dim(\mathbb{P}^n_F) = \dim(V_\lambda)$ and V_λ is smooth. But the dualizing sheaf of V_λ is numerically trivial, whereas the dualizing sheaf of the projective n-space is anti-ample, contradiction. □

6. Cohomology and lattice points

We now assume that the polynomials $Q_0, \ldots, Q_n \in k[T_0, \ldots, T_n]$ used to define our Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$ have degree $d = 1$. After applying an automorphism of the projective n-space, we reduce to the situation $Q_i = T_i$. Tensoring the Euler sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1) \longrightarrow \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \longrightarrow \mathscr{E}_1 \longrightarrow 1$$

with $\mathscr{O}_{\mathbb{P}^n}(1)$ shows $\mathscr{E}_1 = \Theta_{\mathbb{P}^n}(-1)$, and the dual becomes $\mathscr{E}_{-1} = \Omega^1_{\mathbb{P}^n}(1)$. We now use the Bott formula (see [36], Section 1.1)

$$h^s(\Omega^r_{\mathbb{P}^n}(l)) = \begin{cases} \binom{n+l-r}{n}\binom{l-1}{r} & \text{if } s = 0; \\ \binom{r-l}{r}\binom{-l-1}{n-r} & \text{if } s = n; \\ 1 & \text{if } s = r \text{ and } l = 0; \\ 0 & \text{else} \end{cases}$$

to understand the groups $H^i(Y, \mathscr{O}_Y)$ better. Note that the binomial coefficient $\binom{x}{n} = x(x-1)\ldots(x-r+1)/n!$ is defined for natural numbers n and ring elements $x \in R$ whenever the denominator $n!$ is invertible.

Our computation crucially relies on the splitting type of the Frobenius push-forward $F_*(\mathscr{O}_{\mathbb{P}^n})$, which indeed splits by the Horrocks Criterion (see [36], Section 2.3). Understanding the splitting type involves seemingly innocent lattice point counts: For each integer t and l, define multiplicities $\mu_{t,l} \geq 0$ as the number of lattice points $(l_0, \ldots, l_n) \in \mathbb{Z}^{n+1}$ contained in the polytope $P_{t,l} \subset \mathbb{R}^{n+1}$ defined by

$$l_0 + \ldots + l_n = t - pl \quad \text{and} \quad 0 \leq l_0, \ldots, l_n \leq p - 1. \tag{8}$$

This is the intersection of an affine hyperplane with a hypercube. Clearly, the polytope is non-empty if and only if $0 \leq t - pl \leq (n+1)(p-1)$, and we have the recursion formula

$$\mu_{t+p,l+1} = \mu_{t,l}. \tag{9}$$

Note that the $\mu_{t,l} \geq 0$ also depends on n and p, but we neglect this dependence in notation. Applying a result of Achinger ([1], Theorem 2.1) to the toric variety $\mathbb{P}^n$, we get:

Proposition 6.1. *The Frobenius push-forward $F_*(\mathscr{O}_{\mathbb{P}^n}(t))$ splits as a sum of invertible sheaves, and the summand $\mathscr{O}_{\mathbb{P}^n}(l)$ appears with multiplicity $\mu_{t,l} \geq 0$. In other words, the splitting type is of the form*

$$(\underbrace{a, \ldots, a}_{\mu_{t,a}}, \cdots, \underbrace{b, \ldots, b}_{\mu_{t,b}})$$

starting with $a = \lceil (t - (n+1)(p-1))/p \rceil$ and ending with $b = \lfloor t/p \rfloor$.

In preparation for our analysis of $H^i(Y, \mathscr{O}_Y)$, we now express the cohomological invariants of certain locally free sheaves on $\mathbb{P}^n$ in terms of lattice points and binomial coefficients:

Proposition 6.2. *The cohomological invariants of the locally free sheaf $\mathscr{F}_{r,t} = \Lambda^r(F^*(\Omega^1_{\mathbb{P}^n}(1))) \otimes \mathscr{O}_{\mathbb{P}^n}(t)$ are given by the formula*

$$h^s(\mathscr{F}_{r,t}) = \begin{cases} \sum_l \mu_{t,l} \binom{n+l}{n} \binom{l+r-1}{r} & \text{if } s = 0; \\ \sum_l \mu_{t,l} \binom{-l}{r} \binom{-r-l-1}{n-r} & \text{if } s = n; \\ \mu_{t,-r} & \text{if } s = r; \\ 0 & \text{else.} \end{cases}$$

In particular, we have $h^0(\mathscr{F}_{r,t}) = 0$ for $0 \leq t \leq p-1$, and $h^s(\mathscr{F}_{0,0}) = 0$ for $s \geq 1$.

Proof. Set $\mathscr{F} = \mathscr{F}_{r,t}$. We have $H^s(\mathbb{P}^n, \mathscr{F}) = H^s(\mathbb{P}^n, F_*\mathscr{F})$ because the Frobenius map is affine, and the projection formula gives $F_*\mathscr{F} = \Omega^r_{\mathbb{P}^n}(r) \otimes F_*(\mathscr{O}_{\mathbb{P}^n}(t))$. Now combine the Bott formula for $h^s(\Omega^r_{\mathbb{P}^n}(l))$ and Achinger's description of $F_*(\mathscr{O}_{\mathbb{P}^n}(t))$ to get the general formula for $h^s(\mathscr{F})$. In the special case $s = 0$ this reduces to

$$h^0(\mathscr{F}_{r,t}) = \sum_l \mu_{t,l} \binom{n+l}{n} \binom{r+l-1}{r}.$$

The second binomial coefficient vanishes for $l \leq 0$. For $l \geq 1$ and $t \leq p-1$ the multiplicity $\mu_{t,l}$ is zero, because the polytope $P_{t,l} \subset \mathbb{R}^{n+1}$ in (8) becomes empty. Finally, we have $\mathscr{F}_{0,0} = \mathscr{O}_{\mathbb{P}^n}$ and thus $h^s(\mathscr{F}_{0,0}) = 0$ in all degrees $s \geq 1$. □

We now can express the cohomology groups of our Moret-Bailly family as follows:

Proposition 6.3. *Suppose that $p \geq n+1$. For every degree $i \geq 0$, the Leray–Serre spectral sequence for $\varphi : Y \to \mathbb{P}^n$ gives a natural identification*

$$H^i(Y, \mathscr{O}_Y) = \begin{cases} H^j(\mathbb{P}^n, \Lambda^j(\mathscr{E}_{-p})) & \text{if } i = 2j \text{ is even;} \\ H^j(\mathbb{P}^n, \Lambda^j(\mathscr{E}_{-p}) \otimes \mathscr{O}_{\mathbb{P}^n}(1)) & \text{if } i = 2j+1 \text{ is odd.} \end{cases} \tag{10}$$

Moreover, the dimension of these vector spaces are given by the formula

$$h^i(\mathscr{O}_Y) = \begin{cases} \mu_{0,-j} & \text{if } i = 2j \text{ is even;} \\ \mu_{1,-j} & \text{if } i = 2j+1 \text{ is odd,} \end{cases}$$

where $\mu_{t,l} \geq 0$ is the number of lattice points in the polytope $P_{t,l} \subset \mathbb{R}^{n+1}$ as in (8).

Proof. The assertion indeed holds for $i = 0$, because $\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n} = \Lambda^0(\mathscr{E}_{-p})$, and the only lattice point in $P_{0,0}$ has coordinates $l_0 = \ldots = l_n = 0$. Suppose from now on that $i \geq 1$.

The Leray-Serre spectral sequence is $E_2^{i,j} = H^i(\mathbb{P}^n, R^j\varphi_*\mathscr{O}_Y) \Rightarrow H^{i+j}(Y, \mathscr{O}_Y)$. For dimension reasons, the differentials $d_r : E_r^{i,j} \to E_r^{i+r,j-r+1}$ vanish on the E_r-pages whenever $r \geq n+1$. Since $p-1 \geq n$, they must also vanish on the pages with $2 \leq r \leq n$, according to Proposition 2.3. In turn, the associated graded on the abutment is

$$\operatorname{gr} H^i(Y, \mathscr{O}_Y) = \bigoplus_{r+s=i} H^s(\mathbb{P}^n, \Lambda^r(R^1\varphi_*(\mathscr{O}_Y)).$$

In our situation $R^1\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n}(1) \oplus \mathscr{E}_{-p} = F^*(\Omega^1(1)) \oplus \mathscr{O}_{\mathbb{P}^n}(1)$, which shows that $\Lambda^r(R^1\varphi_*(\mathscr{O}_Y)) = \mathscr{F}_{r,0} \oplus \mathscr{F}_{r-1,1}$, in the notation of Proposition 6.2. Here we set $\mathscr{F}_{-1,1} = 0$ for convenience. Thus

$$h^i(\mathscr{O}_Y) = \sum_{s=0}^{i} h^s(\mathscr{F}_{i-s,0}) + \sum_{s=0}^{i} h^s(\mathscr{F}_{i-s-1,1}). \tag{11}$$

The summands for $s = 0$ vanish by Proposition 6.2. Furthermore, for $1 \leq s \leq n-1$ we have

$$s \neq i-s \Rightarrow h^s(\mathscr{F}_{i-s,0}) = 0 \quad \text{and} \quad s \neq i-s-1 \Rightarrow h^s(\mathscr{F}_{i-s-1,1}) = 0.$$

In the boundary case $i = n$, the last summands $h^n(\mathscr{F}_{0,0})$ and $h^n(\mathscr{F}_{-1,1})$ vanish, the former by 6.2, the latter because $\mathscr{F}_{-1,1} = 0$. In turn, the sum (11) simplifies to

$$h^i(\mathscr{O}_Y) = \begin{cases} h^j(\mathscr{F}_{j,0}) & \text{if } i = 2j \text{ is even;} \\ h^j(\mathscr{F}_{j,1}) & \text{if } i = 2j+1 \text{ is odd.} \end{cases}$$

The formula for the vector space dimensions now follows from Proposition 6.2. We also see that the filtration on the abutment $H^i(Y, \mathscr{O}_Y)$ has merely one step, which gives the natural identification (10) of groups. □

Because of the relevance for the Picard scheme, we record:

Corollary 6.4. *We have $h^1(\mathscr{O}_Y) = n+1$, whereas $h^2(\mathscr{O}_Y)$ equals the number of lattice points $(l_0, \ldots, l_n)$ satisfying $l_0 + \ldots + l_n = p$ and $0 \leq l_0, \ldots, l_n \leq p-1$.*

Proof. One easily checks that terms $H^i(\mathbb{P}^n, \varphi_*(\mathscr{O}_Y))$, $i \geq 1$ and also the term $H^3(\mathbb{P}^n, R^1\varphi_*(\mathscr{O}_Y))$ on the E_2-page for the Leray–Serre spectral sequence with respect to $\varphi : Y \to \mathbb{P}^n$ vanish. In turn, the formula for $h^1(\mathscr{O}_Y)$ and $h^2(\mathscr{O}_Y)$ of the Proposition hold regardless to the assumption $p \geq n+1$. In particular, $h^1(\mathscr{O}_Y) = n+1$, because the only lattice points in the polytope $P_{1,0} \subset \mathbb{R}^{n+1}$ are the standard basis vectors. □

We now consider the case that gives varieties with $c_1 = 0$, which means

$$n = p-2, \quad \dim(Y) = 2n+1 = 2p-3 \quad \text{and} \quad \omega_Y = \mathscr{O}_Y.$$

With computer algebra [33], we computed the cohomological invariants for the first six primes in the following table:

p	n	$\dim(Y)$	$h^0(\mathscr{O}_Y),\ldots,h^n(\mathscr{O}_Y)$
2	0	1	1
3	1	3	$1, 2$
5	3	7	$1, 4, 52, 68$
7	5	11	$1, 6, 786, 1251, 6891, 7872$
11	9	19	$1, 10, 167950, 293830, 18480520, 25109950,$ $251849140, 296659645, 859743835, 905642810$
13	11	23	$1, 12, 2496132, 4457256, 825038490, 1149834280, 27258578260,$ $33480335274, 223425722070, 250522227132, 616161367152,$ 639330337978

Note that the running time for $h^i(\mathscr{O}_Y)$ at the prime $p = 13$ and in degree $i = 12$ was about three days.

7. Cohomology and weights

The goal of this section is gain further control on the cohomology of the Moret-Bailly family $Y = (A\times\mathbb{P}^n)/H$, in particular for $H^2(Y,\mathscr{O}_Y)$ and $H^1(Y,\Theta_Y)$, by using automorphisms of the superspecial abelian variety A and their induced representations on cohomology. This is best formulated with the machinery of weights, and requires some preparation. Fix some integer $l\geq 1$, and consider

$$G = \mu_l = \mathbb{G}_m[l] = \operatorname{Spec}\mathbb{Z}[T]/(T^l-1),$$

viewed as a family of finite diagonalizable group schemes over the ground ring $R=\mathbb{Z}$. Let S be a scheme endowed with trivial G-action. Recall that by [14], Exposé I, Proposition 4.7.3, a *G-linearization* for a quasicoherent sheaf $\mathscr{F}$ on S is nothing but a *weight decomposition* $\mathscr{F} = \bigoplus_w \mathscr{F}_w$, where w runs over the character group $\mathbb{Z}/l\mathbb{Z} = \underline{\operatorname{Hom}}(G,\mathbb{G}_m)$. Such characters are also known as *weights*. A weight w is called *trivial* if $\mathscr{F}_w = 0$. We say that the sheaf $\mathscr{F}$ is *pure of weight* w_0 if all other weights $w\neq w_0$ are trivial.

For each ring A, the group elements $\zeta\in G(A)\subset A^\times$ act on $\mathscr{F}_w\otimes_R A$ via multiplication by $\lambda = w(\zeta)$. Note that for every base-change $a:\operatorname{Spec}(k)\to S$ for some field k containing a primitive l-th root of unity ζ, the weight decomposition for $\mathscr{F}$ becomes the *eigenspace decomposition* on the vector space $V=\mathscr{F}(a)$ for the automorphism $\zeta: V\to V$ with respect to the eigenvalues $\lambda = w(\zeta)$. Also note that G-linearizations for the sheaf $\mathscr{F}$ correspond to G-actions on the finite S-scheme stemming from the sheaf of dual numbers $\mathscr{A} = \mathscr{O}_S\oplus\mathscr{F}$, or the vector S-scheme coming from $\mathscr{A} = \operatorname{Sym}^\bullet(\mathscr{F})$.

Now assume we are over a ground field k of characteristic $p>0$. Since μ_l is already defined over $\mathbb{F}_p$ we have a canonical identification $\mu_l^{(p)} = \mu_l$, and the relative Frobenius map $\mu_l\to\mu_l^{(p)} = \mu_l$ given by $\zeta\mapsto\zeta^p$ induces multiplication by p on the character group. In turn, the Frobenius pullback $\mathscr{F}^{(p)}$ has two canonical G-linearizations: One stemming from $\mu_l^{(p)} = \mu_l$ with $(\mathscr{F}^{(p)})_w = (\mathscr{F}_w)^{(p)}$, the other via the Frobenius map, such that $(\mathscr{F}^{(p)})_w = (\mathscr{F}_{pw})^{(p)}$. It turns out that the latter is more important for us.

In what follows we assume that $G = \mu_l$ acts on an abelian variety A of dimension $g \geq 1$ such that the induced representation on $\operatorname{Lie}(A)$ is pure of weight $w_0 \in \mathbb{Z}/l\mathbb{Z}$, and that there is a μ_l-equivariant principal polarization $A \to P$, for the dual abelian variety $P = \operatorname{Pic}^0_{A/k}$ with the induced μ_l-action. Using the equivariant bijections $H^1(A, \mathscr{O}_A) = \operatorname{Lie}(P) \to \operatorname{Lie}(A)$ we immediately obtain:

Lemma 7.1. *The induced μ_l-representation on $H^1(A, \mathscr{O}_A)$ is also pure of weight $w = w_0$.*

Now suppose additionally that A is superspecial, and consider the Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$ formed with the homogeneous polynomials $Q_i = T_i$, as in Section 6. Since the induced action of $G = \mu_l$ on $\operatorname{Lie}(A)$ is pure, each copy $\alpha_p \subset A$ is normalized by G, and we get induced actions on the quotients A/α_p.

Consider the diagonal G-action on $X = A \times \mathbb{P}^n$, with trivial action on the second factor. This also can be seen as an action of the relative group scheme $G \times \mathbb{P}^n = \mu_{l,\mathbb{P}^n}$ that normalizes the action of the family $H \subset X$ of height-one group schemes with $\operatorname{Lie}_{H/\mathbb{P}^n} = \mathscr{O}_{\mathbb{P}^n}(-1)$, and thus induces a G-action on our Moret-Bailly family $Y = X/H$. The structure morphism $\varphi : Y \to \mathbb{P}^n$ is equivariant, with trivial G-action on the base. By Proposition 1.2 we have a four-term exact sequence

$$0 \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-1) \longrightarrow \operatorname{Lie}(A) \otimes_k \mathscr{O}_{\mathbb{P}^n} \longrightarrow \operatorname{Lie}_{Y/k} \longrightarrow \mathscr{O}_{\mathbb{P}^n}(-p) \longrightarrow 0$$

The cokernel for the inclusion on the left is the sheaf $\mathscr{E}_1$, and the above yields the short exact sequence $0 \to \mathscr{E}_1 \to \operatorname{Lie}_{Y/k} \to \mathscr{O}_{\mathbb{P}^n}(-p) \to 0$ of sheaves with G-linearizations. We saw in the proof for Proposition 3.2 that $\operatorname{Ext}^1(\mathscr{O}_{\mathbb{P}^n}(-p), \mathscr{E}_1) = 0$. By [14], Exposé I, Proposition 4.7.4 we may choose a splitting that respects linearizations. In turn, $\operatorname{Lie}_{Y/\mathbb{P}^n} = \mathscr{E}_1 \oplus \mathscr{O}_{\mathbb{P}^n}(-p)$ as sheaves with G-linearization.

Lemma 7.2. *In the above setting, the summand $\mathscr{E}_1$ is pure of weight $w = w_0$, whereas $\mathscr{O}_{\mathbb{P}^n}(-p)$ is pure of weight pw_0.*

Proof. The sheaf $\operatorname{Lie}(A) \otimes_k \mathscr{O}_{\mathbb{P}^n}$ is pure of weight $w = w_0$, hence the same holds for the subsheaf $\mathscr{O}_{\mathbb{P}^n}(-1)$ and the quotient sheaf $\mathscr{E}_1$. The relative Frobenius map $A \to A^{(p)}$ becomes G-equivariant, provided that we take the induced G-action via the Frobenius map $\mu_l \to \mu_l^{(p)} = \mu_l$. In turn, the $\mathscr{O}_{\mathbb{P}^n}(-p) \subset \operatorname{Lie}(A^{(p)}) \otimes_k \mathscr{O}_{\mathbb{P}^n}$ is pure of weight $w = pw_0$. □

We now make a similar analysis for the higher direct images $R^i\varphi_*(\mathscr{O}_Y)$, which also come with induced G-linearizations. To understand their weight decompositions, it suffices to treat the case $i = 1$, according to Proposition 2.3. Now we use the canonical projection $Y = (A \times \mathbb{P}^n)/H \to A/A[F] \times \mathbb{P}^n = A^{(p)} \times \mathbb{P}^n$. This map becomes G-equivariant if the right-hand side is endowed with the μ_l-action coming from the Frobenius map.

According to Proposition 2.1, we have a four-term exact sequence

$$0 \longrightarrow \mathscr{K} \longrightarrow H^1(A, \mathscr{O}_A)^{(p)} \otimes_k \mathscr{O}_{\mathbb{P}^n} \longrightarrow R^1\varphi_*(\mathscr{O}_Y) \longrightarrow \mathscr{K}^{(p)} \longrightarrow 0, \tag{12}$$

where $\mathscr{K} = \mathscr{H}om(\mathscr{E}_1, \mathscr{O}_{\mathbb{P}^n})$. The cokernel for the inclusion on the left is invertible and of degree one, and we identify it with $\mathscr{O}_{\mathbb{P}^n}(1)$. In turn, the above sequence becomes the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^n}(1) \to R^1\varphi_*(\mathscr{O}_Y) \to \mathscr{E}_{-p} \to 0$. We already observed in the proof for Proposition 3.2 that $\operatorname{Ext}^1(\mathscr{E}_{-p}, \mathscr{O}_{\mathbb{P}^n}(1)) = 0$, so there is a direct sum decomposition $R^1\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n}(1) \oplus \mathscr{E}_{-p}$. of sheaves with G-linearizations.

Lemma 7.3. *In the above setting, the summand $\mathscr{O}_P(1)$ is pure of weight $w = pw_0$, whereas $\mathscr{E}_{-p}$ is pure of weight p^2w_0.*

Proof. The Lie algebra $\mathfrak{a} = \operatorname{Lie}(A)$ is pure of weight $w = w_0$. According to Lemma 7.1, the same holds for $H^1(A, \mathscr{O}_A)$. In turn, the Frobenius pullback $H^1(A, \mathscr{O}_A)^{(p)}$ is pure of weight $w = pw_0$, since we use the action stemming from the Frobenius map $\mu_l \to \mu_l^{(p)}$. We now proceed as for Lemma 7.2. □

The G-action on the Moret-Bailly family Y induces G-representations on Hodge groups $H^s(Y, \Omega^r_Y)$ and tangent cohomology $H^s(Y, \Theta_Y)$. We now compute the weights in two relevant cases:

Proposition 7.4. *With respect to the induced G-representation, the following holds:*

(i) *The group $H^2(Y, \mathscr{O}_Y)$ is pure of weight $w = p^2w_0$.*
(ii) *For the tangent cohomology $H^1(Y, \Theta_Y)$ all weights of the form $w = mw_0$ with coefficient $m \notin \{0, p+1, p^2+1, p^2+p\}$ are trivial.*

Proof. According to Proposition 6.3, the Leray–Serre spectral sequence $\varphi : Y \to \mathbb{P}^n$ induces a canonical identification $H^2(Y, \mathscr{O}_Y) = H^1(\mathbb{P}^n, \mathscr{E}_{-p})$. By Lemma 7.3, the sheaf $\mathscr{E}_{-p}$ is pure of weight $w = p^2w_0$, and assertion (i) follows.

Dualizing $0 \to \varphi^*\Omega^1_{\mathbb{P}^n/k} \to \Omega^1_{Y/k} \to \varphi^* \operatorname{Lie}^\vee_{Y/\mathbb{P}^n} \to 0$ yields a short exact sequence $0 \to \varphi^* \operatorname{Lie}_{Y/\mathbb{P}^n} \to \Theta_Y \to \varphi^*\Theta_{\mathbb{P}^n} \to 0$, which gives an exact sequence

$$H^1(Y, \varphi^* \operatorname{Lie}_{Y/\mathbb{P}^n}) \longrightarrow H^1(Y, \Theta_Y) \longrightarrow H^1(Y, \varphi^*\Theta_{\mathbb{P}^n}).$$

The term on the right is pure of weight $w = 0$, because $\varphi : Y \to \mathbb{P}^n$ is G-equivariant, with trivial action on $\mathbb{P}^n$. The term on the left sits in a short exact sequence

$$H^1(\mathbb{P}^n, \operatorname{Lie}_{Y/\mathbb{P}^n}) \longrightarrow H^1(Y, \varphi^*(\operatorname{Lie}_{Y/\mathbb{P}^n})) \longrightarrow H^0(\mathbb{P}^n, R^1\varphi_*(\mathscr{O}_Y) \otimes \operatorname{Lie}_{Y/\mathbb{P}^n})$$

which comes from the Leray–Serre spectral sequence for $\varphi : Y \to \mathbb{P}^n$ and the projection formula. Now recall that

$$R^1\varphi_*(\mathscr{O}_Y) = \mathscr{O}_{\mathbb{P}^n}(1) \oplus \mathscr{E}_{-p} \quad \text{and} \quad \operatorname{Lie}_{Y/\mathbb{P}^n} = \mathscr{E}_1 \oplus \mathscr{O}_{\mathbb{P}^n}(-p) \tag{13}$$

as sheaves with G-linearizations. One easily computes $H^1(\mathbb{P}^n, \operatorname{Lie}_{\mathbb{P}^n/S}) = 0$ using the short exact sequence $0 \to \mathscr{O}_{\mathbb{P}^n}(-1) \to \mathscr{O}_{\mathbb{P}^n}^{\oplus n+1} \to \mathscr{E}_1 \to 0$. In turn, it suffices to understand the possible weights in $H^0(\mathbb{P}^n, \mathscr{F})$ for the sheaf $\mathscr{F} = R^1\varphi_*(\mathscr{O}_Y) \otimes \operatorname{Lie}_{Y/\mathbb{P}^n}$. For this we compute the weights occurring in each of the summands

$$\mathscr{O}_{\mathbb{P}^n}(1) \otimes \mathscr{E}_1, \quad \mathscr{O}_{\mathbb{P}^n}(1) \otimes \mathscr{O}_{\mathbb{P}^n}(-p), \quad \mathscr{E}_{-p} \otimes \mathscr{E}_1 \quad \text{and} \quad \mathscr{E}_{-p} \otimes \mathscr{O}_{\mathbb{P}^n}(-p)$$

inside $\mathscr{F}$. These are pure of weight $w = mw_0$, where the coefficient m is a some of the form $p+1$ and $p+p$ and p^2+1 and p^2+p, respectively. The second case does not contribute, because the global sections for $\mathscr{O}_{\mathbb{P}^n}(1-p)$ vanish. Assertion (ii) follows. □

We now deduce a crucial fact that can be formulated without the machinery of weights:

Corollary 7.5. *Suppose $p \neq 2$. Then the sign involution on the superspecial abelian variety A induces an action of the multiplicative group $G = \{\pm 1\}$ on our Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$, and the induced representation on the cohomology groups $H^2(Y, \mathscr{O}_Y)$ and $H^1(Y, \Theta_Y)$ is multiplication by $\lambda = -1$ and $\lambda = 1$, respectively.*

Proof. We may assume that k is algebraically closed, and regard the abstract group $G = \{\pm 1\}$ as the diagonalizable group scheme $G = \mu_l$ with $l = 2$. Write $A = E_1 \times \ldots \times E_g$ as a product of supersingular elliptic curve, and use the canonical identification $E_i = \operatorname{Pic}_{E_i/k}$ to obtain an equivariant principal polarization.

The induced G-representation on $\operatorname{Lie}(A)$ is multiplication by $\lambda = -1$. This is pure of weight $w_0 = 1$, seen as an element of the character group $\mathbb{Z}/2\mathbb{Z}$. As discussed above, we get an induced action on Y, such that the quotient map $\epsilon : A \times \mathbb{P}^n \to Y$ is equivariant. According to Proposition 7.4, the induced representation on $H^2(Y, \mathscr{O}_Y)$ is pure of weight $w = p^2 w_0 = 1$, because $p \equiv 1$ modulo $l = 2$. In contrast, the only possible non-zero weights on $H^1(Y, \Theta_Y)$ are $w = m w_0$, with coefficient $m \equiv 0$ modulo $l = 2$. In turn, $H^1(Y, \Theta_Y)$ is pure of weight $w = 0$. □

8. Non-existence of formal liftings

We continue to study our Moret–Bailly family $Y = (A \times \mathbb{P}^n)/H$ over the ground field k of characteristic $p > 0$, formed with a superspecial abelian variety A of dimension $g = n + 1$.

Recall that one says that the proper scheme Y *formally lifts to characteristic zero* if there is a proper flat morphism $\mathfrak{Y} \to \operatorname{Spf}(R)$ of formal schemes, together with an identification $\mathfrak{Y} \otimes_R k = Y$, where R is some complete local noetherian ring with residue field $R/\mathfrak{m}_R = k$ such that the canonical map $\mathbb{Z} \to R$ is injective. Note that we may assume that R is integral and one-dimensional, by passing to the residue class ring for some suitable prime ideal.

Note that A, like any abelian variety, projectively lifts to characteristic zero, but for $g \geq 2$ there are formal liftings $\mathfrak{A} \to \operatorname{Spf}(R)$ to characteristic zero that are not algebraizable.

Proposition 8.1. *Suppose $p \geq 3$. If the Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$ formally lifts to characteristic zero, it also projectively lifts to characteristic zero.*

From this we immediately get the main result of this paper, by using Corollary 3.4 and Theorem 5.1:

Theorem 8.2. *Suppose that $p \geq 3$, and that the superspecial abelian variety A has dimension $g = p - 1$. Then the Moret-Bailly family $Y = (A \times \mathbb{P}^n)/H$ is of dimension $2p - 3$, has dualizing sheaf $\omega_Y = \mathscr{O}_Y$, and does not formally lift to characteristic zero.*

In the above situation, let $f : \mathfrak{Y} \to \operatorname{Spc}(R)$ be the miniversal deformation of the Moret–Bailly family $Y = (A \times \mathbb{P}^n)/H$. Then R is a complete local noetherian ring, where the vector space $R/(\mathfrak{m}_R + pR)$ has dimension $h^1(\Theta_Y)$. From Theorem 8.2, we see that the canonical map $\mathbb{Z} \to R$ factors over $\mathbb{Z}/p^\nu\mathbb{Z}$, for some $\nu \geq 1$. It would be interesting to determine this exponent.

The proof for Proposition 8.1 requires some preparation. Suppose we have a formal lifting $\mathfrak{Y} \to \operatorname{Spf}(R)$. Refining the descending chain $\mathfrak{m}_R^j$, we get a descending chain of ideals $\mathfrak{a}_i$ with $\operatorname{length}(\mathfrak{a}_i/\mathfrak{a}_{i+1}) = 1$, such that $R = \varprojlim R_i$, where $R_i = R/\mathfrak{a}_i$. Setting $Y_i = \mathfrak{Y} \otimes_R R_i$, we get an increasing sequence $Y = Y_0 \subset Y_1 \subset \ldots$, where each Y_i is a proper flat R_i-scheme, and the comparison maps $Y_{i-1} \to Y_i \otimes_{R_i} R_{i-1}$ are isomorphisms. In fact, the formal scheme $\mathfrak{Y}$ is nothing but the resulting inverse system $(Y_i)_{i\geq 0}$ of R-schemes.

The group $G = \{\pm 1\}$ acts on the abelian variety A via the sign involution, and induces an action on the Moret–Bailly family $Y = (A \times \mathbb{P}^n)/H$. Consider the cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathfrak{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \longrightarrow & \operatorname{Spec}(R). \end{array}$$

With respect to the trivial G-actions on k and R, the maps on the left and the bottom are equivariant.

Proposition 8.3. *There is a G-action on the formal scheme $\mathfrak{Y}$ making the above diagram equivariant.*

Proof. For this we use Rim's results about equivariant structures on versal deformations [41], in the form obtained in [45]. Choose a Cohen ring Λ with residue field k. Recall that this a complete discrete valuation ring with maximal ideal $\mathfrak{m}_\Lambda = p\Lambda$. Let (Art_Λ) be the category of local Artin rings A over Λ with residue field k, and $\mathscr{F} \to (\mathrm{Art}_\Lambda)^{\mathrm{op}}$ be the category fibered in groupoids whose fiber category over A comprises the pairs $\xi = (X, \varphi)$, where X is a proper flat A-scheme and $\varphi : Y \to X \otimes_R k$ is an isomorphism. This is a *deformation category* in the sense of Talpo and Vistoli [48]. To conform with [45], the symbol A, which otherwise denotes our superspecial abelian variety, is here also used for local Artin rings; this should not cause any confusion.

We now construct by induction on $i \geq 0$ compatible G-actions Y_i, making the structure morphism $Y_i \to \operatorname{Spec}(R)$ equivariant. For $i = 0$ we choose the given action on $Y_0 = Y$. Now suppose that we already defined the action on Y_i. Using the notation from [45], set $A = R_i$ and $A' = R_{i+1}$. Write $\xi \in \mathscr{F}(A)$ for the given family $Y_i = \mathfrak{Y} \otimes_R A$, and $\operatorname{Lif}(\xi, A')$ for the set of isomorphism classes $[f]$ of all cartesian morphisms $f : \xi \to \xi'$ over the inclusion $\operatorname{Spec}(A) \subset \operatorname{Spec}(A')$, in the fibered category $\mathscr{F}$. This set comes with a G-action, namely $\sigma \cdot [f] = [f \circ \sigma^{-1}]$. The family $Y_{i+1} = \mathfrak{Y} \otimes_R A'$ yields an element in $L = \operatorname{Lif}(\xi, A')$, which thus is non-empty. Furthermore, it carries the structure of a torsor for the *tangent space* $T = H^1(Y, \Theta_Y)$ for the deformation category. It is also endowed with a group of operators G, meaning that the action

$$L \times T \longrightarrow L, \quad (\xi, t) \longmapsto \xi \cdot t \tag{14}$$

satisfies ${}^\sigma(\xi \cdot t) = {}^\sigma\xi \cdot {}^\sigma t$. In turn, we obtain a cohomology class $[L] \in H^1(G, T)$. This cohomology group vanishes, because $p \neq 2$, and it follows that L contains a

G-fixed point. See loc. cit. Section 2 for details. By Corollary 7.5, the G-action on T is trivial, and (14) ensures that the G-action on L is trivial as well.

In particular, the object $\xi' \in \mathscr{F}(A')$ corresponding to $Y_{i+1} = \mathfrak{Y} \otimes_R A'$ has G-fixed isomorphism class. By loc. cit., Theorem 3.3 the obstruction for extending the G-action from Y_i to Y_{i+1} lies in $H^2(G, \mathfrak{a}_i/\mathfrak{a}_{i+1} \otimes_k \mathrm{Aut}_{\xi_0}(\xi_{k[\epsilon]}))$. Again this group must vanishes, because $p \neq 2$, and we conclude that the action on Y_i extends to an action on Y_{i+1}. □

Recall that a *G-linearized invertible sheaf* on Y is an invertible sheaf $\mathscr{L}$, together with a G-linearization, that is, a G-action on the line bundle $L = \mathrm{Spec}\,\mathrm{Sym}^\bullet(\mathscr{L}^\vee)$ such that the structure morphism $L \to Y$ is equivariant. Write $\mathrm{Pic}(Y, G)$ for the abelian group of isomorphism classes for G-linearized invertible sheaves. This can also be viewed as the equivariant cohomology group $H^1(Y, G; \mathscr{O}_Y^\times)$, or the Picard group of the quotient stack $[Y/G]$.

Proposition 8.4. *The restriction map* $\mathrm{Pic}(\mathfrak{Y}, G) \to \mathrm{Pic}(Y, G)$ *is surjective.*

Proof. Let $\mathscr{L}_0$ be a G-linearized invertible sheaf on $Y_0 = Y$. We show by induction on $i \geq 0$ that it extends to Y_i. This is obvious for $i = 0$. Suppose now that we have a linearized extension $\mathscr{L}_i$ on Y_i. The ideal sheaf $\mathscr{I}$ for the closed embedding $Y_i \subset Y_{i+1}$ has square zero, and is isomorphic to $\mathscr{O}_Y$ by flatness. In turn, we have a short exact sequence $0 \to \mathscr{O}_Y \to \mathscr{O}_{Y_i}^\times \to \mathscr{O}_{Y_{i-1}}^\times \to 1$, which gives an exact sequence

$$0 \longrightarrow H^1(Y, \mathscr{O}_Y) \longrightarrow \mathrm{Pic}(Y_{i+1}) \longrightarrow \mathrm{Pic}(Y_i) \xrightarrow{\partial} H^2(Y, \mathscr{O}_Y).$$

By naturality of cohomology, this sequence is equivariant with respect to the induced G-actions. The isomorphism class of $\mathscr{L}_i$ is G-invariant, so Corollary 7.5 gives $\partial(\mathscr{L}_i) = -\partial(\mathscr{L}_i)$. With $p \neq 2$ we infer that the obstruction $\partial(\mathscr{L}_i)$ vanishes, hence $\mathscr{L}_i$ extends to some invertible sheaf on Y_{i+1}.

It remains to choose an extension that admits a linearization. To achieve this we use the results form [45], as in the previous proof. Let L be the set of isomorphism classes for pairs $(\mathscr{L}', \varphi')$, where $\mathscr{L}'$ is invertible on Y_{i+1} and $\varphi' : \mathscr{L}_i \to \mathscr{L}'|Y_i$ is an isomorphism, and define $T = H^1(Y, \mathscr{O}_Y)$. Then L carries the structure of an T-torsor with group of operators G, giving a cohomology class $[L] \in H^1(G, T)$. This cohomology group vanishes because $p \neq 2$. In turn, we can choose a lifting $\mathscr{L}_{i+1}$ whose isomorphism class is G-fixed. The group of automorphisms of $\mathscr{L}_{i+1}$ restricting to the identify on $\mathscr{L}_i$ is the vector space $V = \mathrm{Hom}(\mathscr{L}_{i+1}, \mathscr{O}_Y) = \mathrm{Hom}(\mathscr{L}_0, \mathscr{O}_Y)$. According to [45], Theorem 1.2 the obstruction against the existence of a G-linearization lies in the cohomology group $H^2(G, \mathfrak{a}_i/\mathfrak{a}_{i+1} \otimes_k V)$, which again vanishes. Summing up, the linearized sheaf $\mathscr{L}_i$ extends to Y_{i+1}. □

Proof for Proposition 8.1. Fix a formal lifting $\mathfrak{Y} \to \mathrm{Spf}(R)$ to characteristic zero, and choose a very ample invertible sheaf $\mathscr{L}$ on Y. Replacing $\mathscr{L}$ by $\mathscr{L} \otimes \sigma^*(\mathscr{L})$, where $\sigma \in G$ is the generator, we may assume that the isomorphism class is G-invariant. According to [20], Theorem 5.2.1 there is a spectral sequence

$$H^r(G, H^s(X, \mathscr{O}_X^\times)) \Longrightarrow H^{r+s}(X, G, \mathscr{O}_X^\times).$$

The resulting 5-term exact sequence shows that the obstruction for the existence of a G-linearization on $\mathscr{L}$ lies in the elementary abelian group $H^2(G, k^\times) = k^\times/k^{2\times}$. Thus $\mathscr{O}_Y(1) = \mathscr{L}^{\otimes 2}$ can be endowed with a G-linearization, and is very ample.

In light of Proposition 8.3, the G-action on Y extends to $\mathfrak{Y}$, and by Proposition 8.4 there is some invertible sheaf $\mathscr{O}_{\mathfrak{Y}}(1)$ restricting to $\mathscr{O}_Y(1)$. According to Grothendieck's Existence Theorem, $\mathfrak{Y}$ is the formal completion of some flat projective R-scheme ([22], Theorem 5.4.5). In turn, our Moret–Bailly family Y projectively lifts to characteristic zero. □

References

[1] P. Achinger: A characterization of toric varieties in characteristic p. Int. Math. Res. Not. IMRN 2015, 6879–6892.

[2] P. Achinger, M. Zdanowicz: Non-liftable Calabi-Yau varieties in characteristic $p \geq 5$ Preprint, arXiv:1710.08202.

[3] M. Artin: Algebraization of formal moduli I. In: D. Spencer, S. Iyanaga (eds.), Global Analysis, pp. 21–71. Univ. Tokyo Press, Tokyo, 1969.

[4] M. Artin, A. Grothendieck, J.-L. Verdier (eds.): Théorie des topos et cohomologie étale des schémas (SGA 4) Tome 3. Springer, Berlin, 1973.

[5] A. Beauville: Variétés Kähleriennes dont la première classe de Chern est nulle. J. Differential Geom. 18 (1983), 755–782.

[6] P. Berthelot, L. Breen, W. Messing: Théorie de Dieudonné cristalline. II. Springer, Berlin, 1982.

[7] F. Bogomolov: On the decomposition of Kähler manifolds with a trivial canonical class. Math. USSR Sbornik 22 (1974), 580–583.

[8] F. Bogomolov: Hamiltonian Kählerian manifolds. Dokl. Akad. Nauk SSSR 243, 1101–1104 (1978).

[9] J. Brillhart, P. Morton: Class numbers of quadratic fields, Hasse invariants of elliptic curves, and the supersingular polynomial J. Number Theory 106 (2004), 79–111.

[10] S. Cynk, D. van Straten: Small resolutions and non-liftable Calabi-Yau threefolds. Manuscripta Math. 130 (2009), 233–249.

[11] S. Cynk, M. Schütt: Non-liftable Calabi–Yau spaces. Ark. Mat. 50 (2012), 23–40.

[12] P. Deligne: Relèvement des surfaces K3 en caractéristique nulle. In: J. Giraud, L. Illusie, M. Raynaud (eds.), Surface algébrique, pp. 58–79. Springer, Berlin, 1981.

[13] M. Demazure, P. Gabriel: Groupes algébriques. Masson, Paris, 1970.

[14] M. Demazure, A. Grothendieck (eds.): Schemas en groupes I (SGA 3 Tome 1). Springer, Berlin, 1970.

[15] M. Demazure, A. Grothendieck (eds.): Schémas en groupes II (SGA 3 Tome 2). Springer, Berlin, 1970.

[16] J. Eagon, D. Northcott: Ideals defined by matrices and a certain complex associated with them. Proc. Roy. Soc. London 269 (1962), 188–204.

[17] T. Ekedahl: Foliations and inseparable morphisms. In: S. Bloch (ed.), Algebraic geometry, Bowdoin, 1985, pp. 139–149. Amer. Math. Soc., Providence, RI, 1987.

[18] T. Ekedahl, N. Shepherd-Barron: Tangent lifting of deformations in mixed characteristic. J. Algebra 291 (2005), 108–128.

[19] G. Faltings, C.-L. Chai: Degeneration of abelian varieties. Ergeb. Math. Grenzgebiete (3) 22. Springer, Berlin, 1990

[20] A. Grothendieck: Sur quelques points d'algèbre homologique. Tohoku Math. J. 9 (1957), 119–221.

[21] A. Grothendieck, J. Dieudonné: Éléments de géométrie algébrique I: Le langage des schémas. Springer, Berlin, 1970.

[22] A. Grothendieck: Éléments de géométrie algébrique III: Étude cohomologique des faiscaux cohérent. Publ. Math., Inst. Hautes Étud. Sci. 11 (1961).

[23] A. Grothendieck: Les schemas de Hilbert. Seminaire Bourbaki, Exp. 221 (1961).

[24] A. Grothendieck: Les schémas de Picard: Propriétés générales. Séminaire Bourbaki, Exp. 236 (1962), 221–243.

[25] A. Grothendieck: Un théorème sur les homomorphismes de schémas abéliens. Invent. Math. 2 (1966), 59–78.

[26] A. Grothendieck (ed.): Groupe de monodromie en géométrie algébrique (SGA 7 I). Springer, Berlin, 1972.

[27] M. Hirokado: A non-liftable Calabi–Yau threefold in characteristic 3. Tohoku Math. J. 51 (1999), 479–487.

[28] M. Hirokado, H. Ito, N. Saito: Calabi–Yau threefolds arising form fibre products of rational quasi-elliptic surfaces, I. Ark. Mat. 45 (2007), 279–296.

[29] M. Hirokado, H. Ito, N. Saito: Calabi-Yau threefolds arising from fiber products of rational quasi-elliptic surfaces. II. Manuscripta Math. 125 (2008), 325–343.

[30] M. Kashiwara, P. Schapira: Categories and sheaves. Springer, Berlin, 2006.

[31] N. Katz: Serre–Tate local moduli. In: J. Giraud, L. Illusie, M. Raynaud (eds.), Surfaces Algebriques, pp. 138–202 Springer, Berlin, 1981.

[32] Y. Kawamata: Unobstructed deformations. J. Algebraic Geom. 1 (1992), 271-291.

[33] W. Bosma, J. Cannon, C. Fieker, A. Steel (eds.): Handbook of Magma functions, Edition 2.16 (2010).

[34] L. Moret-Bailly: Familles de courbes et de varietes abeliennes sur P^1. In: L. Szpiro (ed.), Séminaire sur les pinceaux de courbes de genre au moins deux, pp. 109–140. Astérisque 86 (1981).

[35] P. Norman, F. Oort: Moduli of abelian varieties. Ann. of Math. 112 (1980), 413–439.

[36] C. Okonek, M. Schneider, H. Spindler: Vector bundles on complex projective spaces. Birkhäuser, Boston, Mass., 1980.

[37] M. Olsson: Algebraic spaces and stacks. American Mathematical Society, Providence, RI, 2016.

[38] F. Oort: Which abelian surfaces are products of elliptic curves? Math. Ann. 214 (1975), 35–47.

[39] Z. Patakfalvi, M. Zdanowicz: On the Beauville–Bogomolov decomposition in characteristic $p \geq 0$. Preprint, arXiv:1912.12742.

[40] Z. Ran: Deformations of manifolds with torsion or negative canonical bundle. J. Algebraic Geom. 1 (1992), 279–291.

[41] D. Rim: Equivariant G-structure on versal deformations. Trans. Amer. Math. Soc. 257 (1980), 217–226.

[42] C. Schoen: Desingularized fiber products of semi-stable elliptic surfaces with vanishing third Betti number. Compos. Math. 145 (2009), 89–111.

[43] S. Schröer: The T^1-lifting Theorem in positive characteristics. J. Algebraic Geom. 12 (2003), 715–739.

[44] S. Schröer: Some Calabi–Yau threefolds with obstructed deformations over the Witt vectors. Compositio Math. 140 (2004), 1579–1592.

[45] S. Schröer, Y. Takayama: On equivariant formal deformation theory. Rend. Circ. Mat. Palermo. 26 (2018), 1113–1122.

[46] J.-P. Serre: Universal morphisms and Albanese varieties. Variétés de Picard. Sém. C. Chevalley 3, 1960.

[47] T. Shioda: Supersingular $K3$ surfaces. In: K. Lonsted (ed.), Algebraic geometry, pp. 564–591. Springer, Berlin, 1979.

[48] M. Talpo, A. Vistoli: Deformation theory from the point of view of fibered categories. In: G. Farkas and I. Morrison (eds.), Handbook of moduli. Vol. III. International Press, Somerville, MA; Higher Education Press, Beijing, 2013.

[49] J. Tate: Algorithm for determining the type of a singular fiber in an elliptic pencil. In: B. Birch, W. Kuyk (eds.), Modular functions of one variable IV, pp. 33–52. Springer, Berlin, 1975.

[50] G. Tian: Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson-Weil metric. In: S. Yau (ed.), Mathematical aspects of string theory, pp. 629–646. World Sci. Publishing, Singapore, 1987.

[51] A. Todorov: Applications of the Kähler–Einstein–Calabi–Yau metric to moduli of $K3$ surfaces. Invent. Math. 61 (1980), 251–265.
[52] B. Totaro: The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen–Macaulay. J. Algebraic Geom. 28 (2019), 751–771.
[53] M. Zdanowicz: Arithmetically rigid schemes via deformation theory of equivariant vector bundles. Preprint, arXiv:1810.10885, to appear in Math. Z.

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