DISCRIMINANT GROUPS OF WILD CYCLIC QUOTIENT SINGULARITIES

DINO LORENZINI AND STEFAN SCHRÖER

21 January 2021

Abstract. Let p be prime. We describe explicitly the resolution of singularities of several families of wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities in dimension two, including families that generalize the quotient singularities of type E_6, E_7, and E_8 from $p = 2$ to arbitrary characteristics. We prove that for odd primes, any power of p can appear as the determinant of the intersection matrix of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. We also provide evidence towards the conjecture that in this situation one may choose the wild action to be ramified precisely at the origin.

Contents

Introduction

The goal of this paper is to study *wild quotient singularities* which arise from actions of $G := \mathbb{Z}/p\mathbb{Z}$ on the formal power series ring $A = k[[u, v]]$ when k is an algebraically closed field of characteristic $p > 0$. Here the term "wild" refers to the fact that the order of the group G is not coprime to the characteristic exponent of the ground field k. The resulting quotient singularity is the ring of invariants A^G or, more precisely, the closed point of $\mathrm{Spec}(A^G)$.

Let $X \to \mathrm{Spec}(A^G)$ be a resolution of the singularity. Let C_i, $i = 1, \ldots, r$, denote the irreducible components of the exceptional divisor, and form the *intersection matrix*

$$N := ((C_i \cdot C_j)_X)_{1 \leq i,j \leq r} \in \mathrm{Mat}_r(\mathbb{Z}).$$

This matrix is negative-definite. The *discriminant group* $\Phi_N := \mathbb{Z}^r/N\mathbb{Z}^r$ attached to N is a finite group of order $|\det(N)|$, independent of the resolution. The group Φ_N appears as a natural

2010 *Mathematics Subject Classification.* 14J17, 14B05, 13A50, 14E15.

quotient of the class group $\mathrm{Cl}(A^G)$ (5.7). Attached to the resolution is its *dual graph* Γ_N, with vertices $v_1, \dots, v_r$, where v_i and v_j are linked by $(C_i \cdot C_j)_X$ distinct edges when $i \neq j$. Our ultimate, long term, goal is to characterize the intersection matrices N, discriminant groups Φ_N, and dual graphs Γ_N, that can arise from such wild quotient singularities.

The *fixed point scheme* of the action of G on $\operatorname{Spec} A$ is defined by the ideal $I := (\sigma(a) - a \mid a \in A, \sigma \in G)$. We say that the action is *ramified precisely at the origin* if the radical of I is the maximal ideal (u, v); in this case, the closed point of $\operatorname{Spec}(A^G)$ is singular. Otherwise, we say that the action is *ramified in codimension* 1. When I is principal, A^G is regular ([21, Theorem 2]), and when A^G is regular, I is conjectured to be principal [21, Conjecture 9].

It is known that when the exceptional divisor has smooth components with normal crossings, all components C_i are smooth projective lines and the dual graph Γ_N is a tree [28, Theorem 2.8]. It is also known that the discriminant group Φ_N is an elementary abelian p-group [28, Theorem 2.6], so that in particular we may write

$$|\Phi_N| = |\det(N)| = p^s$$

for some integer $s \geq 0$. In this article, we consider which exponents $s \geq 0$ can arise in this way. By studying diagonal actions on products of curves, the first author [30, Theorem 3.15] produced wild quotient singularities with $|\Phi_N| = p^s$ for all exponents $s \geq 2$ with $s \not\equiv 1$ modulo p. Mitsui [33] later explicitly resolved all wild quotient singularities arising from product of curves, and showed that the previous list is the complete list of exponents arising from product of curves. The *missing exponents* are then $s = 0$, as well as all s with $s \equiv 1 \mod p$.

0.1. *We conjecture that for p odd, all exponents $s \geq 0$ arise in this way from wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities associated with an action that is ramified precisely at the origin.*

In this article, we prove this conjecture for $s = 0$ and $s = 1$ by explicitly resolving certain wild quotient singularities of independent interest. We also exhibit singularities as in the conjecture that are likely to produce a group Φ_N with $|\Phi_N| = p^s$ for all other missing values $s > 1$ (0.3). When the condition that the action be ramified precisely at the origin is relaxed, we can prove the following result.

Theorem (see 5.5). *For p odd, all missing exponents $s \geq 0$ arise from wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities associated with an action that is ramified in codimension* 1.

Let $c, d, e \geq 2$ be integers. Recall that the equation $x^c + y^d + z^e = 0$ is said to define a *Brieskorn surface singularity.* The missing exponents s are exhibited to arise from wild quotient singularities with the help of well-chosen Brieskorn singularities, as in our next theorem.

Theorem (see 5.1 and 5.3). *Let $B := k[[x, y, z]]/(z^p + x^c + y^d)$. Assume that p does not divide cd. Let $g := \gcd(c, d)$. Any resolution of* $\operatorname{Spec} B$ *has an intersection matrix whose associated discriminant group has order p^{g-1} and is killed by p. When $c = pm + 1$ and $d = pn + 1$ for some $m, n \geq 1$, then* $\operatorname{Spec} B$ *is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

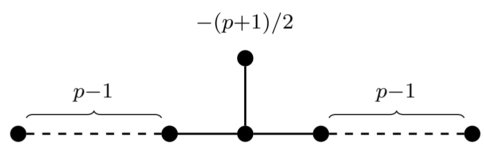
The resolutions of the Brieskorn singularities in the previous theorem are found in Theorem 5.1, and coincide with the known resolutions in characteristic 0 ([18, Theorem, page 232], [37]). The theorem is valid when $p = 2$, but in this case, the order p^{g-1} is always an even power of 2, and thus provides no examples of missing odd exponents. The theorem shows that when $p = 2$ and $\gcd(p, cd) = 1$, all singularities $z^p + x^c + y^d = 0$ are wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities. It would be of interest to determine whether this fails to be the case when $p > 2$.

Let now C_n denote the n-th Catalan number, and let $p \geq 3$. To produce singularities associated with an action that is ramified precisely at the origin and which have $|\Phi_N| = p$, we expand on the work of Peskin [41] and consider the ring $B_\mu := k[[x, y, z]]/(h)$, where $\mu \in k[y]$ and

$$h := z^p + 2y^{p+1} - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n.$$

When $\mu = 1$, this equation defines a wild quotient singularity that can be regarded as an analogue of the E_6^1-singularity (notation as in Artin's classification in [3]). We compute explicitly its resolution in our next theorem. When $p = 3$, the graph Γ_N below reduces to the Dynkin diagram E_6. When drawing a dual graph, we adopt in this article the usual convention that a vertex is adorned with the associated self-intersection number, unless this self-intersection number is -2, in which case it is suppressed.

Theorem (see 6.3). *Let p be an odd prime. Let B_μ be as above. Then* $\operatorname{Spec} B_\mu$ *has a resolution independent of μ, with dual graph Γ_N of the following form*



The associated discriminant group Φ_N has order p.

0.2. To treat the case where Φ_N is the trivial group in Conjecture 0.1, we use a family of hypersurface singularities introduced in [31] and which is of independent interest. Fix a system of parameters a, b in $k[[x, y]]$. Let $\mu \in k[[x, y]]$, and consider the equation

$$z^p - (\mu ab)^{p-1} z - a^p y + b^p x = 0, \tag{0.1}$$

and the associated ring

$$B_\mu = B := k[[x, y, z]]/(z^p - (\mu ab)^{p-1} z - a^p y + b^p x).$$

(a) *Assume that μ is a unit in $k[[x, y]]$.* It is shown in [31], 7.1, that B is isomorphic to the ring of invariants A^G of an explicit wild action of $\mathbb{Z}/p\mathbb{Z}$ on $A = k[[u, v]]$ ramified precisely at the origin. More precisely, after identifying A with the ring

$$k[[x, y]][u, v]/(u^p - (\mu a)^{p-1} u - x, v^p - (\mu b)^{p-1} v - y),$$

the action is determined by the automorphism σ with $\sigma(u) = u + \mu a$ and $\sigma(v) = v + \mu b$. The morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified only at the maximal ideal $\mathfrak{m}$, and we find that the étale fundamental group $\pi_1^{\text{loc}}(A^G)$ of the punctured spectrum $U := \operatorname{Spec} A^G \setminus \{\mathfrak{m}\}$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$. Such actions are called *moderately ramified* in [31], and we refer the reader to [31] for further information on these actions.

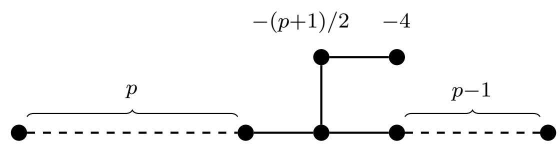
(b) *Assume that $\mu \neq 0$, and that it is coprime to both a and b.* Then B is again isomorphic to the ring of invariants A^G for the action on $A := k[[u, v]]$ described above. However, in this case the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified in codimension 1 and the group $\pi_1^{\text{loc}}(A^G)$ is trivial.

In this article, we restrict our attention to the case where $a = y^n$ and $b = x^m$. The case $\mu = 0$ is then also of interest:

(c) *Assume that* $\mu = 0$, *with* $a = y^n$ *and* $b = x^m$. The resulting hypersurface is a Brieskorn singularity of type $z^p - y^{pn+1} + x^{pm+1}$.

In the specialized case where $a = y^n$ and $b = x^m$, preliminary computations with Magma [6] and Singular [11] show that the resolution of singularities in all three cases above has the same combinatorial type, independent of μ. We prove that this is indeed the case in two instances in this article, when $a = y$ and $b = x$ in Theorem 9.2, and when $a = y^2$ and $b = x$ in Theorem 7.1. In the latter case, Artin [3] (see also [40]) shows when $p = 2$ that the values $\mu = 0$, $\mu = 1$, and $\mu = y$, produce the rational double points E_8^0, E_8^2, and E_8^1, respectively. These singularities are not isomorphic but have the same resolution graph, the Dynkin diagram E_8. Our generalization of these singularities to any odd prime p has a resolution with the following dual graph.

Theorem (see 7.1). *Let* p *be an odd prime. Let* B_μ *be as in* 0.2. *Assume that* $a = y^2$ *and* $b = x$. *Then* $\operatorname{Spec} B_\mu$ *has a resolution of singularities independent of* μ, *with dual graph* Γ_N



The associated discriminant group Φ_N *is trivial.*

0.3. Let p be odd. Recall that when $\mu = 1$, the associated quotient singularity $\operatorname{Spec} B_{\mu=1}$ is induced by an action that is ramified precisely at the origin. It is likely that by varying the exponents m and n in $a = y^n$ and $b = x^m$, one will obtain examples of resolutions of $\operatorname{Spec} B_{\mu=1}$ with associated discriminant group Φ_N of order p^s for any power s with $s \not\equiv -1 \mod p$. In particular, we exhibit in 5.6 the appropriate exponents m and n that would cover all remaining open cases in our conjecture 0.1 (that is, all values of s with $s \equiv 1 \mod p$).

Peskin's singularity with $\mu = 1$ introduced above, and all the singularities considered in [30] or [33], are also induced by an action that is ramified precisely at the origin. When $p = 2$, none of the known explicit resolutions for examples in these classes of singularities produce an associated discriminant group Φ_N with order 2^s and s *odd*. This lack of examples might indicate that there is a serious obstruction to exhibiting such examples. It is natural to wonder whether such examples in fact do not exist for actions ramified precisely at the origin.

Let $p = 2$. The Dynkin diagram E_7, with discriminant group Φ_{E_7} of order 2, might be the most ubiquitous graph with discriminant group of order 2^s with s odd. Many other such examples are exhibited in 8.3. Artin [3] showed that there exists a wild $\mathbb{Z}/2\mathbb{Z}$-action on $A = k[[u, v]]$, ramified in codimension 1, such that $\operatorname{Spec} A^{\mathbb{Z}/2\mathbb{Z}}$ is a rational double point of type E_7. He also showed that any such surface singularity must have a trivial local fundamental group. In other words, there cannot exist a wild $\mathbb{Z}/2\mathbb{Z}$-action on $A = k[[u, v]]$, ramified precisely at the origin, such that $\operatorname{Spec} A^{\mathbb{Z}/2\mathbb{Z}}$ has a resolution of combinatorial type E_7.

Inspired by Artin's considerations, we define in Section 8 some explicit wild $\mathbb{Z}/p\mathbb{Z}$-actions on $A = k[[u, v]]$ ramified in codimension 1. When $p = 2$, we exhibit for each s *odd* an explicit example conjectured to have discriminant group of order 2^s. In Section 9, for any prime p, we exhibit a wild $\mathbb{Z}/p\mathbb{Z}$-action on $A = k[[u, v]]$ ramified in codimension 1 which results in an A_{p-1}-singularity.

Theorem (see 9.4). *Let* k *be a field of characteristic* $p > 0$. *Let* $A := k[[u, v]]$. *Then there exists an automorphism* $\sigma : A \to A$ *of order* p *such that* $\operatorname{Spec} A^{\langle\sigma\rangle}$ *is a rational double point of* *type A_{p-1}, which has discriminant group $\Phi_{A_{p-1}}$ of order p. Any such automorphism induces a morphism* $\operatorname{Spec} A \to \operatorname{Spec} A^{\langle\sigma\rangle}$ *that must be ramified in codimension* 1.

It is natural to wonder whether the same result holds for any *Hirzebruch–Jung chain* whose discriminant group has order p (definition recalled in 1.1). The last statement in the above theorem follows from a result of Ito and Schröer [19], which states that if the action is ramified precisely at the origin, then the resolution of the resulting quotient singularity has a dual graph Γ_N which must have a vertex of valency at least 3.

Artin shows in [2] that in characteristic $p = 2$, all wild quotient singularities A^G with $\operatorname{Spec} A \to \operatorname{Spec} A^G$ ramified precisely at the origin can be described by an equation of the form (0.1) with $\mu = 1$. In particular, all such singularities are complete intersection. We show in Proposition 10.1 that when $p = 2$, any wild quotient singularity A^G is a complete intersection, even when $\operatorname{Spec} A \to \operatorname{Spec} A^G$ ramifies in codimension 1. When A^G is a complete intersection, it is then also Gorenstein, with an intersection matrix which is *numerically Gorenstein*. The purely linear algebraic definition of numerically Gorenstein is recalled in 10.2, and it is natural to wonder whether this condition imposes a new restriction on intersection matrices associated with $\mathbb{Z}/2\mathbb{Z}$-quotient singularities. The answer to this question is negative, and we show in Proposition 10.5 that any intersection matrix N such that Φ_N is killed by 2 is always numerically Gorenstein.

The paper is organized as follows. Section 1 contains several useful facts concerning the linear algebra of intersection matrices N, in particular formulas for the order of Φ_N when the dual graph Γ_N is star-shaped. Sections 2 and 3 are preparatory sections, where we recall basic facts regarding how to compute self-intersection numbers on a resolution of a singularity using data coming from intermediate blow-ups. This will be applied in later sections to the resolution of $\operatorname{Spec} B_\mu$, where we found it useful, instead of starting the resolution process by blowing up the maximal ideal, to first blow up an ideal naturally related to the ideal defining the fixed scheme of the action. We provide in Section 4 the explicit resolution of certain weighted homogeneous singularities of the form $W^q - U^aV^b(V^d - U^c) = 0$, with p, q, a, b, c, d subject to certain mild restrictions. Over $\mathbb{C}$, such resolution has already been obtained by Orlik and Wagreich ([37], [38], [39]) in full generality. The proofs of the theorems presented in this introduction are found in Sections 5 to 10.

Acknowledgement. The authors gratefully acknowledge funding support from the Research and Training Group in Algebraic Geometry, Algebra, and Number Theory at the University of Georgia, from the National Science Foundation RTG grant DMS-1344994 and from the Simons Collaboration Grant 245522, as well as from the Research and Training Group GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology, funded by the Deutsche Forschungsgemeinschaft.

1. Intersection matrices

Let B be a complete noetherian local ring that is two-dimensional and normal. Let C_i, $i = 1, \dots, n$, denote the irreducible components of the exceptional divisor of a resolution of singularities of $\operatorname{Spec} B$, with associated intersection matrix $N := ((C_i \cdot C_j))_{1 \le i,j \le n}$. This section collects some facts that depend only on the linear algebra of the matrix N and which are used in later sections.

An $n \times n$ *intersection matrix* $N = (c_{ij})$ is a symmetric negative-definite integer matrix with negative coefficients on the diagonal, and non-negative coefficients off the diagonal. The *discriminant group* $\Phi = \Phi_N$ is defined as the finite abelian group $\mathbb{Z}^n/N\mathbb{Z}^n$, which has order $|\det(N)|$. The associated *graph* $\Gamma = \Gamma_N$ arises as follows: Introduce vertices $v_1, \ldots, v_n$ corresponding to the standard basis vectors in $\mathbb{Z}^n$. Two vertices $v_i \neq v_j$ are linked by exactly $c_{ij} \geq 0$ edges. If not stated otherwise, we tacitly assume that Γ is connected.

The *degree* or *valency* of a vertex $v \in \Gamma$ is the number of edges attached to v. A vertex v with valency at least three is called a *node*, and a vertex v with valency one is called *terminal*. A graph is a *chain* if it is connected and does not contain any node. It is called *star-shaped* if it is a tree with a unique node. Given a star-shaped graph Γ with node v_0, we can consider the subgraph $\Gamma \setminus \{v_0\}$ obtained by removing the vertex v_0 and all the edges containing v_0. This complement is the disjoint union of $m \geq 3$ chains $\Delta_1, \ldots, \Delta_m$ that we call the *terminal chains* of Γ.

1.1. Suppose that N is an intersection matrix whose graph Γ_N is a chain, with $\ell \geq 1$ consecutive vertices $v_1, \ldots, v_\ell$. For convenience, we label the diagonal entries of N by $c_{ii} = -s_i$, and we assume below that $s_i \geq 2$ for $i = 1, \ldots, \ell$. We associate to N with this ordering of the vertices a unique sequence of positive integers $1 = r_\ell < \cdots < r_1 < r_0$ such that the following matrix equality holds, where the square matrix on the left is N:

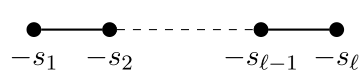
$$\begin{pmatrix} -s_1 & 1 & & \\ 1 & -s_2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -s_\ell \end{pmatrix} \begin{pmatrix} r_1 \\ \vdots \\ r_{\ell-1} \\ r_\ell \end{pmatrix} = \begin{pmatrix} -r_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

When needed, we will denote $R = R_N$ the transpose of the vector $(r_1, \ldots, r_\ell)$, so that NR is the transpose of $(-r_0, 0, \ldots, 0)$. It is known that $|\det(N)| = r_0$, and that Φ_N is cyclic of order r_0 ([28], 3.13). To be able to refer to r_0 and r_1 without indices, we will relabel them as $r_0 := a$ and $r_1 := b$. Note that by construction, $\gcd(a, b) = 1$, and that we can express the reduced fraction a/b completely in terms of $s_1, \ldots, s_\ell$ as a continued fraction

$$\frac{a}{b} = [s_1, s_2, \ldots, s_\ell] := s_1 - \cfrac{1}{s_2 - \cfrac{1}{\ddots - \cfrac{1}{s_\ell}}}. \tag{1.1}$$

Clearly, any reduced fraction a/b with $a > b$ determines an intersection matrix N as above. The reduced fraction $a/b = 1/1$ determines the matrix $N = (-1)$. We note that $-a/b = \det(N)/\det(N')$, where N' is obtained from N by removing its first line and first column (recall that the determinant of the empty matrix is 1 by convention).

As is customary, the vertices of the graph Γ_N of an intersection matrix $N = (c_{ij})$ are labeled with the *self-intersection numbers* $-s_i := c_{ii}$, and self-intersection numbers $-s_i = -2$ are usually omitted. For a chain Γ_N as above, we get the following drawing:



We call such chain a *Hirzebruch–Jung chain*. Recall that $p/(p-1) = [2, \ldots, 2]$ and that the corresponding intersection matrix of size $p-1$ and determinant $(-1)^{p-1}p$ is denoted by A_{p-1}. This intersection matrix will be shown to arise in the context of $\mathbb{Z}/p\mathbb{Z}$-singularities in 9.4.

1.2. Let $m \geq 3$. Let $a_1/b_1, \ldots, a_m/b_m$ be reduced fractions with $a_i/b_i \geq 1$ for $i = 1, \ldots, m$. Let $s_0 \geq 1$ be any integer. We denote by $N = N(s_0 \mid a_1/b_1, \ldots, a_m/b_m)$ the following matrix. Its graph $\Gamma = \Gamma_N = \Gamma(s_0 \mid a_1/b_1, \ldots, a_m/b_m)$ is star-shaped with m terminal chains attached to a central node v_0 having self-intersection number $-s_0$. Let $\Delta_1, \ldots, \Delta_m$ be the Hirzebruch–Jung chains determined by the fractions $a_1/b_1, \ldots, a_m/b_m$. The graph Γ is obtained by attaching to v_0 with a single edge the initial vertex of each chain Δ_i. In this article, when referring to a matrix of the form $N = N(s_0 \mid a_1/b_1, \ldots, a_m/b_m)$, we will always assume that it is an intersection matrix, i.e., that N is negative-definite.

Proposition 1.3. *Let $N = N(s_0 \mid a_1/b_1, \ldots, a_m/b_m)$ be an $n \times n$ intersection matrix as above, with star-shaped graph Γ_N. Then $s_0 > \sum_{j=1}^m b_j/a_j$, and the following hold:*

(i) *We have* $\det(N) = (-1)^n (\prod_j a_j)(s_0 - \sum_j b_j/a_j)$. *In particular, there is an integer factorization*

$$|\det(N)| = \left(\frac{\prod_j a_j}{\operatorname{lcm}(a_1, \ldots, a_m)} \right) \left(\operatorname{lcm}(a_1, \ldots, a_m)(s_0 - \sum_j b_j/a_j) \right).$$

(ii) *In the discriminant group Φ_N, the class of the standard basis vector $e_{v_0} \in \mathbb{Z}^n$ corresponding to the central node v_0 has order* $\operatorname{lcm}(a_1, \ldots, a_m)(s_0 - \sum_j b_j/a_j)$.

(iii) *Let w_j denote the terminal vertex of the chain Δ_j. Then Φ_N is generated by the classes of e_{w_j}, $j = 1, \ldots, m$. Moreover, the class of e_{v_0} is equal to the class of $a_j e_{w_j}$, and the group Φ_N is killed by* $\operatorname{lcm}(a_1, \ldots, a_m)^2(s_0 - \sum_j b_j/a_j)$.

(iv) *If $a_j = p$ for all j and $ps_0 - \sum_j b_j = 1$, then Φ_N is killed by p and has order p^{m-1}.*

(v) *Assume that Φ_N is killed by a prime p. If p divides a_j for some j, then the class of e_{v_0} is trivial in Φ_N.*

Proof. Without loss of generality, we may assume that N equals the block matrix

$$N = \begin{pmatrix} -s_0 & * & \cdots & * \\ * & N_1 & & \\ \vdots & & \ddots & \\ * & & & N_m \end{pmatrix} \in \operatorname{Mat}_n(\mathbb{Z}),$$

where N_i is the intersection matrix with graph Δ_i, with vertices numbered consecutively starting from the vertex adjacent to the node v_0. The $*$'s in the above matrix stand for sequences of appropriate size, starting with 1 followed by zeros. Let R_i denote the positive integer vector associated to N_i, such that

$$N_i R_i = {}^t(-a_i, 0, \ldots, 0).$$

Form the block column integer vector R in $\mathbb{Z}^n$ given as

$$R := \operatorname{lcm}(a_1, \ldots, a_m)\ {}^t(1, a_1^{-1} R_1, \ldots, a_m^{-1} R_m).$$

By construction, the greatest common divisor of the entries in R is 1, since, given a prime p such that p^s exactly divides $\operatorname{lcm}(a_1, \ldots, a_m)$, there exists at least one index i such that a_i is exactly divisible by p^s. In particular, the coefficient of R corresponding to the last vertex on the chain Δ_i is coprime to p. Let $x := s_0 - \sum_j b_j/a_j$. Then

$$NR = \operatorname{lcm}(a_1, \ldots, a_m)\ {}^t(-x, 0, \ldots, 0).$$

Note that $x > 0$, because N is negative-definite, so the integer ${}^tRNR$ must be negative. By negative-definiteness, we also know that $\det(N)$ has sign $(-1)^n$. Using [28], Theorem 3.14 with the matrix N and the vector R, we get

$$\det(N) = (-1)^n (s_0 - \sum_j b_j/a_j) \cdot (\prod_j a_j)$$

and the assertion (i) follows. The assertion in (ii) follows immediately from the equality

$$NR = \mathrm{lcm}(a_1, \ldots, a_m)\ {}^t(-x, 0, \ldots, 0)$$

and the fact that the greatest common divisor of the coefficients of R is 1. For (iii), to show that $e_{v_0} - a_j e_{w_j}$ is in the image of N, consider the unique positive vector S_j whose first component is 1 and such that $N_j S_j$ is equal to the transpose of $(0, \ldots, 0, -a_j)$. Extend this vector to a vector $\overline{S}_j \in \mathbb{Z}^n$ by setting all other components to 0. Then $N\overline{S}_j = e_{v_0} - a_j e_{w_j}$. The proof that for any vertex w on the chain Δ_j, there exists an integer c_w such that $e_w - c_w e_{w_j}$ is in the image of N is similar, and is left to the reader. Using (ii) to find the order of the class of e_{v_0}, it follows immediately that the class of e_{w_j} is killed by $\mathrm{lcm}(a_1, \ldots, a_m)^2(s_0 - \sum_j b_j/a_j)$, for all j. Part (iv) is immediate from (i) and (iii). In Part (v), assume that p divides a_j. As the class of e_{w_j} is killed by p by hypothesis, we find from (iii) that the class of e_{v_0} is trivial.

□

2. Computation of self-intersections

Let B be a complete local noetherian ring that is two-dimensional and normal. It is known that a resolution of singularities $X \to \mathrm{Spec}(B)$ exists, and that it can be obtained from the sequence

$$X = Y_t \longrightarrow Y_{t-1} \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow Y_0 = \mathrm{Spec}(B),$$

where each $Y_i \to Y_{i-1}$ is the normalization of the blow-up of the finitely many singular points of Y_{i-1} (see, e.g., [26], Theorem on page 151 and Remark B on page 155). In this section we develop a method for computing the self-intersection of particular irreducible components of the exceptional divisor on X. This information is needed in the proofs of each of our explicit computation of resolutions in Theorems 4.4, 6.3, 7.1, and 9.2. For the sake of exposition, we assume that the residue field $k = B/\mathfrak{m}_B$ is algebraically closed.

Note that the process described above usually does not produce the minimal desingularization, as some irreducible components of the exceptional divisor on X might be (-1)-curves, and thus contract to smaller resolutions of singularities. This may even happen for the strict transforms of the exceptional divisors on the first blow-up Y_1 (see Example in [25, page 205]).

2.1. Let $X \to \mathrm{Spec}(B)$ be any resolution of singularities, and write $C_1, \ldots, C_n$ for the irreducible components of the exceptional divisor. We then have intersection numbers

$$c_{ij} = (C_i \cdot C_j)_X := \chi(\mathscr{O}_{C_j}(C_i)) - \chi(\mathscr{O}_{C_j}) = \deg(\mathscr{O}_{C_j}(C_i)),$$

and can form the resulting intersection matrix $N = (c_{ij})_{1 \le i,j \le n}$. Associated with N is the connected graph $\Gamma = \Gamma_N$ with vertices $v_1, \ldots, v_n$, and a pair of vertices $v_i \neq v_j$ is linked by exactly c_{ij} edges. We call Γ the *resolution graph* or the *dual graph* attached to $X \to \mathrm{Spec}\, B$.

Now consider a factorization $X \to Y \to \mathrm{Spec}\, B$, where $\pi : X \to Y$ is the contraction of certain exceptional curves, say $C_{s+1} \cup \ldots \cup C_n$. We regard the induced morphism $Y \to \mathrm{Spec}(B)$ as a partial resolution of singularities, and by definition of contraction, Y is *normal*. Write $D_1, \ldots, D_s \subset Y$ for the images in Y of the non-contracted curves $C_1, \ldots, C_s \subset X$. These images are Weil divisors which are not necessarily Cartier. Following Mumford [36], page 17 (see also [14], 7.1.16, or [46], Theorem 1.2), one has *rational intersection numbers* $(D_i \cdot D_j)_Y \in \mathbb{Q}$ obtained as follows: First define the rational pull-back $\pi^*(D_i) := C_i + \sum_{k>s} \lambda_k C_k$, where $\lambda_{s+1}, \ldots, \lambda_n \in \mathbb{Q}$ are the fractions uniquely determined by the conditions $(\pi^*(D_i) \cdot C_k)_X = 0$ for all $s < k \leq n$. One then sets

$$(D_i \cdot D_j)_Y := (\pi^*(D_i) \cdot C_j)_X = (\pi^*(D_i) \cdot \pi^*(D_j))_X.$$

These numbers actually do not depend on the choice of resolution $\pi : X \to Y$.

Suppose now that $\pi : X \to Y$ is the contraction of all but the first curve C_1. Assume furthermore that Γ is a tree. Let v be the vertex corresponding to C_1, and consider the graph $\Gamma \smallsetminus \{v\}$ obtained from Γ by removing the vertex v and all the edges attached to v. The graph $\Gamma \smallsetminus \{v\}$ decomposes into connected components $\Gamma \smallsetminus \{v\} = \Delta_1 \cup \ldots \cup \Delta_r$, with corresponding intersection matrices $N_1, \ldots, N_r$ for each component. Since Γ is a tree, there exists a unique vertex $w_i \in \Delta_i$ which is adjacent to v in Γ. Define $\Delta_i' := \Delta_i \smallsetminus \{w_i\}$, with intersection matrix N_i'. We call

$$\delta_i := -\frac{\det(N_i')}{\det(N_i)} \in \mathbb{Q}_{>0}$$

the *correction term* at w_i (recall that the determinant of the empty matrix is 1, and we use this convention if Δ_i is reduced to the single vertex w_i). The correction terms δ_i are indeed positive, since the signs of $\det(N_i)$ and $\det(N_i')$ are given by $(-1)^{r_i}$ and $(-1)^{r_i-1}$, where r_i is the number of vertices of Δ_i. When Δ_i is a chain as in 1.1 corresponding to a fraction a_i/b_i, we have $\delta_i = b_i/a_i$. The geometric meaning of the correction terms is as follows:

Proposition 2.2. *In the above situation, the integral self-intersection and the rational self-intersection are related by the formula*

$$(C_1 \cdot C_1)_X = (D_1 \cdot D_1)_Y - \sum_{i=1}^{r} \delta_i.$$

Proof. For ease of notation, we let in this proof $C = C_1$ and $D = D_1$. Let N_0 denote the lower-right principal minor of N. Recall from our earlier description that N_0 is a block diagonal matrix with $\det(N_0) = \prod_{i=1}^r \det(N_i)$. Then

$$(\lambda_2, \ldots, \lambda_n) = -((C \cdot C_2)_X, \ldots, (C \cdot C_n)_X) N_0^{-1}.$$

It follows that

$$(D \cdot D)_Y = (\pi^*(D) \cdot C)_X = (C \cdot C)_X + \sum_{i=2}^{n} \lambda_i (C_i \cdot C)_X.$$

Since Γ_N is a tree, we find that if $(C \cdot C_i)_X \neq 0$, then $(C \cdot C_i)_X = 1$. We only need to compute explicitly λ_i when $(C \cdot C_i)_X \neq 0$. According to our definitions, there are r such indices i, and in each case, the coefficient λ_i is the top left corner of the corresponding matrix N_i^{-1}, that is, $\det(N_i')/\det(N_i)$, as desired. ☐

We will use Proposition 2.2 in the following situation. Let $\mathfrak{b}$ be an ideal in B, and let $Z \to \operatorname{Spec} B$ denote the blowing-up with center $V(\mathfrak{b})$. Denote by $E \subset Z$ the schematic preimage of the center. Let $\nu : Y \to Z$ be the normalization map and denote by $D = \nu^{-1}(E)$ the schematic preimage of E. *Assume that D, and hence E, are irreducible.* Let D_{red} denote the support of D endowed with its induced reduced structure. Letting D_{red} play the role of D_1 in Proposition 2.2, we find a formula for the rational intersection number $(D_{\text{red}} \cdot D_{\text{red}})_Y$ in term of data from a resolution $X \to Y$. Our next proposition shows how to obtain $(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y$ from data associated with the blowing-up $Z \to \operatorname{Spec} B$.

The exceptional divisor $E \subset Z$ is given by the sheaf of ideals $\mathscr{O}_Z(1) \subset \mathscr{O}_Z$. The reduction E_{red} is a projective curve over the residue field k, allowing us to define the integral intersection number

$$(E \cdot E_{\mathrm{red}})_Z := \chi(\mathscr{O}_{E_{\mathrm{red}}}(E)) - \chi(\mathscr{O}_{E_{\mathrm{red}}}) = \deg \mathscr{O}_{E_{\mathrm{red}}}(-1).$$

In practice, $(E \cdot E_{\mathrm{red}})_Z$ can often be computed, and such computation is done for instance in Proposition 3.6.

Proposition 2.3. *In the above situation where D, and hence E, are assumed irreducible, write $E = mE_{\mathrm{red}}$ and let $d \geq 1$ be the degree of the induced map $\nu : D_{\mathrm{red}} \to E_{\mathrm{red}}$. Then we have*

$$(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = \frac{d^2}{m}(E \cdot E_{\mathrm{red}})_Z.$$

Proof. First, we check that $(D \cdot \nu^{-1}(F))_Y = (E \cdot F)_Z$ for every effective Cartier divisor $F \subset Z$ that does not contain the support of E. The two intersection numbers are the k-degrees of the finite schemes $D \cap \nu^{-1}(F)$ and $E \cap F$, respectively. Fix a point $z \in E \cap F$, consider the local ring $A = \mathscr{O}_{F,z}$ and choose an element $t \in \mathfrak{m}_A$ defining $F \cap E \subset F$ locally. Then A is a local noetherian ring of dimension one without embedded components, and $M = \mathscr{O}_{\nu^{-1}(F),z}$ is a finite A-module of rank one for which the multiplication map $t : M \to M$ is injective. According to [17], Chapter IV, Lemma 21.10.13, the modules A/tA and M/tM have the same A-length, hence also the same k-vector space dimension. Applying this with a difference $F - F'$ of effective Cartier divisors that are linearly equivalent to E, we conclude $(D \cdot D)_Y = (E \cdot E)_Z$.

To simplify notation write $E' = E_{\mathrm{red}}$ and $D' = D_{\mathrm{red}}$. With $D = hD'$ we get

$$h^2(D' \cdot D')_Y = (D \cdot D)_Y = (E \cdot E)_Z = m(E \cdot E')_Z. \tag{2.1}$$

We now use Kleiman's theory of rational degrees $\deg(V'/V) \in \mathbb{Q}_{\geq 0}$ for morphisms $V' \to V$ between irreducible proper schemes that are not necessarily integral ([22], Definition on page 277). According to [22, Lemma 2], the commutative diagram

$$\begin{array}{ccc} D' & \longrightarrow & D \\ \downarrow & & \downarrow \\ E' & \longrightarrow & E \end{array}$$

gives the equation $\deg(D'/E') \cdot \deg(E'/E) = \deg(D'/D) \cdot \deg(D/E)$, and furthermore we have $\deg(E'/E) = 1/m$ and $\deg(D'/D) = 1/h$. Thus $\deg(D'/E') = m/h$. Inserting this into (2.1) yields the assertion. $\square$

3. Blowing up non-reduced centers

We begin this section with some general facts on the computation of blowing-ups, needed for instance to fully justify the explicit computations done in Proposition 3.6. Let B be a noetherian ring, and let $\mathfrak{b} \subset B$ be an ideal. Endow the associated *Rees ring*

$$B[\mathfrak{b}T] := B \oplus \mathfrak{b}T \oplus \mathfrak{b}^2T^2 \oplus \ldots \subset B[T]$$

with the grading induced by the standard grading on $B[T]$. The morphism $\operatorname{Proj}(B[\mathfrak{b}T]) \to \operatorname{Spec} B$ is called the *blowing-up* of $\operatorname{Spec}(B)$ with center $\operatorname{Spec}(B/\mathfrak{b})$. We denote $\operatorname{Proj}(B[\mathfrak{b}T])$ by $\operatorname{Bl}_{\mathfrak{b}}(B)$ or, when no confusion may ensue, simply by Z. Let E denote the schematic preimage in Z of the center of the blowing-up.

Assume now that R is a noetherian ring with a surjection $R \to B$. Let $\mathfrak{a}$ denote the preimage in R of the ideal $\mathfrak{b}$. Consider the blowing-up $Z' := \mathrm{Bl}_{\mathfrak{a}}(R)$ with center $V(\mathfrak{a})$, and the commutative diagram induced by the surjection $R[\mathfrak{a}T] \to B[\mathfrak{b}T]$ of Rees rings:

$$\begin{array}{ccc} Z & \longrightarrow & Z' \\ \downarrow & & \downarrow \\ \operatorname{Spec} B & \longrightarrow & \operatorname{Spec} R. \end{array}$$

The horizontal morphisms are closed immersions.

Recall that an element $f \in R$ is called *regular* if multiplication by f on R is an injective map. Assume now that the kernel of $R \to B$ is generated by a regular element $f \in R$. Then $\mathrm{Spec}(B)$ is an effective Cartier divisor in $\mathrm{Spec}(R)$, and our next proposition provides a criterion for checking whether the closed subscheme Z is an effective Cartier divisor in Z', when Z' and $V(\mathfrak{a})$ are 'nice'. This criterion is explicit and in general not very difficult to verify.

Each element $g \in \mathfrak{a}$ defines a basic open set $D_+(g) := \operatorname{Spec} R[\mathfrak{a}T]_{(gT)}$ of Z' called the *g-chart*. When $\mathfrak{a} = (g_1, \dots, g_r)$, the union $\cup_{i=1}^r D_+(g_i)$ is an affine open cover of Z'.

Proposition 3.1. *Let R be a noetherian ring, locally of complete intersection*[1]*. Let $g_1, \dots, g_r \in R$ be a regular sequence, and set $\mathfrak{a} := (g_1, \dots, g_r)$. Let $f \in R$ be a regular element contained in $\mathfrak{a}$, and set $B := R/(f)$ and $\mathfrak{b} := \mathfrak{a}B$. Consider as above the blowing-ups $Z \to \operatorname{Spec} B$ and $Z' \to \operatorname{Spec} R$.*

For each $i = 1, \dots, r$, choose a factorization $f/1 = (g_i/1)^{s_i} h_i$ in $R[\mathfrak{a}T]_{(g_iT)}$, with $s_i \geq 0$ and $h_i \in R[\mathfrak{a}T]_{(g_iT)}$. Assume that for each i, the closed subscheme $V(h_i, g_i/1)$ of $D_+(g_i)$ has codimension two in $D_+(g_i)$. Then

(a) *The closed subscheme Z of Z' is an effective Cartier divisor. Its restriction on the g_i-chart $D_+(g_i)$ is the closed subscheme $V(h_i)$.*
(b) *The scheme Z is locally of complete intersection.*

Proof. Part (a) follows from Proposition 3.2. Part (b) follows from Proposition 3.4. ☐

Proposition 3.2. *Keep the notation introduced at the beginning of this section. Let $g \in \mathfrak{a}$. Suppose that we have a factorization $f/1 = (g/1)^s h$ in $R[\mathfrak{a}T]_{(gT)}$, for some $s \geq 0$ and some element $h \in R[\mathfrak{a}T]_{(gT)}$. Suppose also that the following two assumptions hold:*

(i) The closed subscheme $V(h, g/1)$ of $D_+(g)$ has codimension at least two.
(ii) The basic open set $D_+(g) \subset Z'$ satisfies Serre's Condition (S_2).

Then $Z \cap D_+(g) = V(h)$ as closed subschemes of the g-chart $D_+(g)$.

Proof. By hypothesis, $g/1$ and h define two closed subschemes $V(g/1)$ and $V(h)$ in $D_+(g)$. All schemes below are viewed as subschemes in $Z' := \mathrm{Bl}_{\mathfrak{a}}(R)$. The conclusion of the proposition is implied by the following two claims:

(a) The open subsets $D_+(g) \cap (Z \setminus E)$ and $V(h) \setminus V(g/1)$ are equal.
(b) The closed subscheme $V(h) \cap V(g/1)$ is an effective Cartier divisor on $V(h)$.

Then, on one hand the schematic closure of the inclusion $D_+(g) \cap (Z \setminus E) \to D_+(g) \cap Z$ is equal to $D_+(g) \cap Z$ by Lemma 3.3, and on the other hand the schematic closure of the inclusion $V(h) \cap V(g) \to V(h)$ is equal to $V(h)$, also by Lemma 3.3.

---

[1]Recall that $g_1, \dots, g_d \in R$ is called a *regular sequence* if the class of g_i is a regular element in the ring $R/(g_1, \dots, g_{i-1})$, for each $1 \leq i \leq d$. The ring R is called *locally of complete intersection* if for each $\mathfrak{p} \in \operatorname{Spec} R$, the completion of $R_{\mathfrak{p}}$ is isomorphic to a ring of the form $A/(a_1, \dots, a_s)$, where A is a regular complete local ring, and $a_1, \dots, a_s$ is a regular sequence.

We leave it to the reader to verify (a). To prove (b), note that since f is regular in R, the element $f/1$ is regular in $R[\mathfrak{a}T]_{(gT)}$. Thus $V(h)$ and $V(g/1)$ are two Cartier divisors in $D_+(g)$. We need to show that the image of $g/1$ is not a zero-divisor in $R[\mathfrak{a}T]_{(gT)}/(h)$. Assumption (ii) implies that any effective Cartier divisor on $D_+(g)$ satisfies Serre's Condition (S_1). In particular, the ring $R[\mathfrak{a}T]_{(gT)}/(h)$ has no embedded primes and thus the zero divisors in $R[\mathfrak{a}T]_{(gT)}/(h)$ are contained in the minimal primes ideals. Krull's Principal Ideal Theorem shows the irreducible components of $V(h)$ all have codimension one in $D_+(g)$. Assumption (i) implies then that $g/1$ cannot be contained in a minimal prime ideal of $R[\mathfrak{a}T]_{(gT)}/(h)$. Thus $g/1$ is regular in $R[\mathfrak{a}T]_{(gT)}/(h)$. □

Lemma 3.3. *Let V be the complement of an effective Cartier divisor F on a noetherian scheme Y. Then the schematic image in Y of the open embedding $V \to Y$ coincides with Y.*

Proof. (a) The assertion is local, so we may assume that $Y = \operatorname{Spec}(A)$ and $F = V(g)$, where $g \in A$ is a regular element. The schematic image is defined by the kernel of the localization map $A \to A_g$, with $a \mapsto a/1$. Since g is regular, this kernel is the zero ideal. □

In the context of Proposition 3.2, we say that the equation $h = 0$ is the *strict transform* of $f = 0$ on the g-chart. One easily sees that condition (i) ensures that the exponent $s \geq 0$ is the maximal exponent. Note that in any case there is a factorization $f/1 = (g/1)^s h$ with maximal $s \geq 0$, by Krull's Intersection Theorem, and the resulting factor h is unique because $g/1$ is regular. In light of Krull's Principal Ideal Theorem, when $V(h, g/1)$ of $D_+(g)$ has codimension at least two in $D_+(g)$, it has codimension exactly two. This condition depends only on the radical ideal $\sqrt{(h, g/1)}$, a remark which usually substantially simplifies the computations.

Proposition 3.4. *Suppose the ideal $\mathfrak{a} \subset R$ is generated by a regular sequence $g_1, \ldots, g_d \in R$. If the scheme $S = \operatorname{Spec}(R)$ satisfies Serre's Condition (S_m), or is locally of complete intersection, the same holds for the blowing-up $\operatorname{Bl}_{\mathfrak{a}}(R)$.*

Proof. The canonical module surjection $R^{\oplus d} \to \mathfrak{a}$ coming from the regular sequence yields a closed embedding $\operatorname{Bl}_{\mathfrak{a}}(R) \subset \mathbb{P}^{d-1}_R$. Consider the short exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{O}_P^{\oplus d} \stackrel{(g_iT)}{\longrightarrow} \mathscr{O}_P(1) \longrightarrow 0$$

of locally free sheaves on $P = \mathbb{P}^r_R$. The kernel has $\operatorname{rank}(\mathscr{F}) = d - 1$. Using the inclusion $\mathscr{O}_P(1) \subset \mathscr{O}_P$, we get a composite map $\mathscr{F} \to \mathscr{O}_P$. According to [4], Exposé VII, Proposition 1.8, the image is the quasicoherent ideal corresponding to the closed subscheme $X = \operatorname{Bl}_{\mathfrak{a}}(R)$. Moreover, for each point $x \in X$, the image of any basis in $\mathscr{F}_x$ in the local ring $\mathscr{O}_{P,x}$ is a regular sequence contained in the maximal ideal $\mathfrak{m}_x$. More explicitly, we have

$$R[\mathfrak{a}T]_{(Tg_j)} = R[S_1, \ldots, S_d]/(S_1 g_j - g_1, \ldots, S_d g_j - g_d), \tag{3.1}$$

where the identification is given by $S_i = g_iT/g_jT$, and the generators in the above ideal form a regular sequence in the polynomial ring. This result is due to Micali [32], Theorem 1. It follows that the scheme $\operatorname{Bl}_{\mathfrak{a}}(R)$ is locally of complete intersection if this holds for the ring R.

Fix a point $x \in X$ and consider the local ring $A = \mathscr{O}_{X,x}$. It remains to show that $\operatorname{depth}(A) \geq m$ or $\operatorname{depth}(A) = \dim(A) < m$. For this we may assume that $S = \operatorname{Spec}(R)$ is local, and that x lies over the closed point $s \in S$. Set $c = d-1$. The local ring $A' = \mathscr{O}_{P,x}$ has $\dim(A') = \dim(R)+c$ and $\operatorname{depth}(A') = \operatorname{depth}(R) + c$. Moreover, the residue class ring A has $\dim(A) = \dim(A') - c$ and $\operatorname{depth}(A) = \operatorname{depth}(A') - c$, the former by Krull's Principal Ideal Theorem, the latter by [], Proposition. The assertion on the Serre Condition is immediate. □

Note that the relation $S_j g_j - g_j = 0$ is equivalent to $S_j = 1$, because g_j is regular. In other words, in (3.1) one may simply omit the indeterminate S_j. Also note that if R is integral, so is the Rees ring, and we may regard (3.1) as the R-subalgebra in $\mathrm{Frac}(R)$ generated by the fractions $g_1/g_j, \ldots, g_d/g_j$.

3.5. Let us return now to the wild quotient singularities recalled in 0.2. Let $R = k[[x, y, z]]$ be a formal power series ring over a field k of characteristic $p > 0$, and consider the element

$$f := z^p - (\mu ab)^{p-1} z - a^p y + b^p x.$$

Here $a, b \in k[[x, y]]$ is a system of parameters, and $\mu \in k[[x, y]]$ is a non-zero element that is coprime to both a and b. Let $B := R/(f)$.

Let $\mathfrak{a} := (a, b, z) \subset R$. We call $Z := \mathrm{Bl}_{\mathfrak{a}B}(B) \to \mathrm{Spec}\, B$ the *initial blowing-up*. In Theorem 7.1 and Theorem 9.2, we will later compute a complete resolution $X \to Z \to \mathrm{Spec}\, B$ of this initial blowing-up in two special cases. Recall that the exceptional divisor $E \subset Z$ is given by the sheaf of ideals $\mathscr{O}_Z(1) \subset \mathscr{O}_Z$. Our next proposition computes the term $(E \cdot E_{\mathrm{red}})_Z$, needed for instance when applying Proposition 2.3.

Proposition 3.6. *Keep the assumptions of 3.5. Then the following holds:*

(i) The reduction E_{red} is isomorphic to the projective line $\mathbb{P}^1_k$.
(ii) The z-chart on Z is disjoint from the exceptional divisor, and thus is regular.
(iii) The scheme Z is locally of complete intersection.
(iv) We have $(E \cdot E_{\mathrm{red}})_Z = -1$.
(v) The local ring $\mathscr{O}_{E,\eta}$ at the generic point η of E has length $p \dim_k k[[x, y]]/(a, b)$.

Proof. The blowing-up $\mathrm{Bl}_{\mathfrak{a}}(R)$ is covered by the a-chart, the b-chart and the z-chart. We start by examining the a-chart, which is the spectrum of the ring

$$R[\mathfrak{a}T]_{(aT)} = R[b/a, z/a]/(b/a \cdot a - b, z/a \cdot a - z).$$

Consider the factorization $f = a^p h$ with

$$h := \left(\frac{z}{a}\right)^p - \mu^{p-1} a^{p-1} \left(\frac{b}{a}\right)^{p-1} \left(\frac{z}{a}\right) - y + \left(\frac{b}{a}\right)^p x.$$

The radical J of the ideal generated by h and a in $R[\mathfrak{a}T]_{(aT)}$ clearly contains b. It thus also contains x and y, because a, b is a system of parameters in $k[[x, y]]$. Hence, J also contains z/a and z. It follows that the subscheme $V(h, a)$ of the a-chart is one-dimensional. According to Proposition 3.1, the scheme $\mathrm{Bl}_{\mathfrak{a}B}(B)$ coincides on the a-chart with the effective Cartier divisor defined by the equation $h = 0$. The exceptional divisor is given by the additional equation $a = 0$, thus equals $\mathrm{Spec}\, A$, where A is the quotient of $k[[x, y, z]][b/a, z/a]$ modulo the ideal generated by a, b, z, and $(z/a)^p - y + (b/a)^p x$. Let $Q := (x, y, z/a) \subset A$. Since the classes of $x, y, z/a$ are nilpotent, and since the quotient A/Q is isomorphic to the domain $k[b/a]$, we find that Q is the minimal prime ideal of A.

One easily sees that the z-chart on $\mathrm{Bl}_{\mathfrak{a}}(R)$ is disjoint from the exceptional divisor. The situation for the b-chart is similar to the a-chart and it follows that $\mathrm{Bl}_{\mathfrak{a}B}(B)$ is locally of complete intersection. Moreover, the reduced exceptional divisor $E_{\mathrm{red}} = \mathrm{Spec}\, k[b/a] \cup \mathrm{Spec}\, k[a/b]$ is a copy of $\mathbb{P}^1_k$.

The restriction to E_{red} of the invertible sheaf $\mathscr{O}_Z(1) = \mathscr{O}_Z(-E)$ is generated by the elements $aT/1$ and $bT/1$ on the two charts, respectively. Viewing $a/b \in k[a/b, b/a]^\times$ as a cocycle, one deduces that $\mathscr{O}_Z(1)$ has degree 1 on E_{red}, so that $(E \cdot E_{\mathrm{red}})_Z = -1$.

It remains to compute the length of $\mathscr{O}_{E,\eta}$. The coordinate ring of the exceptional divisor E on the a-chart is given by

$$R[b/a, z/a]/(b/a \cdot a - b, z/a \cdot a - z, h, a).$$

Clearly, the ideal on the right is also generated by b, z, h, a. In turn, the above ring is isomorphic to $k[x, y, b/a, z/a]/(a, b, h)$. Regard the latter as $\Lambda[z/a]/(h)$, where Λ is the polynomial ring in the indeterminate b/a over the local Artin ring $k[x, y]/(a, b)$. The ring extension $\Lambda \subset \Lambda[z/a]/(h)$ if finite and free, because h is a monic in z/a. All coefficients of h except the leading one are nilpotent in Λ, consequently z/a becomes nilpotent modulo h. It follows that $\Lambda \subset \Lambda[z/a]/(h)$ induces bijections on all residue fields. Clearly, the minimal prime $\mathfrak{p} \subset \Lambda$ is generated by x and y. In turn, the local Artin ring $\Lambda_{\mathfrak{p}}$ has length $\dim_k k[x, y]/(a, b)$, whereas the local Artin ring $\mathscr{O}_{E,\eta} = \Lambda_{\mathfrak{p}}[z/a]/(h)$ has length $\deg(h) \cdot \mathrm{length}(\Lambda_{\mathfrak{p}}) = p \cdot \dim_k k[x, y]/(a, b)$. □

Remark 3.7. The ring $B = k[[x, y, z]]/(f)$ can be identified with the ring of invariants A^G for an action of the group $G = \mathbb{Z}/p\mathbb{Z}$ on the ring $A := k[[u, v]]$, as recalled in 0.2, where the generator acts via $u \mapsto u + \mu a$ and $v \mapsto v + \mu b$. We note below that the initial blowing-up $\mathrm{Bl}_{\mathfrak{a}B}(B) \to \mathrm{Spec}(B)$ considered in 3.6 is canonically associated to the action.

Indeed, the fixed scheme of the action is by definition the largest closed subscheme of $\mathrm{Spec}\, A$ on which the action is trivial, and we find that for the above action it corresponds to the ideal $I := (\sigma(u) - u, \sigma(v) - v) = (\mu a, \mu b)$ in A. Under the above identification $B = A^G$ we have $z = ub - va$, and therefore $\mu z \in I$. We find that $(\mu a, \mu b, \mu z) \subseteq I \cap B$. The reverse inclusion also holds since A is flat over $k[[x, y]]$ (same proof as in [45], Lemma 1.5, when $p = 2$ and a similar choice of initial blow-up was also used). Thus the ideals $I \cap B$ and $\mathfrak{a}B = (a, b, z)$ coincide up to the factor μ and, hence, the total spaces of the resulting blowing-ups coincide.

4. Some weighted homogeneous singularities

Let k be an algebraically closed field of characteristic exponent $p \geq 1$. The goal of this section is to describe a resolution of the singularity at the origin on the hypersurface given by the equation

$$W^q - U^a V^b (V^d - U^c) = 0$$

when the integers $p, q, a, b, c, d \geq 1$ are subject to certain mild restrictions. This is achieved in Theorem 4.4. Note that this singularity is not necessarily isolated. The above polynomial is weighted homogeneous, and resolutions of such singularities were already studied by Orlik and Wagreich in [37], [38] and [39], exploiting $\mathbb{G}_m$-actions corresponding to the weights. The former two papers rely on transcendental methods, and the latter mainly treats the case of isolated singularities. Our method is completely algebraic, and relies on the theory of toric varieties and Hirzebruch–Jung singularities.

To compute a resolution of our surface singularity, we first make an initial blow-up that separates the irreducible components of the plane curve $U^a V^b (V^d - U^c) = 0$. We then pass to certain nicer subrings of the charts, and identify their formal completions with suitable monoid rings. This necessitates taking roots of power series along the way, requiring some restrictions on the integers p, q, a, b, c, d as in 4.3.

4.1. Let us start with a brief review of the theory of *Hirzebruch–Jung singularities*. Suppose that $t, r \geq 1$ and $s \geq 0$ are integers such that $\rho := \gcd(t, r, s)$ is prime to p. Consider the ring

$$R := k[U, V, W]/(W^t - U^r V^s).$$

We have a factorization $W^t - U^r V^s = \prod (W^{t/\rho} - \zeta U^{r/\rho} V^{s/\rho})$, where the product runs over the ρ-th roots of unity ζ in k. The corresponding minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_\rho \subset R$ define a partial normalization $R \subset \prod R/\mathfrak{p}_i$, and it usually suffices to understand the rings $R/\mathfrak{p}_i$.

Assume from now on that $\rho = 1$, so that R is an integral domain. Let R' be its normalization. Let D_U and D_V denote the preimages in $\operatorname{Spec} R'$ of the closed subsets of $\operatorname{Spec} R$ defined by $U = W = 0$ and $V = W = 0$, respectively. To describe the resolution of the singularity of $\operatorname{Spec} R'$ at the maximal ideal (U, V, W) when $\operatorname{Spec} R'$ is singular at this point, it is standard to first express R' as the normalization of a different domain R_0, as we now recall. Given the triple (t, r, s), we associate below a unique new triple $(t', 1, s')$ such that R' can be identified with the normalization of the ring $R_0 := k[u, v, w]/(w^{t'} - uv^{s'})$. *This identification is such that the closed subsets D_U and D_V on $\operatorname{Spec} R'$ are again equal to the preimages under the new normalization map $\operatorname{Spec} R' \to \operatorname{Spec} R_0$ of the closed subsets of $\operatorname{Spec} R_0$ defined by $u = w = 0$ and $v = w = 0$, respectively.* We leave it to the reader to check this claim using the explicit description of R_0 recalled below.

Write $r = r_0 + ct$ and $s = s_0 + dt$ for some integers $r_0, s_0, c, d \geq 0$ with $r_0, s_0 < t$. Then the fraction $W/(U^c V^d)$ is integral over R since it satisfies the equation $(W/(U^c V^d))^t = U^{r_0} V^{s_0}$. We can thus replace R by $R[W/(U^c V^d)]$. In particular, if either r or s is divisible by t, then R' is regular above (U, V, W). We define in this case the *fraction type* of R or R' to be 0. If R' is not regular, then upon replacing R with $R[W/(U^c V^d)]$ we may assume that $0 < r, s < t$.

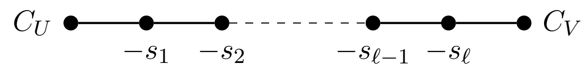
Let $h := \gcd(t, r)$ and $h' := \gcd(t, s)$. Since $\gcd(t, r, s) = 1$, we find that $\gcd(r, h') = \gcd(s, h) = 1$. Thus we can write $ar = 1 + bh'$ and $cs = 1 + dh$ for some non-negative integers a, b, c, d. Let $U_1 := W^{at/h'}/(U^{(ar-1)/h'} V^{as/h'})$ and $V_1 := W^{ct/h}/(U^{cr/h} V^{(cs-1)/h})$. We find that $U_1^{h'} = U$ and $V_1^h = V$. In the integral extension $R[U_1, V_1]$, we find that $W^{t/(hh')} = U_1^{r/h} V_1^{s/h'}$. If t/h' divides r, or if t/h divides s, we find that R' is regular above (U, V, W), and we define again in this case the *fraction type* of R or R' to be 0.

Assume then that R' is not regular. Replacing R with $R[U_1, V_1]$, we may assume now that $h = h' = 1$, and upon replacing R by a larger integral extension if necessary, we can also assume that $0 < r, s < t$.

There exists a unique integer e with $0 < e < t$ and $er = s + ct$ for some integer c. Since $s < t$ by assumption, we find that $c \geq 0$. Consider the ring $R_1 := k[U, V, Z]/(Z^t - U^r V^{s+ct})$. We find that this ring has two natural integral extensions. Indeed, $R_1[Z/V^c]$ is isomorphic to the ring R. Writing $r\rho = 1 + ft$ for some integers $\rho, f \geq 0$, we find that $w := Z^\rho/(UV^e)^f$ is such that $w^r = Z$ and $w^t = UV^e$. Thus $R_1[w]$ is integral over R_1 and isomorphic to $R_0 := k[U, V, W]/(W^t - UV^e)$. We define in this case the *fraction type* of R or R' to be $(t - e)/t$, with $0 < (t - e)/t < 1$ and $\gcd(e, t) = 1$.

Given a resolution of singularities $X \to \operatorname{Spec} R'$, we write $C \subset X$ for the exceptional curve, and C_U and C_V for the strict transforms in X of the Weil divisors D_U and D_V on $\operatorname{Spec} R'$, respectively. We endow all these closed subsets with the induced reduced structure of scheme. The following theorem is well-known (see, e.g., the pictures in [20, page 37] or [8, Theorem 2.4.1]), but we did not find a suitable reference in the literature which also proved the statement regarding the divisors C_U and C_V. We include a sketch of proof below, with references, for the convenience of the reader.

Theorem 4.2. *Let s and t be coprime integers with $0 < s < t$. Let $R := k[U, V, W]/(W^t - UV^s)$ and denote by R' its normalization. There is a resolution of singularities $X \to \operatorname{Spec} R'$ such* *that $C_U \cup C \cup C_V$ is a divisor with simple normal crossings having the following dual graph:*



The integer $\ell \geq 1$ and the self-intersection numbers $-s_i$ are computed from the continued fraction expansion $t/(t-s) = [s_1, \ldots, s_\ell]$ as described in (1.1). *Moreover, the irreducible components of C are isomorphic to $\mathbb{P}^1_k$.*

Proof. The proof relies on the theory of toric varieties, and we refer the reader to the monographs [9], [10], or [20], for the general theory. The book [9] assumes from the onset that the characteristic of k is 0, but the proofs of the results quoted below are valid in all characteristics and can be applied to our purposes. We identify $Z = \operatorname{Spec} R$ as an explicit (non-normal) toric variety, and use the general theory of toric varieties to describe the normalization $Y \to Z$ and the toric resolution $X_\Sigma \to Y$ attached to an explicit fan Σ.

Consider the lattices $N := \mathbb{Z}^2$ and $M := \operatorname{Hom}(N, \mathbb{Z})$. Write $e_1, e_2 \in N$ for the standard basis of N, and $e_1^*, e_2^* \in M$ for the dual basis. Let $\sigma \subset N_\mathbb{R} := N \otimes_\mathbb{Z} \mathbb{R}$ be the closed convex cone generated by the vectors e_2 and $te_1 - (t-s)e_2$. The dual cone $\sigma^\vee \subset M_\mathbb{R}$ is generated by $\alpha := (t-s)e_1^* + te_2^*$ and $\beta := e_1^*$. Let $\gamma := e_1^* + e_2^*$, and let $S \subset M$ be the submonoid generated by α, β, γ. We have the relation $t\gamma = \alpha + s\beta$, and can identify $k[U,V,W]/(W^t - UV^s)$ with the monoid ring $k[S]$ via $U \mapsto \alpha$, $V \mapsto \beta$, and $W \mapsto \gamma$.

Let $S' := \sigma^\vee \cap M$. Clearly, the abelian group M is generated by β and γ. It follows that $\alpha \in M$ and, hence, $S \subseteq S'$. Since S' is always saturated, S' is equal to the saturation of the monoid S. It follows that the normal toric variety Y attached to N and σ, namely $Y := \operatorname{Spec} k[\sigma^\vee \cap M]$, is the normalization of the non-normal toric variety $Z := \operatorname{Spec} k[S]$.

The cone σ is in normal form, and when $t > s > 0$, [9, Theorem 10.2.3] provides an explicit description of a refinement fan Σ of σ such that the induced morphism $X_\Sigma \to Y$ is a toric resolution of singularities. Using the Hirzebruch-Jung continued fraction $[s_1, \ldots, s_\ell]$ of $t/(t-s)$, one constructs a sequence of vectors $u_0 := e_2$, $u_1, \ldots, u_\ell$, $u_{\ell+1} := te_1 - (t-s)e_2$ such that $\sigma = \cup_{i=1}^{\ell+1} \sigma_i$ with σ_i the cone generated by u_{i-1} and u_i. The fan Σ consists in the cones σ_i and their faces.

Using the Orbit-Cone Correspondence [9, Theorem 3.2.6], we find that the ray generated by u_i, $i = 0, \ldots, \ell+1$, corresponds to a curve C_i on X_Σ. Since Σ is a simplicial fan, the intersection products $(C_i \cdot C_j)_{X_\Sigma}$ with $0 \leq i \neq j \leq \ell+1$ can be computed as in [9, Corollary 6.4.3], and are found to equal 1. The self-intersections $(C_i \cdot C_i)_{X_\Sigma}$ for $i = 1, \ldots, \ell$ are computed to equal $-s_i$ using [9, Theorem 10.2.5] along with [9, Theorem 10.4.4].

The curve $C_1 \cup \cdots \cup C_\ell$ is the exceptional divisor of the toric desingularization $X_\Sigma \to Y$. Using the Orbit-Cone Correspondence for the surface Y, we let D and D' denote the curves on Y corresponding to the rays in the cone σ generated e_2 and $te_1 - (t-s)e_2$, respectively. The natural properties of the map $X_\Sigma \to Y$ implies that D is the image of C_0, and D' is the image of $C_{\ell+1}$. It remains to show that D is the reduced preimage of the Weil divisor $U = W = 0$ on Z, and that similarly, D' is the reduced preimage of the Weil divisor $V = W = 0$ on Z. $\square$

4.3. Let $q, a, b, c, d \geq 1$ be integers. Set

$$m := ad + bc + cd \quad \text{and} \quad g := \gcd(c, d).$$

Noting that m/g is an integer, we further set

$$w := \gcd(q, m/g), \quad \text{and} \quad w_a := \gcd(q, m/g, a), \quad w_b := \gcd(q, m/g, b).$$

In our main result below on the resolution of the hypersurface singularity $W^q - U^aV^b(V^d - U^c) = 0$, we assume that

$$\gcd(a, c/g) = \gcd(b, d/g) = 1 \quad \text{and} \quad \gcd(p, wg) = 1. \tag{4.1}$$

Note that the latter condition automatically holds when $p = 1$. The reader will easily check that the condition $\gcd(a, c/g) = 1$ is equivalent to the condition $\gcd(m/g, c/g) = 1$. Similarly, $\gcd(b, d/g) = 1$ if and only if $\gcd(m/g, d/g) = 1$.

Denote by $\alpha, \beta, \gamma \in \mathbb{Q}_{<1}$ the fraction types of the normal Hirzebruch–Jung singularities associated with the triples (t, r, s) given by

$$(\frac{qc}{gw_a}, \frac{m}{gw_a}, \frac{a}{w_a}), \quad (\frac{qd}{gw_b}, \frac{m}{gw_b}, \frac{b}{w_b}), \quad \text{and} \quad (q, \frac{m}{g}, 1),$$

respectively. Finally, set

$$s_0 := \frac{w^2g^2}{qcd} + w_a\alpha + w_b\beta + g\gamma. \tag{4.2}$$

We are now ready to state the main result of this section. Three complements to Theorem 4.4 are given in 4.7, 4.8, and 4.9.

Theorem 4.4. *Set $B := k[U, V, W]/(W^q - U^aV^b(V^d - U^c))$, and assume that the conditions (4.1) holds. With the above notation, we have the following:*

(i) The fraction $s_0 > 0$ is an integer.

(ii) The hypersurface singularity has a resolution of singularities $X \to \operatorname{Spec}(B)$ where, using the notation in 1.2, the exceptional divisor $C \subset X$ has star-shaped dual graph

$$\Gamma = \Gamma(s_0 \mid \underbrace{\alpha^{-1}, \dots, \alpha^{-1}}_{w_a}, \underbrace{\beta^{-1}, \dots, \beta^{-1}}_{w_b}, \underbrace{\gamma^{-1}, \dots, \gamma^{-1}}_{g}).$$

when $\alpha, \beta, \gamma > 0$. When one of α, β, γ, equals 0 (e.g., when q divides m/g), the graph Γ is as above, except that the corresponding w_a chains (resp. w_b or g chains) are removed.

(iii) The curve C has simple normal crossings. All irreducible components of C are copies of $\mathbb{P}^1_k$, except possibly for the central node. When $w = 1$, the central node is also isomorphic to $\mathbb{P}^1_k$.

Proof. Since our ground field k is algebraically closed, we can rewrite the defining polynomial for our hypersurface singularity as

$$f = W^q - U^aV^b \prod_\zeta (V^{d/g} - \zeta U^{c/g}),$$

where the product runs over the g-th roots of unity $\zeta \in k$. Assumption (4.1) ensures that we have exactly $g \geq 1$ distinct factors in the product.

To construct the desired resolution of singularities $X \to \operatorname{Spec}(B)$, we first make an initial blowing-up $Z = \operatorname{Bl}_{\mathfrak{a}B}(B) \to \operatorname{Spec}(B)$, for the ideal $\mathfrak{a} := (U^{c/g}, V^{d/g})$ in the polynomial ring $R = k[U, V, W]$. The ambient blowing-up $\operatorname{Bl}_{\mathfrak{a}}(R)$ has two charts, the $U^{c/g}$-chart and the $V^{d/g}$-chart. The former is given by four generators $U, V, W, V^{d/g}/U^{c/g}$ subject to the single relation

$$\left(\frac{V^{d/g}}{U^{c/g}}\right) \cdot U^{c/g} = V^{d/g}, \tag{4.3}$$

as recalled in Proposition 3.4. On this chart we rewrite the defining polynomial as

$$f = W^q - U^{a+c}V^b \cdot \prod_\zeta (V^{d/g}/U^{c/g} - \zeta). \tag{4.4}$$

Clearly, the radical of the ideal generated by f and $U^{c/g}$ contains U, V, and W. Hence, its zero-locus is one-dimensional, and according to Proposition 3.1 the blowing-up $Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ on the $U^{c/g}$-chart of $\mathrm{Bl}_{\mathfrak{a}}(R)$ is the effective Cartier divisor with equation $f = 0$. In other words, write A' for the coordinate ring of the blowing-up $Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ on the $U^{c/g}$-chart. Then this ring is generated by four indeterminates $U, V, W, V^{d/g}/U^{c/g}$ subject to the two relations (4.3) and $f = 0$ with f as in (4.4).

4.5. The exceptional divisor $E \subset Z$ is given by $f = U^{c/g} = 0$ on this chart. The reduction E_{red} is defined by $U = V = W = 0$, and $V^{d/g}/U^{c/g}$ can be regarded as a coordinate function. The situation on the $V^{d/g}$-chart is symmetric, and we conclude that $E_{\mathrm{red}} = \mathbb{P}^1_k$ is a projective line. This description also yields the intersection number: Recall that the ambient $\mathrm{Bl}_{\mathfrak{a}}(R)$ is the homogeneous spectrum of the Rees ring $R[\mathfrak{a}T]$, so the invertible sheaf $\mathscr{O}_Z(1)$ is generated by $TU^{c/g}$ and $TV^{d/g}$ on our two charts. In turn, the restriction to $E_{\mathrm{red}} = \mathbb{P}^1_k$ is given by the cocycle $U^{c/g}/V^{d/g}$, and it follows that $(E \cdot E_{\mathrm{red}})_Z = -1$.

4.6. Let us note here also that the multiplicity of E is qcd/g^2. This can be seen as follows. On the $U^{c/g}$-chart, the scheme E_{red} is defined by the ideal $Q := (U, V, W)$. Thus the multiplicity of E can be computed as the length of the ring $(A'/(U^{c/g}))_Q$. It is easy to verify that the ring $A'/(U^{c/g})$ is k-isomorphic to the ring $\big(k[U, V, W]/(U^{c/g}, V^{d/g}, W^q)\big)\,[V^{d/g}/U^{c/g}]$, and the claim follows.

The ring A' is locally of complete intersection, but usually fails to be normal. Let $\nu : Y \to Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ denote the normalization morphism. To understand the normalization and minimal resolution of the singularities of the chart $\operatorname{Spec} A'$ of Z, we pass to a subring A of A' with only three generators and one relation that has the same normalization as A'. It turns out that on formal completions, the resolution of singularities of A is given by the theory of toric surface (i.e., Hirzebruch–Jung) singularities. This formal passage to toric varieties requires the existence of certain roots of formal power series. When $p > 1$, their existence follows from Hensel's Lemma together with the conditions (4.1), which imply that $\gcd(m/g, c/g)$ and $\gcd(m/g, d/g)$ are coprime to p.

We proceed as follows: Let A be the k-subalgebra of A' generated by the three elements U, W, and $V^{d/g}/U^{c/g}$. The ring extension $A \subset A'$ is finite, because $A' = A[V]$ and the generator V satisfies the integral equation $V^{d/g} - U^{c/g}(V^{d/g}/U^{c/g}) = 0$ in (4.3). Clearly, $V^{d/g} \in A$, and the relation (4.4) shows that $V^b \in \mathrm{Frac}(A)$. Since we assume that $\gcd(b, d/g) = 1$ in (4.1), we find that V can be written as rational function in V^b and $V^{d/g}$ and, hence, $V \in \mathrm{Frac}(A)$. It follows that the rings A and A' have the same integral closure in $\mathrm{Frac}(A)$. The reduced exceptional divisor on $\mathrm{Spec}(A')$ is defined by the ideal (U, V, W), and the restriction of $\mathrm{Spec}(A') \to \mathrm{Spec}(A)$ to it is a closed embedding, because $V^{d/g}/U^{c/g} \in A$ and thus the map $A \to A'/(U, V, W) = k[V^{d/g}/U^{c/g}]$ is surjective.

It turns out that the subring A has a much nicer description than A', in particular when passing to formal completions along the exceptional divisor. Recall that $m := ad + cd + bc$.

Taking the d/g-power of (4.4) and using equation (4.3) we get a single relation

$$W^{qd/g} = U^{m/g}\left(\frac{V^{d/g}}{U^{c/g}}\right)^b \cdot \prod_\zeta (V^{d/g}/U^{c/g} - \zeta)^{d/g}. \tag{4.5}$$

Since b and d/g are coprime by assumption (4.1), we find that $w^{qd/g} = u^{m/g}z^b\prod_\zeta(z-\zeta)^{d/g}$ is an irreducible polynomial in $k[u,w,z]$. By abuse of notation, we will also say that the *equation* (4.5) *is irreducible*. Using Krull's Principal Ideal Theorem, we conclude that the algebra A is generated by $U, W, V^{d/g}/U^{c/g}$ subject to the single relation (4.5).

To understand the normalization of A, we pass to formal completions $\widehat{A_\mathfrak{m}}$ with respect to maximal ideals $\mathfrak{m}$ of the form $(U, W, V^{d/g}/U^{c/g} - \xi)$ for various scalars $\xi \in k$. Note that these maximal ideals correspond to points on the exceptional divisor.

Let us start with the simplest case where ξ is neither zero nor a g-th root of unity; here it turns out that the normalization of $\widehat{A_\mathfrak{m}}$ is regular. Indeed, the relation (4.5) now takes the form

$$W^{qd/g} = U^{m/g} \cdot \delta \tag{4.6}$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$. To proceed, we first verify that $\gcd(qd/g, m/g, p) = 1$. This is clear when $p = 1$, so let us assume that $p \geq 2$ is prime. Suppose that p divides both qd/g and m/g. Since p does not divide $w := \gcd(q, m/g)$ by hypothesis, we have $p \nmid q$ and, hence, $p \mid d/g$, contradicting $\gcd(d/g, m/g) = 1$, which we also assume in (4.1).

We conclude that there exist positive integers r and s such that $\ell := r(m/g) - s(qd/g)$ is coprime to $p \geq 1$. With Hensel's Lemma we find roots $\delta_1 := \delta^{r/\ell}$ and $\delta_2 := \delta^{s/\ell}$ in $\widehat{A_\mathfrak{m}}$, and obtain a factorization $\delta = \delta_1^{m/g}/\delta_2^{qd/g}$. It follows that $\widehat{A_\mathfrak{m}}$ is isomorphic to the complete local ring described by the same three generators, but with a modified relation (4.6) in which $\delta = 1$. This shows that $\widehat{A_\mathfrak{m}}$ is isomorphic to a complete local ring for a point on the product of a plane curve with the affine line. Consequently, the normalization is indeed regular. Note that the plane curve is usually reducible, and the number of irreducible components is our integer $w := \gcd(q, m/g) = \gcd(qd/g, m/g)$.

Next, assume that $\xi = \zeta$ is one of the g-th root of unity. Rewrite (4.5) as

$$W^{qd/g} = U^{m/g}\left(\frac{V^{d/g}}{U^{c/g}} - \zeta\right)^{d/g} \cdot \delta \tag{4.7}$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$. As in the preceding paragraph, one reduces to the situation $\delta = 1$. Since $\gcd(d/g, m/g) = 1$, the above relation is then irreducible.

Consider the triple $(t, r, s) = (qd/g, m/g, d/g)$. We can identify $\widehat{A_\mathfrak{m}}$ with the completion of $k[u,v,w]/(w^t - u^r v^s)$ at (u, v, w). Using the results reviewed in 4.1 and 4.2 regarding the desingularization of $\operatorname{Spec} k[u,v,w]/(w^t - u^r v^s)$, we find that the singularity on $\widehat{A_\mathfrak{m}}$ is a Hirzebruch–Jung singularity of fraction type γ.

Finally, assume that $\xi = 0$. Our relation becomes

$$W^{qd/g} = U^{m/g}\left(\frac{V^{d/g}}{U^{c/g}}\right)^b \cdot \delta$$

for some unit $\delta \in \widehat{A_\mathfrak{m}}$, and again we reduce to the situation $\delta = 1$. The above equation is usually not irreducible, and the number of irreducible factors is our integer $w_b := \gcd(q, m/g, b)$, which also equals $\gcd(qd/g, m/g, b)$ since we noted in 4.3 that $\gcd(d/g, m/g) = 1$. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_{w_b} \subset \widehat{A_\mathfrak{m}}$ be the resulting minimal prime ideals.

Consider the triple $(t, r, s) = (qd/(gw_b), m/(gw_b), b/(gw_b))$. We can identify $\widehat{A_\mathfrak{m}}/\mathfrak{p}_i$ with the completion of $k[u, v, w]/(w^t - u^r v^s)$ at (u, v, w). Using the results reviewed in 4.1 and 4.2 regarding the desingularization of $\operatorname{Spec} k[u, v, w]/(w^t - u^r v^s)$, we find that the singularity on $\widehat{A_\mathfrak{m}}/\mathfrak{p}_i$ has the resolution of a Hirzebruch–Jung singularity of fraction type β. The number of such singularities on the normalization of $\widehat{A_\mathfrak{m}}$ is $w_b \geq 1$.

The situation on the $V^{d/g}$-chart is symmetric, where $w_a \geq 1$ Hirzebruch–Jung singularities of fraction type α appear. Summing up, we have described the singularities appearing on the normalization $\nu : Y \to Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$.

Recall from 4.5 that the exceptional divisor $E \subset Z$ has reduction $E_{\mathrm{red}} = \mathbb{P}^1_k$, with coordinate rings $k[V^{d/g}/U^{c/g}]$ and $k[U^{c/g}/V^{d/g}]$. Write $D := \nu^{-1}(E)$ for the preimage of the exceptional divisor under the map ν. We now analyze the induced morphism $D_{\mathrm{red}} \to E_{\mathrm{red}}$. This morphism is flat, because E_{red} is regular. The formal description of the normalization $\nu : Y \to Z$ via inclusions $k[[S]] \subset k[[S']]$ of monoid rings shows that D_{red} is regular. Equation (4.6) implies that

$$\deg(D_{\mathrm{red}}/E_{\mathrm{red}}) = w = \gcd(q, m/g). \tag{4.8}$$

In a similar way, Equation (4.7) tells us that $D_{\mathrm{red}} \to E_{\mathrm{red}}$ is completely ramified over the points where $V^{d/g} = \xi$ is a g-th root of unity. Hence, the curve D_{red} is connected. Since it is also regular, it is in fact irreducible. We can then apply Proposition 2.3 along with 4.5 and 4.6 and obtain that

$$(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = \frac{w^2}{(qcd/g^2)} (E \cdot E_{\mathrm{red}})_Z = -w^2 g^2/qcd.$$

Let $X \to Y$ be the resolution of singularities obtained by resolving the Hirzebruch–Jung singularities of fraction types α, β and γ occurring on Y. The resulting dual graph Γ is star-shaped, with the central node corresponding to the strict transform $C_0 \subset X$ of $D_{\mathrm{red}} \subset Y$. When $\gamma > 0$, there are g terminal chains obtained from the continued fraction development of $1/\gamma = [s_1, \ldots, s_\ell]$. Using the identification of $\widehat{A_\mathfrak{m}}$ with the completion of $k[u, v, w]/(w^{qd/g} - u^{m/g} v^{d/g})$ at (u, v, w) discussed above, as well as Theorem 4.2 and the identifications reviewed in 4.1, one sees that the vertex of the terminal chain adjacent to the central node has self-intersection $-s_1$. The situation for the other Hirzebruch–Jung singularities is similar.

It is now an easy matter to compute the self-intersection $(C_0 \cdot C_0)_X$ using Proposition 2.2, which asserts that $(C_0 \cdot C_0)_X = (D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y - \sum_i \delta_i$. There are w_a correcting terms α, w_b correcting terms β, and g correcting terms γ (see just before 2.2 for the correcting term of a chain). Hence, $|(C_0 \cdot C_0)_X| = s_0$, as desired. Since $(C_0 \cdot C_0)_X$ is the self-intersection of a curve on a regular surface, we find that it must be a negative integer, proving (i). To complete the proof of Theorem 4.4 it remains to show in (iii) that the central node C_0 is a rational curve when $w = 1$. This is done using the following proposition. □

Proposition 4.7. *Keep the hypotheses of Theorem* 4.4. *Let $v_0 \in \Gamma$ be the central node, and let $C_0 \subset X$ be the corresponding curve on the resolution $X \to \operatorname{Spec} B$. We have $h^1(\mathscr{O}_{C_0}) = (g(w-1) + 2 - w_a - w_b)/2$. In particular, when $w = 1$, $h^1(\mathscr{O}_{C_0}) = 0$.*

Proof. Consider the ramified covering $C_0 \to E_{\mathrm{red}} = \mathbb{P}^1_k$ induced from the morphism $X \to Y$. It follows from (4.8) that the degree of this map is w. Assumption (4.1) ensures that this degree is coprime to the characteristic exponent, so that the map is separable. Let us regard the closed points on $\mathbb{P}^1_k$ as elements $\xi \in k \cup \{\infty\}$. The description of the normalization of the rings $\widehat{A_\mathfrak{m}}$ in the preceding proof shows that $C_0 \to \mathbb{P}^1_k$ is totally ramified over each of the g-th roots of unity in k, and therefore the ramification indices are coprime to p. Furthermore, there are w_a points in C_0 over $\xi = 0$ and all these points have the same ramification index w/w_a. Similarly, there are w_b points in C_0 over $\xi = \infty$ with ramification index w/w_b. Applying the Riemann–Hurwitz Formula $2h^1(\mathscr{O}_{C_0}) - 2 = w(2h^1(\mathscr{O}_{\mathbb{P}^1_k}) - 2) + \sum_x (e_x - 1)$, we get the desired formula. □

The scheme $\operatorname{Spec}(B)$ contains two copies of the affine line, given by the equations $U = W = 0$ and $V = W = 0$. Write C_U and C_V for their respective strict transforms in X with respect to the resolution $X \to \operatorname{Spec}(B)$. For a later application in Theorem 7.1, we explicitly determine below how these curves intersect the exceptional divisor $C \subset X$ *when* $w = 1$. Under this additional hypothesis, the partial resolution $Y \to \operatorname{Spec} B$ contains exactly one Hirzebruch–Jung singularity of fraction type α and one of type β. Let Δ_α and Δ_β be the terminal chains of Γ resulting from resolving these two singularities. Write C_α and C_β for the irreducible components of C corresponding to the terminal vertices of Γ lying on Δ_α and Δ_β, respectively.

Proposition 4.8. *Keep the hypotheses of Theorem* 4.4*. Assume that* $w = 1$*. Then the strict transform* C_V *intersects the exceptional divisor* C *only in* C_β*, with intersection number* $(C_V \cdot C_\beta)_X = 1$*. Likewise,* C_U *intersects* C *only in* C_α*, with* $(C_U \cdot C_\alpha)_X = 1$*.*

Proof. By symmetry, it suffices to verify the first assertion. Let us first work with the effective Cartier divisor on $\operatorname{Spec}(B)$ given by $V^{d/g} = 0$. Its strict transform $C'_V \subset X$ has the same support as C_V. Using the notation from the proof of Theorem 4.4, we see that its image on $\operatorname{Spec}(A)$ is given by $V^{d/g}/U^{c/d} = 0$. Using Theorem 4.2 one infers that C'_V intersects only C_β, and that its reduction has intersection number $(C_V \cdot C_\beta)_X = 1$. □

Proposition 4.9. *Keep the hypotheses of Theorem* 4.4*, and suppose furthermore that* $p = q$*. Set* $a_p := 1$ *if* $p \mid a$*, and* $a_p = 0$ *otherwise. Similarly, set* $b_p := 1$ *if* $p \mid b$*, and* $b_p = 0$ *otherwise. Let* N *denote the intersection matrix of the resolution of the hypersurface singularity*

$$W^p - U^a V^b (V^d - U^c) = 0$$

described in Theorem 4.4*. Then* $|\Phi_N| = p^{g+1-a_p-b_p}$*, and the group* Φ_N *is killed by* p*.*

Proof. First note that for $q = p = 1$ the assertion is trivially true, because then our hypersurface singularity is actually regular. So we may assume that $q = p \geq 2$ is a prime number. The triples (t, r, s) in 4.3 specialize to $(pc/g, m/g, a)$, $(pd/g, m/g, b)$, and $(p, m/g, 1)$. From our assumptions (4.1) one easily sees that m/g is coprime to p, pc/g and pd/g. In particular we have $w = w_a = w_b = 1$. Furthermore, the resulting reduced fractions $\alpha, \beta, \gamma \in \mathbb{Q}$ have as denominators the integers $p^{1-a_p}c/g$, $p^{1-b_p}d/g$, and p, respectively. According to Theorem 4.4, the graph Γ_N is star-shaped. Thus we may compute the determinant of the intersection matrix with Proposition 1.3 and obtain

$$|\det(N)| = (p^{1-a_p}c/g)(p^{1-b_p}d/g)p^g(s_0 - \alpha - \beta - g\gamma).$$

The last factor is g^2/pcd in light of the formula (4.2) for the self-intersection $-s_0$ of the central node in Theorem 4.4. Thus $|\Phi_N| = |\det(N)| = p^{g+1-a_p-b_p}$.

The group structure of Φ_N can be obtained by computing the Smith Normal Form of the matrix N, using a row and column reduction of N. Reducing the intersection matrix of each terminal chain as in [27] Lemma 2.5, we find that the matrix N is equivalent to a block diagonal matrix with two blocks, a square matrix A of size $(g+3)\times(g+3)$ that we describe below, and an identity matrix:

$$A := \begin{pmatrix} -s_0 & * & * & * & \dots & * \\ 1 & -p^{1-a_p}c/g & 0 & 0 & & 0 \\ 1 & 0 & -p^{1-b_p}d/g & 0 & & 0 \\ 1 & 0 & 0 & -p & & 0 \\ \vdots & & & & \ddots & \vdots \\ 1 & & & & & -p \end{pmatrix}.$$

The matrix $N \otimes \mathbb{F}_p$ has $g + a_p + b_p$ rows equal to $(1, 0, \dots, 0)$, and we see that its rank is at most $r = 1 + b_p + a_p + 1$. In turn, the vector space dimension of the cokernel is at least $g + 3 - r = g + 1 - a_p - b_p$. It follows that $\Phi_N = \Phi_N \otimes \mathbb{F}_p$. □

Remark 4.10. The explicit resolution of $W^p - UV(V - U^p) = 0$ is needed in the proof of Theorem 7.1. In this case, the intersection matrix is $N = N(2 \mid p/(p-1), p/(p-1), p^2/(2p-1))$, with $|\Phi_N| = p^2$. When p is odd, we do not know if this intersection matrix can occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. When $p = 2$, this equation defines the singularity D_6^0 with trivial local fundamental group [3]. The singularity D_6^1 is a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity (8.6).

More generally, one might wonder whether every intersection matrix arising in Proposition 4.9 can occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. We discuss the case of $W^p - U^pV^p(V^{pm+1} - U^{pn+1})$ and $W^p - UV(V^{pm-1} - U^{pn-1})$ in Theorem 5.3. We note in 8.6 how the intersection matrix of the resolution of the singularity defined by $W^p - UV(V^{pm} - U^{pn-1}) = 0$ might occur as the intersection matrix of the resolution of a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

Remark 4.11. The resolution $X \to Y \to \operatorname{Spec} B$ provided in Theorem 4.4 is not always minimal. This can be seen already in the case where $q = 1$, in which case $\operatorname{Spec} B$ is regular, but the exceptional divisor C on X is not reduced to a point. Indeed, the graph Γ consists in this case of a central node of self-intersection -1 with two terminal chains obtained by resolving Hirzebruch–Jung singularities associated with the triples $(c/g, m/g, a)$ and $(d/g, m/g, b)$. The fraction types of these triples are independent of a and b. Indeed, let $\alpha, \beta > 0$ be the unique positive integers such that $\alpha(d/g) + \beta(c/g) = 1 + (c/g)(d/g)$. Then the triple $(c/g, m/g, a)$ reduces to $(c/g, 1, \alpha)$, and $(d/g, m/g, b)$ reduces to $(d/g, 1, \beta)$.

Other examples where the resolution is not minimal can also be obtained when $q > 1$; for instance, when $p = 2$, the resolution graphs appearing in Example 8.2 are the resolution graphs associated with the cases $W^2 - U^2V^2(V^7 - U^3) = 0$ and $W^2 - U^2V(V^4 - U^3) = 0$, respectively.

5. Brieskorn singularities

Let k be an algebraically closed field of characteristic exponent $p \geq 1$. Let $q, c, d \geq 2$ be integers, with q coprime to cd. Let

$$B := k[[x, y, z]]/(z^q + x^c + y^d).$$

We study in this section properties of the singularity $\operatorname{Spec} B$. Let $g := \gcd(c, d)$.

Theorem 5.1. *Assume that* $\gcd(p, g) = 1$. *Then* $\operatorname{Spec} B$ *admits a star-shaped resolution of singularities* $X \to \operatorname{Spec} B$ *whose associated intersection matrix is*

$$N = N(s_0 \mid a_1/b_1, a_2/b_2, \underbrace{a_0/b_0, \ldots, a_0/b_0}_{g \text{ entries}}),$$

where N *is specified as follows (notation as in* 1.2*). Let*

$$a_1 := c/g, \quad a_2 := d/g, \text{ and } a_0 := q.$$

Set $\ell_0 := cd/g$, $\ell_1 := dq/g$, *and* $\ell_2 := cq/g$, *and define* b_i *by* $b_i\ell_i \equiv -1 \mod a_i$. *Finally, set*

$$s_0 := g^2/cdq + b_1/a_1 + b_2/a_2 + gb_0/q.$$

In case $a_1 = 1$ *(resp.* $a_2 = 1$*), in which case* $b_1 = 0$ *(resp.* $b_2 = 0$*), we remove the term* a_1/b_1 *(resp.* a_2/b_2*) from the matrix* N.

When $q = p$, *the associated discriminant group* Φ_N *is killed by* p *and has order* p^{g-1}.

Proof. Theorem 4.4 provides a resolution of the weighted homogeneous singularity

$$C := k[[x, y, Z]]/(Z^q - x^q y^q (y^d - x^c))$$

when q is coprime to cd. The scheme $\operatorname{Spec} C$ is not normal, and the natural map $C \to B$, with $Z \mapsto zxy$, induces a finite birational morphism $\operatorname{Spec} B \to \operatorname{Spec} C$. Hence, $\operatorname{Spec} B$ has the same resolution as $\operatorname{Spec} C$. The reader will check that the matrix N_C associated to the resolution of $\operatorname{Spec} C$ in Theorem 4.4 is the same as the matrix N appearing in the statement of Theorem 5.1. The discriminant group Φ_N is computed in Proposition 4.9. □

Remark 5.2. A resolution of the Brieskorn singularity of the form $x^c + y^d + z^e = 0$ is known over the complex numbers thanks to the work of [18, Theorem, page 232] when c, d, and e are pairwise coprime, and [37] in general. An explicit description for the intersection matrix N and dual graph Γ_N of a resolution is found for instance in [47], page 284, with a formula giving the self-intersection $-s_0$ of the node given on page 287.

Let now $p > 1$ be prime. When p is coprime to cd, the intersection matrix for the resolution of $z^p + x^c + y^d = 0$ obtained in Theorem 5.1 is the same as the intersection matrix obtained in characteristic 0. Some characteristic $p > 1$ examples appear explicitly already in the literature, such as the case of $z^p + x^2 + y^{p+2} = 0$ when p is odd, treated in [34], Lemma 3.13.

Assume that $p > 1$ is prime and divides cd. The Brieskorn singularity $z^p + x^c + y^d = 0$ has then a resolution in characteristic p which is quite different than in characteristic 0. Indeed, assume that $c = p\gamma$ for some integer γ, and $\gcd(p, d) = 1$. Then in characteristic p, $z^p + x^c + y^d = (z + x^\gamma)^p + y^d$. It follows that the normalization of $k[[x, y]][z]/(z^p + x^c + y^d)$ is regular when $\operatorname{char}(k) = p$. On the other hand, in the case for instance of $z^2 + x^3 + y^6 = 0$ in characteristic 0 (a case which is not covered by Theorem 5.1), the minimal resolution is a smooth elliptic curve of self-intersection -1. This explicit example of a resolution in characteristic 0 (and many others) is found for instance in [23, page 1290].

Theorem 5.3. *Let* $B := k[[x, y, z]]/(f)$, *where* $f(x, y, z)$ *is a weighted homogeneous polynomial of the following form, with* $n, m \geq 1$*:*

(i) $z^p + x^{pm+1} + y^{pn+1}$,
(ii) $z^p + xy(x^{pm-1} - y^{pn-1})$, *and*
(iii) $z^p - x^2 + 2y^{p+1}$ *when* $p \geq 3$.

Then $\operatorname{Spec} B$ *is a wild* $\mathbb{Z}/p\mathbb{Z}$*-quotient singularity. Moreover, the fundamental group of the punctured spectrum* $\operatorname{Spec} B \setminus \{\mathfrak{m}_B\}$ *is trivial.*

Proof. The proof of the theorem is similar for each of the three types of homogeneous polynomials. In each case, there exists a family of rings B_μ, μ homogeneous in $k[x,y]$, such that the ring B can be identified with the ring $B_{\mu=0}$, and such that when $\deg(\mu)$ is large enough, there is an isomorphism between $B_{\mu=0}$ and B_μ. The family B_μ is constructed such that when $\mu \neq 0$ is chosen adequately, the ring B_μ is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

For the weighted homogeneous form in (iii), we use the family B_μ (with $\mu \in k[y]$) described in Theorem 6.3. For the weighted homogeneous forms in (i) and (ii), we use the families discussed in [31] and recalled in 0.2. More precisely, fix a system of parameters a, b in $k[[x,y]]$. Consider the family of hypersurface singularities $\operatorname{Spec} B_\mu$, $\mu \in k[[x,y]]$, with

$$B_\mu := k[[x,y,z]]/(z^p - (\mu ab)^{p-1}z - a^p y + b^p x).$$

Let $G = \mathbb{Z}/p\mathbb{Z}$. When μ is a unit in $k[[x,y]]$, B_μ is isomorphic to the ring of invariants A^G of an action of G on $A = k[[u,v]]$, and in this case the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified precisely at the closed point. When μ is not zero and is coprime to a and coprime to b, then again B_μ is isomorphic to the ring of invariants A^G of an action of G on $A = k[[u,v]]$, but in this case the morphism $\operatorname{Spec} A \to \operatorname{Spec} A^G$ is ramified in codimension 1. The cases (i) and (ii) are obtained by setting $a = y^n$ and $b = x^m$, and $a = x^m$ and $b = y^n$, respectively.

We now claim that it is possible to find a homogeneous polynomial μ of large enough degree such that $B := k[[x,y,z]]/(f)$ is isomorphic over k to B_μ. In the cases (i) and (ii), we note that the homogeneous polynomial $\mu := x^t + y^t$ is coprime to both a and b, so that the corresponding $\operatorname{Spec} B_\mu$ is a quotient singularity associated with an action that is ramified in codimension 1.

To prove the existence of a k-isomorphism from $B := k[[x,y,z]]/(f)$ to B_μ, we use the Lemma in [16], 2.6, page 345. For the details of the proof of this Lemma, the authors of [16] refer the reader to the paper [5]. Recall that the Tjurina ideal of f is $j(f) := (f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, and that there exists an integer $s > 0$ such that $(x,y,z)^s \subseteq j(f)$ if and only if the Tjurina number $\tau := \dim_k(k[[x,y,z]]/j(f))$ is finite. Then the Lemma in [16], 2.6, implies that if $\deg(\mu g) > 2\tau$ (with $g \in k[[x,y,z]]$), then $B := k[[x,y,z]]/(f)$ is isomorphic over k to $k[[x,y,z]]/(f + \mu g)$.

In each case above, we have shown that $\operatorname{Spec} B$ is isomorphic to a quotient singularity $\operatorname{Spec} B_\mu$ such that B_μ is the ring of invariants of an action of $\mathbb{Z}/p\mathbb{Z}$ on the ring $A := k[[u,v]]$ such that the morphism $\operatorname{Spec} A \to \operatorname{Spec} B_\mu$ is ramified in codimension 1. Corollary 1.2 (ii) in [3] shows that the fundamental group of the punctured spectrum of $\operatorname{Spec} B$ is trivial. □

Remark 5.4. Consider the equation $f := z^q + x^c + y^d$ with q, c, d three distinct primes. Let k be a field of characteristic p. Let $B := k[[x,y,z]]/(f)$. Theorem 4.4 shows that the intersection matrix of the resolution of $\operatorname{Spec} B$ is the same in all three characteristics $p = q, c, d$, and has determinant 1. It is natural to wonder whether this matrix can occur in more than one characteristic as the intersection matrix attached to a resolution of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.

Consider the intersection matrix with resolution graph E_8. In Artin's notation in [3], $f := z^2 + x^3 + y^5$ defines the singularity $\operatorname{Spec} B$ denoted by E_8^0, with resolution graph E_8. This singularity is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity when $p = 2$ (see Theorem 5.3, (i)). When $p = 5$, a different singularity, denoted by E_8^1 in [3], also has resolution graph E_8 and is a wild $\mathbb{Z}/5\mathbb{Z}$-quotient singularity.

Theorem 5.5. *Let p be prime. Let $s \geq 0$.*

(a) *Assume that either $s \not\equiv 1 \mod p$, or that p is odd and $s = 1$. Then there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity $\operatorname{Spec} A^G$ with associated action ramified precisely at the origin, and such that the discriminant group of a resolution of the singularity has order p^s.*

(b) *Assume that either p is odd and that $s \equiv 1 \mod p$, or that $p = 2$ and $s = 1$. Then there exists a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity* $\operatorname{Spec} A^G$ *with associated action ramified in codimension* 1 *and such that the discriminant group of a resolution of the singularity has order p^s.*

Proof. (a) The cases $s = 0$ and $s = 1$ are covered by Theorem 7.1 and Theorem 6.3, respectively. The cases with $s \geq 2$ and $s \not\equiv 1 \mod p$ were obtained earlier in the papers [30] and [33].

(b) When $s \equiv 1 \mod p$ and $s \geq p+1$, we use the Brieskorn singularities exhibited in Lemma 5.6, and apply Theorem 5.1 and Theorem 5.3. The case $p = 2$ and $s = 1$ was noted by Artin and is discussed in section 8. The case $s = 1$ is treated in Theorem 9.4. □

Lemma 5.6. *Let p be an odd prime, and r be any positive integer. Then there are integers $m, n > 0$ such that the discriminant group Φ_N of the intersection matrix N associated with the Brieskorn singularity $z^p + x^{pm+1} + y^{pn+1} = 0$ described in* 5.1 *is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{pr+1}$.*

Proof. In view of Theorem 5.1, we need to produce integers n and m such that $\gcd(pn+1, pm+1) = pr+2$. For this, it suffices to take $n := (pr+r+2)/2$, so that $pn+1 = (pr+2)(p+1)/2$, and to set $m = n + (pr+2)$ so that $m := (3pr+r+6)/2$. □

Note that not all elementary abelian p-groups appear as discriminant groups Φ_N attached to the intersection matrix N associated with a Brieskorn singularity $z^p + x^{pm+1} + y^{pn+1} = 0$. Indeed, for all $m, n > 0$, the integer $g = \gcd(pm+1, pn+1)$ is never divisible by p. Thus in the above setting Φ_N cannot be isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{pr-1}$ for any $r > 0$.

Remark 5.7. Let B be a complete noetherian local ring that is two-dimensional and normal, with algebraically closed residue field. Consider a resolution of singularities $X \to \operatorname{Spec} B$, with associated intersection matrix N. Recall that there is a natural surjection $\operatorname{Cl}(B) \to \Phi_N$ (see [25], 14.4). In particular, when $\det(N) \neq 1$, we obtain a natural non-trivial finite quotient of $\operatorname{Cl}(B)$ from the computation of a resolution of $\operatorname{Spec} B$.

The study of the class group $\operatorname{Cl}(B)$ of $B := k[[x,y]][z]/(z^p - x^c - y^d)$ was initiated by Samuel in [44], Proposition (3) in section 6 (see also [13], Chapter IV, section 17). When $p = 2$, Samuel is able to exhibit by a completely algebraic method a finite quotient of $\operatorname{Cl}(B)$ of order p^{g-1}, where $g := \gcd(c,d)$. Under the hypothesis of Theorem 5.1, p^{g-1} would also be the order of the corresponding group Φ_N.

6. Analogues of the E_6 singularities

Let k be an algebraically closed field of characteristic $p \geq 3$. Let $\mu \in k[y]$, $\mu \neq 0$. Consider the automorphism σ of the polynomial ring $k[u,v,y]$ given by

$$u \longmapsto u + \mu v, \quad v \longmapsto v + \mu y, \quad \text{and} \quad y \longmapsto y.$$

This automorphism has order p. We exclude the case $p = 2$ in this section because when $p = 2$, σ has order 4. Let

$$N_u := \operatorname{Norm}(u) = \prod_{d=0}^{p-1} \sigma^d(u) = \prod_{d=0}^{p-1} \left(u + d\mu v + \frac{d(d-1)}{2}\mu^2 y \right),$$

and

$$x := \operatorname{Norm}(v) = v^p - (\mu y)^{p-1} v.$$

Finally, let

$$z := v^2 - \mu y v - 2yu.$$

Let $G := \mathbb{Z}/p\mathbb{Z}$ act on $k[u,v,y]$ through σ. When $\mu = 1$, the ring of invariants $k[u,v,y]^G$ is known to be generated by x, y, z, and N_u, subject to a single relation (see e.g., [7], 4.10). This relation was made explicit by Peskin who showed in [41], Lemma 5.6, that $h_{\mu=1}(x,y,z,N_u) = 0$, where,

$$h_{\mu=1}(x,y,z,N_u) := z^p + 2y^pN_u - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} y^{2p-2n} z^n.$$

Here $C_{n-1} := (2n-2)!/n!(n-1)!$ are the Catalan numbers.

When $\mu \neq 1$, the above result can be used to show that x, y, z, and N_u, are subject to the relation

$$h(x,y,z,N_u) := z^p + 2y^pN_u - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n = 0.$$

Indeed, the morphism $k[u,v,y] \to k[U,V,y]$, which sends $u \mapsto \mu^2 U$, $v \mapsto \mu V$, and $y \mapsto y$, is G-equivariant when $k[u,v,y]$ is endowed with the action of σ, and $k[U,V,y]$ is endowed with the action of σ_1, with $\sigma_1(U) = U + V$ and $\sigma_1(V) = V + y$.

For any choice of $c(y) \in yk[y]$, we can consider the ring

$$A_0 := k[u,v,y]/(N_u - c(y)).$$

We will slightly abuse notation and denote again by x, y, z, u, v, the classes of these elements in A_0. Clearly, the automorphism σ fixes the polynomial $N_u - c(y)$, and thus induces an automorphism on A_0, again denoted by σ. This endows A_0 with an action of G. Let A denote the formal completion $\widehat{A_0}$ of the ring A_0 at the maximal ideal (u,v,y).

The fixed scheme of the G-action on $\mathrm{Spec}(A_0)$ is given by the ideal $I := (\mu v, \mu y)$. When $\mu \in k^*$, $I = (v,y) = (u^p, v, y)$, and thus its radical is the maximal ideal (u,v). Hence, the morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is ramified precisely at the origin. When $\mu \neq 0$ is not a unit, the morphism $\mathrm{Spec}\, A \to \mathrm{Spec}\, A^G$ is ramified in codimension 1.

The study of the singularities of the rings $\mathrm{Spec}\, A^G$ when $\mu = 1$ was initiated by Peskin ([40], Chapter III, §4, and [41], Section 5). In this section we treat the case where $c(y) = y$, and obtain in this way a family of wild quotient singularities A^G of multiplicity 2 whose discriminant groups have order $|\Phi| = p$. For $p > 3$, these singularities can be viewed as analogues of the rational double point of type E_6^1 in characteristic $p = 3$, which was shown to be a wild $\mathbb{Z}/3\mathbb{Z}$-quotient singularity by Artin [3].

Proposition 6.1. *Let $c(y) := y$. Let $\mu \in k[y]$. Then the ring A_0 is a domain, the formal completion A is regular, and the canonical map $k[[u,v]] \to A$ is bijective.*

Proof. The expression $f(u,v,y) := N_u - y$ is a monic polynomial of degree p in the variable u over the factorial ring $k[v,y]$, with constant term $f(0,v,y) = -y$. Its Newton polygon with respect to the y-adic valuation is thus the straight line from $(0,0)$ to $(p,1)$ in $\mathbb{R}^2$, and we conclude with the Eisenstein–Dumas Theorem [35] that f is irreducible as a polynomial over $k(v,y)$, and with the Gauß Lemma that it is also irreducible as a polynomial over $k[v,y]$.

The ring A_0 and its formal completion A are thus two-dimensional domains. To see that the local ring A is regular, we have to check that the cotangent space $\mathfrak{m}_A/\mathfrak{m}_A^2$ has vector space dimension at most two. Indeed, this vector space is generated by u, v, y. In light of the relation $N_u - y = 0$, the class of y vanishes. In turn, the canonical map $k[[u,v]] \to A$ between complete local rings induces a bijection on cotangent spaces, and is thus bijective. ☐

Let $\mu \in k[y]$. Abusing notation slightly, we let $h(x,y,z) \in k[x,y,z]$ be defined as

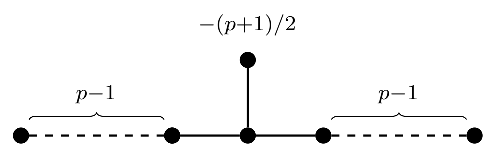
$$h(x,y,z) := z^p + 2y^{p+1} - x^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} (\mu y)^{2p-2n} z^n. \tag{6.1}$$

We let $B_\mu := k[[x,y,z]]/(h)$.

Proposition 6.2. *Let $\mu \in k[y]$, $\mu \neq 0$. Then the canonical map $k[[x,y,z]]/(h) \to A^G$ is bijective. In particular, the wild quotient singularity A^G is a complete intersection of multiplicity two.*

Proof. Both local rings $k[[x,y,z]]/(h)$ and A^G are Cohen–Macaulay, and finite $k[[x,y]]$-algebras of rank p. One easily sees that $h(x,y,z) = 0$ defines an isolated singularity, by using the relations $h_x = -2x$ and $2z(\mu + y\mu_y)h_z + yh_y = 2y^{p+1}$ between partial derivatives. It follows that $k[[x,y,z]]/(h)$ is normal, and that the canonical map induces a bijection on the field of fractions. The map in question is thus bijective, by Zariski's Main Theorem. Clearly, the monomial x^2 is the lowest term in $h(x,y,z)$, and it follows that the complete intersection A^G has multiplicity two. ☐

Theorem 6.3. *Let $\mu \in k[y]$. Let $X \to \operatorname{Spec}(B_\mu)$ be the minimal resolution of singularity, with associated intersection matrix N. Then the dual graph Γ_N is independent of μ, and takes the form:*



The associated discriminant group Φ_N has order p.

Proof. Consider the blow-up $Z \to \operatorname{Spec}(B_\mu)$ of $\operatorname{Spec}(B_\mu)$ with respect to the ideal (x,y,z). Let $Y \to Z$ denote the normalization of Z. Let E denote the exceptional divisor of the blow-up, and let D denote its schematic preimage in Y.

The blow-up Z is covered by three charts that we call the x-chart, y-chart, and z-chart. We consider in detail below the y-chart and show that its normalization contains a unique singular point y_0. Proceeding in an analogous way as for the y-chart, the reader will check that the normalizations of the x-chart and the z-chart are regular.

On the y-chart, the strict transform of $h(x,y,z) = 0$ becomes

$$\left(\frac{z}{y}\right)^p y^{p-2} + 2y^{p-1} - \left(\frac{x}{y}\right)^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} \mu^{2p-2n} y^{2p-n-2} \left(\frac{z}{y}\right)^n = 0.$$

The fraction $x/y^{(p-1)/2}$ satisfies the integral equation

$$\left(\frac{z}{y}\right)^p y + 2y^2 - \left(\frac{x}{y^{(p-1)/2}}\right)^2 + \sum_{n=2}^{(p+1)/2} (-1)^n C_{n-1} \mu^{2p-2n} y^{p-n+1} \left(\frac{z}{y}\right)^n = 0. \tag{6.2}$$

Write $g = (z/y)^p y + 2y^2 - (x/y^{(p-1)/2})^2 + \ldots$ for the polynomial on the left. Up to radical, its jacobian ideal contains y, because this defines the exceptional divisor on the y-chart and there are no singularities outside. Obviously it also contains $x/y^{(p-1)/2}$. Using the partial derivative

$g_y = (z/y)^p + \ldots$, we see that it furthermore contains z/y. Thus the normalization of the y-chart is given by the three variables $z/y, y, x/y^{(p-1)/2}$ and the equation $g = 0$.

We claim that $D_{\rm red}$ is a smooth rational curve, and that $(D_{\rm red} \cdot D_{\rm red})_Y = -1/2$. For this it suffices to check analogously as in Proposition 3.6 that the curve $E_{\rm red}$ is regular, and that $(E \cdot E_{\rm red})_Z = -1$. Then one checks that the natural map $D_{\rm red} \to E_{\rm red}$ is an isomorphism. Finally, noting that the multiplicity of E is $\ell = 2$, we apply the formula $(D_{\rm red} \cdot D_{\rm red})_Y = (E \cdot E_{\rm red})_Z/\ell$ in Proposition 2.3 to obtain the claim.

Regarded as a formal power series, the initial term of g is the quadratic polynomial $2y^2 - (x/y^{(p-1)/2})^2$, which is thus a product of two linear factors since k is algebraically closed. According to Lemma 6.4 below, the singularity must be a rational double point of type A_m for some integer $m \geq 1$. To determine this integer, we compute the Tjurina number of the singularity, which is the colength of the ideal generated by g and its partial derivatives. Setting $x' = x/y^{(p-1)/2}$ and $z' = z/y$, the partial derivatives take the form

$$g_{x'} = 2x', \quad g_y = z'^p + y \cdot \text{unit} \quad \text{and} \quad g_{z'} = \sum_{n=2}^{(p+1)/2} (-1)^n n C_{n-1} \mu^{2p-2n} y^{p-n+1} z'^{n-1}.$$

We now use $g_y = 0$ to substitute for y in the equations $g(0, y, z') = 0$ and $g_{z'}(0, y, z') = 0$, and infer that the jacobian ideal has colength $\tau = 2p$. Recall that the Tjurina number for the A_m-singularity, which is formally isomorphic to $Z^{m+1} - XY = 0$, is given by

$$\tau = \begin{cases} m & \text{if } p \text{ does not divide } m+1; \\ m+1 & \text{else.} \end{cases}$$

It follows that either $m = 2p - 1$ or $m = 2p$, and we shall see below that m is odd.

Write $X \to Y$ for the minimal resolution of singularities of the rational double point, such that the composite map $X \to Y \to \operatorname{Spec}(B_\mu)$ is a resolution of the singularity. The dual graph of this resolution contains a chain $C_1, \ldots, C_m$ of (-2)-curves, together with the strict transform C_0 of the divisor $D_{\rm red}$ on Y.

Suppose that C_0 intersects two distinct exceptional curves $C_i \neq C_j$. Then $(\bigcup_{i\geq 1} C_i) \cap C_0$ is an Artin scheme of length ≥ 2 on C_0. We claim that this is not possible. Indeed, consider the blow-up $X \to Y$. The induced morphism $C_0 \to D_{\rm red}$ is an isomorphism since we have shown above that the point y_0 is a regular point of $D_{\rm red}$. The scheme $(\bigcup_{i\geq 1} C_i)$, which is proper, has schematic image in Y the reduced closed point y_0. The same is true for any closed subscheme of the exceptional divisor, including the subscheme $(\bigcup_{i\geq 1} C_i) \cap C_0$. This is a contradiction since we have on the other hand an isomorphism $C_0 \to D_{\rm red}$, and a closed subscheme of length bigger than one in the source cannot be sent to a closed subscheme of length 1 in the target. Thus C_0 hits precisely one divisor C_i. If $(C_0 \cdot C_i)_X > 1$, a similar argument leads again to a contradiction, and thus we must have $(C_0 \cdot C_i)_X = 1$.

Consider now the involution on B_μ given by $x \mapsto -x$, $y \mapsto y$ and $z \mapsto z$. This involution fixes Peskin's equation (6.1), and induces an involution on the initial blow-up Z and its normalization Y. There the equation $z/y = 0$ defines an invariant Cartier divisor on the A_m-singularity $\operatorname{Spec} \mathcal{O}_{Y,y_0}$, which is the union of two regular Weil divisors D_1 and D_2, and these divisors are interchanged by the involution. The blow-up $Y' \to Y$ of the singular point $y_0 \in Y$ with reduced structure introduces two exceptional curves F_1 and F_2, and the strict transforms of D_1 and D_2 in Y' are disjoint. The intersection $F_1 \cap F_2$ consists in a single point y_0', and the local ring $\mathcal{O}_{Y',y_0'}$ is a rational double point of type A_{m-2}.

We now show that m is odd. First, suppose that the strict transforms of D_1 and D_2 in Y' do not intersect the same exceptional component of the blow-up $Y' \to Y$. It then follows that the involution acts non-trivially on the dual graph attached to the resolution of singularities $X \to Y$. If $m = 2p$ was even, the curve C_0 would pass through the sole fixed point $C_p \cap C_{p+1}$ of the exceptional divisor, and as we have seen above, this is a contradiction. It follows that $m = 2p-1$ must be odd in this case, and that $(C_0 \cdot C_p)_X = 1$. The assertion on the dual graph Γ_N follows.

Suppose now that the strict transforms of D_1 and D_2 in Y' intersect the same exceptional component of the blow-up $Y' \to Y$. We are going to show that this case cannot happen. Indeed, then the Weil divisors $D_1, D_2 \subset Y$ define the same class in the class group $\mathrm{Cl}(\mathcal{O}_{Y,y_0}) = \mathbb{Z}/(m+1)\mathbb{Z}$ of the rational double point of type A_m. Since the curves D_i are regular, the divisors $D_i \subset Y$ are not Cartier. It follows that D_i has order two in $\mathrm{Cl}(\mathcal{O}_{Y,y_0})$ since the sum of D_1 and D_2 is a Cartier divisor on Y. On the other hand, the strict transform of D_i in X intersects a terminal vertex of the exceptional divisor of $X \to Y$, and this fact along with a computation using the intersection matrix of the chain of m curves implies that D_i has order $m+1$ in the class group. This gives $m = 1$, contradicting $m \geq 2p-1 \geq 5$.

To completely determine the intersection matrix N of the resolution $X \to \mathrm{Spec}(B_\mu)$, it remains to compute the self-intersection number $(C_0 \cdot C_0)_X$. We have already observed above that $(D_0 \cdot D_0)_Y = -1/2$, and Proposition 2.2 shows that $(C_0 \cdot C_0)_X = (D_0 \cdot D_0)_Y - \delta$, where the correcting term δ is computed as follows. The determinant of the intersection matrix of the full chain of length $2p-1$ is $-2p$. Removing the vertex adjacent to C_0 from this chain yields two chains of length $p-1$. The determinant of the associated intersection matrix is then p^2. It follows that $\delta = p^2/2p = p/2$. Hence,

$$(C_0 \cdot C_0)_X = -1/2 - p/2 = -(p+1)/2.$$

Proposition 1.3 shows that $|\Phi_N| = p$. □

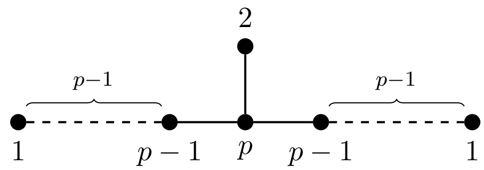
In the course of the proof we have used the following well-known general observation:

Lemma 6.4. *Let $f \in k[[x,y,z]]$ by a power series over an arbitrary field k. Write $f = \sum_{j=0}^{\infty} f^{(j)}$, where $f^{(j)}$ is a homogeneous polynomial of degree j. Suppose that $f^{(0)} = f^{(1)} = 0$, and that f defines an isolated singularity. Assume also that the quadratic part $f^{(2)}$ is the product of two non-associated linear forms. Then $k[[x,y,z]]/(f)$ is isomorphic to $k[[x,y,z]]/(z^{m+1} - xy)$ for some integer $m \geq 2$. In other words, the singularity in question is a rational double point of type A_m.*

Proof. After a linear change of coordinates, we may assume that $f = xy + O(3)$, where we denote by $O(d)$ an element of $\mathfrak{m}^d$. By induction on $d \geq 3$, one makes further coordinate changes of the form $x' := x + a(x,y,z)$, $y' := y + b(x,y,z)$ with $a, b \in \mathfrak{m}^{d-1}$ sending f to a power series of the form $x'y' + \sum_{i=3}^{d} \lambda_i z^i + O(d+1)$. This shows that we may assume $f = xy + \sum_{i=3}^{\infty} \lambda_i z^i$. If all coefficients λ_i vanish, the singularity would not be isolated. Thus our equation is of the form $xy + z^{m+1}\epsilon$ for some $m \geq 2$ and unit ϵ. Multiplying with ϵ^{-1}, we get the equation $(\epsilon^{-1}x)y + z^{m+1}$ for the rational double point of type A_m. □

The description of the fundamental cycle $\mathbf{Z}$ in our next proposition follows from the general description in [47] of the fundamental cycle of a star-shaped dual graph. We leave the proof of the following proposition to the reader.

Proposition 6.5. *The multiplicities in the fundamental cycle* $\mathbf{Z}$ *of the resolution of* $\operatorname{Spec} B_\mu$ *are indicated below next to the corresponding vertex.*



We have $\mathbf{Z}^2 = -2$*, and the fundamental genus is* $h^1(\mathscr{O}_{\mathbf{Z}}) = (p-3)/2$*. The canonical cycle is given by* $K = -\frac{p-3}{2}\mathbf{Z}$*.*

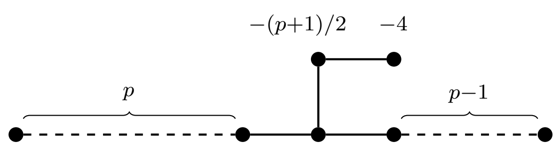
7. Analogues of the E_8 singularities

Let k be an algebraically closed field of characteristic $p > 0$. We compute in this section the resolution of the singularity of $\operatorname{Spec} B_\mu$ introduced in 0.2, for any value of the parameter $\mu \in k[[x, y]]$ when $a = y^2$ and $b = x$. The ring B_μ is given in this case by

$$B_\mu := k[[x, y]][z]/(z^p - (\mu x y^2)^{p-1} z - x^{p+1} + y^{2p+1}).$$

When $p = 2$, the resolution of $\operatorname{Spec} B_\mu$ is known to have dual graph E_8 when $\mu = 0$, $\mu = 1$ and $\mu = y$: these values produce the rational double points E_8^0, E_8^2, and E_8^1, respectively ([3]; see also [40]). The index of determinacy of a singularity E_8^r in characteristic 2 is computed to be 5 in [16], page 346. It follows that when $\mu \in (x, y)^2$, then $\operatorname{Spec} B_\mu$ is isomorphic to E_8^0. For $\mu \in k^\times$, we find that B_μ is isomorphic to E_8^2 through the change of variables $X = \mu^{15/11} x$, $Y = \mu^{10/11} y$, and $Z = \mu^{6/11} z$.

Theorem 7.1. *Let* $p \geq 3$*. Then* $\operatorname{Spec} B_\mu$ *has a resolution of singularities independent of* μ *with the following dual graph* Γ_N*:*



The associated discriminant group Φ_N *is trivial.*

Proof. Set $R = k[[x, y, z]]$ and $f = z^p - (\mu x y^2)^{p-1} z - x^{p+1} + y^{2p+1}$ and write $B = R/(f)$. We start with an initial blowing-up $Z = \mathrm{Bl}_{\mathfrak{a}B}(B)$ for the ideal $\mathfrak{a} = (x, y^2, z)$, as in 3.6. As usual, let $E \subset Z$ denote the exceptional divisor of the blow-up, and E_{red} its reduction. Proposition 3.6 shows that E_{red} is a smooth rational curve, that $E = 2pE_{\mathrm{red}}$, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. One checks that the blow-up is regular on the y^2-chart and the z-chart, and contains a unique singularity, which is located at the origin of the x-chart.

The x-chart is given by four variables $x, y, y^2/x, z/x$ modulo the two relations

$$y^2 = \left(\frac{y^2}{x}\right) x \quad \text{and} \quad \left(\frac{z}{x}\right)^p - \mu^{p-1} x^{p-1} \left(\frac{y^2}{x}\right)^{p-1} \frac{z}{x} - x + \left(\frac{y^2}{x}\right)^p y = 0.$$

The exceptional divisor is given by $x = 0$. Its reduction is defined by $x = y = z/x = 0$. Let us rewrite the second equation above as

$$\left(\frac{z}{x}\right)^p + \left(\frac{y^2}{x}\right)^p y = x\left(\mu^{p-1} x^{p-2} \left(\frac{y^2}{x}\right)^{p-1} \frac{z}{x} + 1\right). \tag{7.1}$$

On the formal completion along the exceptional divisor, $1+\mu^{p-1}x^{p-2}(y^2/x)^{p-1}(z/x)$ is invertible, and we denote by ϵ its inverse. The unit ϵ admits a $(p+1)$-st root δ (with $\delta^{p+1}=\epsilon$). After extracting an expression for x from (7.1) and substituting it in the expression $y^2=\frac{y^2}{x}x$, we find that

$$y^2=\frac{y^2}{x}\left(\left(\frac{z}{x}\right)^p+\left(\frac{y^2}{x}\right)^p y\right)\epsilon.$$

This is formally isomorphic to the equation

$$y^2-U^{p+1}y-UW^p=0$$

in the new set of variables y,U,W, via the map given by $y\mapsto y$, $U\mapsto(y^2/x)\delta$ and $W\mapsto(z/x)\delta$. Note that the reduced exceptional divisor is given by $x=y=z/x=0$ in the old coordinates, and by $y=W=0$ in the new ones. Let

$$B':=k[[y,U,W]]/(y^2-U^{p+1}y-UW^p).$$

We now make a second blow-up $Z'\to\operatorname{Spec}(B')$, with nonreduced center given by (y,U,W^p). Let E' denote the exceptional divisor of this blow-up. Using Proposition 3.1, we infer that the U-chart of Z' is described by four variables $U,W,y/U,W^p/U$ and two relations

$$W^p=\left(\frac{W^p}{U}\right)U\quad\text{and}\quad\left(\frac{y}{U}\right)^2-U^p\left(\frac{y}{U}\right)-\frac{W^p}{U}=0.$$

Substituting the latter in the former and renaming y/U by V gives

$$W^p=UV\left(V-U^p\right). \tag{7.2}$$

The origin (U,V,W) is obviously singular on this chart, and this is a singularity analyzed in Theorem 4.4. The reader will check that Z' has no further singularities on other charts, and that the only singularity on the U-chart is located at the origin. On this chart, the exceptional divisor is given by $U=0$. Its reduction has $U=W=0$. The reader will check that the exceptional divisor E' of this blow-up is a smooth projective line. Note also that the strict transform of the exceptional divisor from the initial blow-up is given by $V=0$, with reduction $V=W=0$, and that this strict transform is also a smooth projective line.

Theorem 4.4 lets us describe explicitly the intersection matrix $N(s_0\mid\alpha^{-1},\beta^{-1},\gamma^{-1})$ of the unique singularity in the U-chart. Using the notation from 4.3, we set $q=p$, $a=b=1$, $c=p$ and $d=1$, and find that $g:=\gcd(c,d)=1$ and $(ad+bc+cd)/g=2p+1$. It follows that

$$\alpha^{-1}=p^2/(2p-1)\quad\text{and}\quad\beta^{-1}=\gamma^{-1}=p/(p-1).$$

Recall that $p\geq 3$ and set $e:=(p+1)/2$. The reader will check that the continued fraction expansion of $\alpha^{-1}=p^2/(2p-1)$ is $\alpha^{-1}=[e,5,2,\dots,2]$ with $2+(p-3)/2$ overall entries, starting with the relations

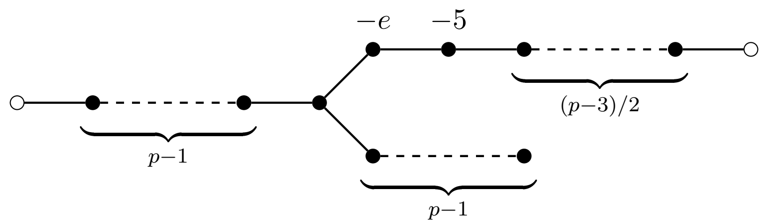
$$p^2=e(2p-1)-(p-1)/2,\quad\text{and}\quad(2p-1)=5(p-1)/2-(p-3)/2.$$

The self-intersection $-s_0$ of the node of the star-shaped graph is computed as:

$$s_0=\frac{1}{p^2}+\frac{2p-1}{p^2}+2\frac{p-1}{p}=2.$$

Having resolved the singularity (7.2), we get a resolution for our original singularity $\operatorname{Spec} B_\mu$ with the following resolution graph:

(7.3)



According to Proposition 4.8, the white terminal vertex to the left corresponds to the strict transform of the exceptional divisor on the initial blow-up, whereas the white terminal vertex on the top right corresponds to the strict transform of the exceptional divisor on the second blow-up.

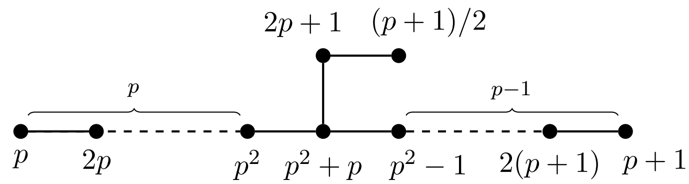
It remains to determine the self-intersection of both of these strict transforms in the resolution of $\operatorname{Spec} B$. Recall that E' is the exceptional divisor for the second blow-up $Z' \to \operatorname{Spec}(B')$. Computing in the affine charts, one sees that E'_{red} is a projective line, with $E' = pE'_{\text{red}}$ and $(E' \cdot E'_{\text{red}})_{Z'} = -2$. Since the U-chart is regular away from the origin, we can conclude using Proposition 2.3 that the self-intersection of the strict transform of E'_{red} in the normalization of Z' is $-2/p$. Proposition 2.2 shows that the strict transform C' of E'_{red} in X has thus $(C' \cdot C')_X = -2/p - \delta$ for some correction term $\delta \in \mathbb{Q}_{>0}$. The term δ is computed as follows. Let Γ_1 be the star-shaped subgraph in (7.3) consisting of all the black vertices, and let $\Gamma_1' \subset \Gamma_1$ be the star-shaped subgraph obtained from Γ_1 by removing the terminal black vertex in the top right position. Let N_1 and N_1' be the resulting intersection matrices. According to Proposition 2.2, we have $\delta = -\det(N_1')/\det(N_1)$. Using Proposition 1.3, we compute that $|\det(N_1)| = p^2$ and $|\det(N_1')| = p^2 - 2p$. Hence, $\delta = -(p^2 - 2p)/p^2 = -1 + 2/p$, and it follows that the white terminal vertex on the top right has self-intersection -1. We can thus contract this divisor. Successively contracting (-1)-curves from the right, we get the desired graph as in the statement of Theorem 7.1 with a terminal vertex of self-intersection number $-4 = -5 + 1$ on the top right.

Recall that we denoted by E the exceptional divisor of $Z \to \operatorname{Spec} B$, and determined using Proposition 3.6 that E_{red} is a smooth rational line, that $E = 2pE_{\text{red}}$, and that $(E \cdot E_{\text{red}})_Z = -1$. As above, Proposition 2.2 shows that the strict transform C of E_{red} in X has $(C \cdot C)_X = -1/2p - \delta$ for some correction term $\delta \in \mathbb{Q}_{>0}$. Let Γ_2 be the star-shaped subgraph in (7.3) consisting of all the black vertices and the terminal white vertex (of self-intersection (-1)) in the top right position. Let Γ_2' be the star-shaped subgraph obtained from Γ_2 by removing the terminal black vertex of Γ_2 attached to the terminal white vertex on the left corresponding to E. Let N_2 and N_2' be the resulting intersection matrices. According to Proposition 2.2, we have $\delta = -\det(N_2')/\det(N_2)$. The matrix N_2 has the same determinant as $N(2 \mid p/(p-1), p/(p-1), (2p+1)/4)$, and N_2' has the same determinant as $N(2 \mid (p-1)/(p-2), p/(p-1), (2p+1)/4)$. Using Proposition 1.3, we compute that $|\det(N_2)| = 2p$ and $|\det(N_2')| = 4p - 1$. Hence, $(C \cdot C)_X = -2$.

Now that the intersection matrix N of the resolution has been determined, with $N = N(2 \mid p/(p-1), (p+1)/p, (2p+1)/4)$, Proposition 1.3 can be used to show that $|\det(N)| = 1$. □

The description of the fundamental cycle $\mathbf{Z}$ in our next proposition follows from the general description in [47] of the fundamental cycle of a star-shaped dual graph. We leave the proof of the following proposition to the reader.

Proposition 7.2. *Keep the assumptions of Theorem 7.1. The multiplicities in the fundamental cycle* $\mathbf{Z}$ *of the resolution of* $\operatorname{Spec} B_\mu$ *are indicated below next to the corresponding vertex.*



We have $\mathbf{Z}^2 = -(p+1)/2$*, and the fundamental genus is* $h^1(\mathscr{O}_\mathbf{Z}) = (p^2-p+2)/2$*. The canonical cycle for the singularity is given by*

$$K = -(2p-4)\mathbf{Z} + \frac{p-3}{2}E_j,$$

where $E_j \in \Gamma_N$ *is the terminal vertex on the top right.*

8. Analogues of the E_7 singularities

When $p = 2$, the blow-up at the maximal ideal of the $\mathbb{Z}/2\mathbb{Z}$-quotient singularity E_8^2 given by

$$z^2 + xy^2z + x^3 + y^5 = 0$$

has a new singularity, namely the singularity E_7^1 given by the equation

$$z^2 + xy^2z + yx^3 + y^3 = 0$$

(see for instance [43], 1.1). The singularity E_8^2 has resolution graph the Dynkin diagram E_8 with trivial discriminant group, while the resolution of E_7^1 has resolution graph E_7 with discriminant group of order 2.

Artin ([3], bottom of page 18, or [40], (2.16), page 104) shows that the Dynkin diagram E_7 cannot be obtained as the resolution graph of a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity whose associated action is ramified precisely at the origin. He shows however that the singularity E_7^1 does occur as the resolution graph of a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity for an action that is ramified in codimension 1.

When $p = 2$, we have not been able to exhibit any wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity whose action is *ramified precisely at the origin* and whose associated intersection matrix has discriminant group of order 2^s with s odd. We suggest in 8.4 for each s odd the existence of explicit examples with group $(\mathbb{Z}/2\mathbb{Z})^s$. In each case, these wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities are associated to actions that are ramified in codimension 1.

The above considerations have analogues for any prime p. Indeed, consider the singularity at the maximal ideal of $\operatorname{Spec} B_n$, where

$$B_n := k[[x,y,z]]/(z^p - (xy^n)^{p-1}z - y^{pn+1} + x^{p+1}).$$

This singularity is a special case of the singularity recalled in 0.2, where we have set $a = y^n$ and $b = x$. In particular, this singularity is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose moderately ramified action is ramified precisely at the origin. When $n = p = 2$, this singularity is E_8^2.

Consider the blow-up of $\operatorname{Spec} B_n$ at the maximal ideal (x,y,z). Then the chart defined by the variables $y, x/y, z/y$, has a singular point whose local ring is isomorphic to the local ring C_n, where

$$C_n := k[[x,y,z]]/(z^p - (xy^n)^{p-1}z - y^{(n-1)p+1} + yx^{p+1}). \tag{8.1}$$

When $n > 1$, the closed point of $\operatorname{Spec} C_n$ is singular, and we show below in 8.1 that the singularity of $\operatorname{Spec} C_n$ is again a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity, but for an action that is ramified in codimension 1.

When $n = 2$, the singularity of $\operatorname{Spec} B_2$ is treated in Theorem 7.1 and generalizes the E_8^2-singularity. The singularity of $\operatorname{Spec} C_2$ is the E_7^1-singularity when $p = 2$, and thus $\operatorname{Spec} C_2$ is a natural generalization for all primes p of the E_7^1-singularity. Our educated guess for the resolution of $\operatorname{Spec} C_2$ is discussed in Example 8.5. In the examples that we were able to compute, the discriminant groups Φ_{B_n} and Φ_{C_n} of the intersection matrices of the resolutions of $\operatorname{Spec} B_n$ and $\operatorname{Spec} C_n$ when $n > 1$ satisfy $|\Phi_{C_n}| = p|\Phi_{B_n}|$.

We can further generalize the ring C_n as follows. Let $a, b \in k[[x, y]]$, not both 0. Set

$$A_0 := k[[x, y]][U, V]/(U^p - (ay)^{p-1}U - y, V^p - (by)^{p-1}V - xy).$$

Let L denote the field of fractions of A_0. The ring A_0 and the field L are endowed with an automorphism σ of order p fixing $k[[x, y]]$ and with

$$\begin{aligned} \sigma(U) &:= U + ay, \\ \sigma(V) &:= V + by. \end{aligned}$$

As usual, we set $G := \langle\sigma\rangle$. Let $z := aV - bU$. Then $\sigma(z) = z$, and we find that

$$z^p - (aby)^{p-1}z - a^p xy + b^p y = 0. \tag{8.2}$$

Let B denote the subring $k[[x, y]][z]$ of A_0. Let A denote the subring $A_0[\frac{V}{U}]$ of L. The group G acts on A, since $\sigma(V/U) = (V/U + by/U)(1 + ay/U)^{-1}$ and $1 + ay/U$ is a unit in A_0.

Proposition 8.1. *Keep the above notation. The ring homomorphism $A \to k[[u, v]]$, which sends U to u and V/U to v, is a k-isomorphism. In the special case where either $a = x^m$ and $b = y^n$, or $a = y^n$ and $b = x^m$ for some integers $m, n \geq 1$, then the ring of invariants A^G is equal to the ring B. In particular, $\operatorname{Spec} C_n$ is a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity.*

Proof. The equation $U^p - (ay)^{p-1}U - y = 0$ first shows that y/U is in the maximal ideal of A_0, and then that y/U^p is in A_0 and is a unit. The ring A_0 is not integrally closed, since it is clear from the equation $V^p - (by)^{p-1}V - xy = 0$ that

$$(\frac{V}{U})^p - (\frac{by}{U})^{p-1}(\frac{V}{U}) - \frac{y}{U^p}x = 0$$

is an integral relation for $\frac{V}{U}$ over A_0. Since x and y can be expressed in terms of U and V/U, we find that $A := A_0[\frac{V}{U}]$, viewed as a subring of L, is in fact isomorphic to the power series ring $k[[u, v]]$, with $u := U$ and $v := V/U$.

Consider the ring $B' := k[[x, y]][Z]/(Z^p - (aby)^{p-1}Z - a^p xy + b^p y)$ and the natural map $\varphi : B' \to A^G$ which sends Z to z. Assume that either $a = x^n$ and $b = y^m$, or that $a = y^n$ and $b = x^m$ for some integers $m, n \geq 1$. We claim that φ is an isomorphism. One can show that B' is an integral domain, and that its field of fractions injects in $\operatorname{Frac}(A)$, and has image by degree considerations equal to $\operatorname{Frac}(A^G)$. The ring B' is Cohen–Macaulay since it is free as a module over the regular ring $k[[x, y]]$. Thus B' is normal as soon as it is regular in codimension 1. This can be shown, because of the special forms of a and b, using the Jacobian criterion. Let $f := Z^p - (aby)^{p-1}Z - a^p xy + b^p y$. Then if a prime ideal $\mathfrak{p}$ of B' contains the classes of f, and of the partial derivatives f_x, f_y, f_Z, then $\mathfrak{p}$ contains (x, y, Z).

The reader will check that the ring C_n is isomorphic to B when $a = -x$ and $b = -y^{n-1}$. When $p = n = 2$, the proposition is proved in [40], (2.16), page 104. □

Example 8.2. Let $p = 2$. Computations with Magma and Singular indicate that $\operatorname{Spec} B_3$ admits a resolution with smooth rational curves and dual graph drawn on the left below, with trivial discriminant group,

while $\operatorname{Spec} C_3$ admits a resolution with smooth rational curves and dual graph drawn on the right above, with discriminant group of order 2.

Example 8.3. We show in this example that there are (many) intersection matrices N with Φ_N killed by 2 and of order 2^s with s odd. Since our interest is to provide evidence that there may exist wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities whose resolutions have discriminant groups of order 2^s with s odd, we note that any such resolution must also have an intersection matrix N whose fundamental cycle Z satisfies $|Z^2| \leq 2$ ([28], 2.4). This is a non-trivial restriction on the possible matrices N, and we exhibit below matrices that also satisfy this restriction.

Recall that a star-shaped graph with $n \geq 4$ vertices is called a *star*, or the *complete bipartite graph* $K_{1,n-1}$, if it consists in a single node and $n-1$ terminal vertices attached to the node. We write the intersection matrix N of a star on n vertices as $N = N(s_0 \mid s_1/1, \ldots, s_{n-1}/1)$, where $-s_0$ denotes the self-intersection of the node, and $-s_i$ denotes the self-intersection of the i-th terminal vertex when $i > 0$. The Dynkin diagram D_4 is a star on 4 vertices, and so are the two graphs in Example 8.2.

Consider any intersection matrix $N = N(s_0 \mid s_1/1, \ldots, s_{n-1}/1)$ such that one of the s_j with $j \geq 1$ is even and at most one of the s_j with $j \geq 1$ is divisible by 4. Assume in addition that Φ_N is killed by 2, and that the fundamental cycle Z of N satisfies $|Z^2| \leq 2$. Define the matrix $N_i(s_0 \mid s_1/1, \ldots, s_{n-1}/1, s_n/1)$, $i = 1, 2$, by

$$s_n := i + \left(\prod_{j=1}^{n-1} s_j\right)/|\Phi_N|.$$

We claim that the two intersection matrices N_1 and N_2 have graphs that are stars on $n+1$ vertices with $|\det(N_i)| = i|\det(N)|$. Moreover, both groups Φ_{N_i} are killed by 2, and both fundamental cycles Z_i of N_i satisfy $|Z_i^2| \leq 2$.

Proof. Let $\ell_{n-1} := \operatorname{lcm}(s_1, \ldots, s_{n-1})$. Then the order of the node in Φ_N is equal to $\ell_{n-1}(s_0 - \sum_{j=1}^{n-1} 1/s_j)$ (use 1.3 (ii)). This order equals 1 since we assume that one of the s_j is even (use 1.3 (v)). It follows that $|\Phi_N| = (\prod_{j=1}^{n-1} s_j)/\ell_{n-1}$ (use 1.3 (i)). In particular, $(\prod_{j=1}^{n-1} s_j)/|\Phi_N| = \ell_{n-1}$ is an integer. The equality $|\det(N_i)| = i|\det(N)|$ follows from an easy computation.

We find that $\operatorname{lcm}(s_1, \ldots, s_{n-1}, \ell_{n-1} + i) = \operatorname{lcm}(\ell_{n-1}, \ell_{n-1} + i)$, which equals $\ell_{n-1}(\ell_{n-1} + 1)$ when $i = 1$, and $\ell_{n-1}(\ell_{n-1}/2 + 1)$ when $i = 2$. Hence, the node is trivial in Φ_{N_i} since its order is

$$\operatorname{lcm}(s_1, \ldots, s_{n-1}, \ell_{n-1} + i)(s_0 - \sum_{j=1}^{n-1} 1/s_j - 1/(\ell_{n-1} + i)) = 1.$$

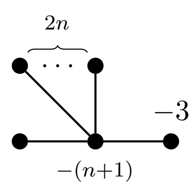
Let $R \in \mathbb{Z}^{n+1}$ denote the transpose of vector $(\ell_{n-1}, \ell_{n-1}/s_1, \ldots, \ell_{n-1}/s_{n-1}, 1)$. Then $N_i R = -ie_{n+1}$. Since all coefficients of R are positive and $N_i R$ has non-positive coefficients, we find that R is an upperbound for the fundamental cycle Z_i of N_i. Then $|Z_i^2| \leq |R^2| \leq i$, as desired.

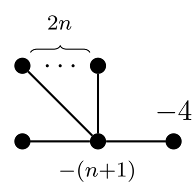
To show that Φ_{N_i} is killed by 2, it suffices to show that the classes of the standard vectors have order 1 or 2 in Φ_{N_i} for each terminal vertex of the graph. This is clear for a terminal vertex v_j with s_j odd or exactly divisible by 2, since the column of N_i corresponding to v_j shows that the class of $s_j v_j$ is equal to the class of the node. We note now that the construction implies that there can be at most one terminal vertex v_j with s_j divisible by 4. If the corresponding class in Φ_{N_i} has order divisible by 4, we would find using the first column of the matrix N_i that this unique class is equal to the sum of classes which all have order 1 or 2, a contradiction. This ends the proof of the claim. □

The sequence $\{s_n\}_{n\geq 1}$ with $s_1 = 2$ and $s_n := \mathrm{lcm}(s_1, \dots, s_{n-1}) + 1$ is called *Sylvester's sequence* $\{2, 3, 7, 43, \dots\}$ in the literature, and produces the only intersection matrices $N(1 \mid s_1/1, \dots, s_{n-1}/1)$ with trivial group Φ_N in the above construction. An example of a star with intersection matrix N such that Φ_N is killed by 2 but $|Z^2| > 2$ is given by $N = N(1 \mid 2/1, 3/1, 10/1, 16/1)$, with group $\Phi_N = (\mathbb{Z}/2\mathbb{Z})^2$ and $|Z^2| = 4$.

Example 8.4. Let $p = 2$. Fix an integer $n \geq 1$. Consider the star graph with a central node of self-intersection $-(n+1)$ attached to $2n+1$ terminal vertices of self-intersection -2. Denote by N_0 its intersection matrix. Proposition 1.3 (iv) shows that $\Phi_{N_0} = (\mathbb{Z}/2\mathbb{Z})^{2n}$. We remark in passing that this matrix does occur as the intersection matrix attached to a quotient singularity (use the equation $z^2 - xy(x^{2n-1} - y^{2n-1})$ and Theorem 5.3 (ii)).

Starting with N_0, the construction in Example 8.3 produces two intersection matrices, the matrix $N_1(n) := N(n+1 \mid 2/1, \dots, 2/1, 3/1)$ with group of order 2^{2n} and whose graph is represented on the left below, and the matrix $N_2(n) := N(n+1 \mid 2/1, \dots, 2/1, 4/1)$ with group of order 2^{2n+1} and whose graph is represented below on the right.





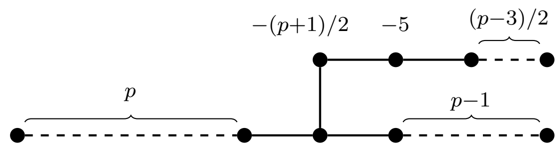
When $n = 1$, the intersection matrices N_1 and N_2 are the matrices of the resolutions of the wild quotient singularities $\operatorname{Spec} B_4$ and $\operatorname{Spec} C_4$, respectively.

When $n \geq 1$, consider the equation $f := z^p - (aby)^{p-1}z - a^p xy + b^p y$ introduced in (8.2), and set $a := x^n$ and $b := y^{2n+1}$. Let $B := k[[x, y, z]]/(f)$. Proposition 8.1 shows that this equation defines a wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularity. We conjecture that $\operatorname{Spec} B$ has a resolution $X \to \operatorname{Spec} B$ with a dual graph equal to the dual graph of $N_2(n)$ represented on the right above. The conjecture thus provides examples of wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities with discriminant group of order 2^{2n+1} for all $n \geq 1$. These quotient singularities are associated with actions that are ramified in codimension 1.

Example 8.5. (Analogues of E_7.) Let p be prime. Computations suggest that the resolution of the wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity $\operatorname{Spec} C_2$ (see (8.1)) has intersection matrix (notation as in 1.2)

$$N = N(2 \mid \frac{p}{p-1}, \frac{p+1}{p}, \frac{p^2}{2p-1})$$

with group $\Phi_N = \mathbb{Z}/p\mathbb{Z}$. When p is odd, the intersection matrix N has the following graph:



The resolution of $\operatorname{Spec} B_2$ is discussed in Theorem 7.1.

Remark 8.6. Consider the equation $z^p - (aby)^{p-1}z - a^pxy + b^py = 0$ introduced in (8.2), and set $a = y^n$ and $b = x^m$ for some integers $m, n \geq 1$. Proposition 8.1 shows that this equation defines a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity. Computations with Magma [6] suggest that for such a and b, the resolution of the singularity at the origin of $z^p - (aby)^{p-1}z - a^pxy + b^py = 0$ has the same intersection matrix as the resolution of the singularity of $z^p - a^pxy + b^py = 0$.

When $a = y^n$ and $b = x^m$, this latter singularity has the form $z^p - xy(y^{pn} - x^{pm-1}) = 0$, and Theorem 4.4 provides an explicit resolution for it. When $p = 2$, we find that $g := \gcd(pn, pm-1)$ is always odd, so the discriminant group of this resolution, which has order 2^{g+1} by 4.9, is always of the form $|\Phi_N| = 2^s$ with s even. Thus the quotient singularity (8.2) in this case is unlikely to provide examples of discriminant groups of order $|\Phi_N| = 2^s$ with s odd.

When $p = 2$, (8.2) in the case $b = x$ and $a = y^n$ gives the equation of the singularity $D^n_{2(2n+1)}$ with resolution graph the Dynkin diagram $D_{2(2n+1)}$ (notation as in [3], section 3).

9. D_4 AND A_{p-1}

We compute in this section the resolution of the singularity of $\operatorname{Spec} B_\mu$ introduced in 0.2, for any value of the parameter μ when $a = y$ and $b = x$. The ring B_μ is given in this case by

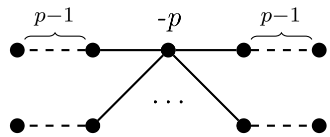
$$B_\mu := k[[x, y]][z]/(z^p - (\mu xy)^{p-1}z - x^{p+1} + y^{p+1}).$$

Let $Z \to \operatorname{Spec}(B_\mu)$ be the blow-up of the ideal $\mathfrak{b} = (x, y, z)$, as in 3.6. We note in Theorem 9.4 that Z has $p+1$ singularities, each again $\mathbb{Z}/p\mathbb{Z}$-quotient singularities, with resolution graph A_{p-1} and associated discriminant group $\mathbb{Z}/p\mathbb{Z}$.

Remark 9.1. When k contains a third root of unity ζ with $\zeta^2 + \zeta + 1 = 0$, the change of variables $X := x + \zeta y$ and $Y := x + \zeta^2 y$ produces $x^3 + y^3 = -\zeta XY(X + \zeta Y)$. In particular, the singularity $z^q - (x^3 + y^3) = 0$ is always isomorphic over k to the singularity $z^q - (x^2y - xy^2) = 0$. When in addition $p = 2$, we find that $B_{\mu=0}$ is isomorphic over $\mathbb{F}_4$ to the singularity D_4^0, given by the equation $z^2 + x^2y + xy^2 = 0$. The dual graph of its resolution is the Dynkin diagram D_4. The Tjurina number of this singularity is equal to 8.

The resolution of $\operatorname{Spec} B_{\mu=1}$ when $p = 2$ is also known to have dual graph D_4 over an algebraically closed field. Indeed, the equation when $\mu = 1$ is stated to be equivalent to D_4^1 in [40], page 102, where D_4^1 is given by the equation $z^2 + xyz + x^2y + xy^2 = 0$. This can be seen indirectly as follows. By Theorem 9.2, the resolution of $\operatorname{Spec} B_{\mu=1}$ is of type D_4. According to Artin's classification [3], there are only two possible isomorphism types of singularities with resolution D_4 when $p = 2$, namely D_4^0 and D_4^1. Since the Tjurina number of $B_{\mu=1}$ is equal to 6, this ring must then be isomorphic to the D_4^1 singularity over the algebraic closure of k. The quotient singularity $\operatorname{Spec} B_{\mu=1}$ when $p > 2$ can thus be considered as a generalization of D_4^1.

Theorem 9.2. Spec B_μ *has a resolution of singularities with star-shaped dual graph* Γ_N *having* $p+1$ *identical terminal chains with* $p-1$ *vertices as follows:*



The associated discriminant group Φ_N *has order* p^p.

Proof. Let $Z \to \mathrm{Spec}(B_\mu)$ be the blow-up of the ideal $\mathfrak{a}B = (a,b,z) = (x,y,z)$, as in Proposition 3.6. Let as usual E denote the exceptional divisor. We find from Proposition 3.6 that E_{red} is a smooth rational curve over k, and that $(E \cdot E_{\mathrm{red}})_Z = -1$. In addition, $E = pE_{\mathrm{red}}$, and the z-chart is regular.

The blow-up Z is covered by three affine charts, and we see that the x-chart is generated by the expressions $x, y/x, z/x$ modulo the relation

$$\left(\frac{z}{x}\right)^p - x\left(1 + \mu^{p-1}x^{p-2}\left(\frac{y}{x}\right)^{p-1}\frac{z}{x} - \left(\frac{y}{x}\right)^{p+1}\right) = 0.$$

Clearly, this chart is regular at the origin. Let $Y \to Z$ denote the normalization of Z. Let D denote as usual the pull-back of the exceptional divisor of Z. It follows from the regularity at the origin that the induced morphism $D_{\mathrm{red}} \to E_{\mathrm{red}}$ is an isomorphism. Hence, we can conclude from Proposition 2.3 that $(D_{\mathrm{red}} \cdot D_{\mathrm{red}})_Y = -1/p$.

Using partial derivatives, one sees that the singular locus on the x-chart is given by $x = z/x = 0$ and $(y/x)^{p+1} = 1$. Let ζ denote a primitive root of the equation $u^{p+1} = 1$. When rewriting the above equation defining the x-chart in terms of the expressions x, $y/x - \zeta^j$, and z/x, we obtain a polynomial of the form $x(y/x - \zeta^j) + O(3)$. Using the changes of variables discussed in the proof of 6.4, we find that the singularity is in fact a rational double point of type A_{p-1}.

Let $X \to Y$ denote a resolution of the singularities of Y. Let C denote the strict transform of D_{red} in X. It follows from Proposition 2.2 that $(C \cdot C)_X = -1/p - (p+1)\delta$, where δ is the correcting term associated with the rational double point A_{p-1}. As noted in 1.1, $\delta = (p-1)/p$, and we find that $(C \cdot C)_X = -p$. The associated discriminant group is computed with Proposition 1.3. □

Let $R := k[[x,y]]$. As recalled in 0.2, let

$$A := k[[u,v]] = R[u,v]/(u^p - (\mu y)^{p-1}u - x, v^p - (\mu x)^{p-1}v - y),$$

and let σ be the automorphism defined by $\sigma(u) = u + \mu y$ and $\sigma(v) = v + \mu x$. Let $G := \langle \sigma \rangle$. The element $z := xu - yv$ is invariant, and we can identify the ring B_μ with A^G.

Let $Z' \to \mathrm{Spec}(A)$ be the blow-up of the induced ideal $\mathfrak{a}A$, and let $Y' \to Z'$ denote the normalization of Z'. We have the commutative diagram

$$\begin{array}{ccccccc} & & Y' & \longrightarrow & Z' & \longrightarrow & \mathrm{Spec}(A) \\ & & \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \mathrm{Spec}(A^G). \end{array}$$

Let y_i, $i = 1, \ldots, p+1$, denote the rational double points in Y of type A_{p-1}. We show below that these points are in fact $\mathbb{Z}/p\mathbb{Z}$-quotient singularities.

Lemma 9.3. *The scheme Y' is regular, and the morphism $Y' \to \operatorname{Spec}(A)$ coincides with the blow-up of the maximal ideal $\mathfrak{m}_A = (u, v)$.*

Proof. Indeed, using the relations

$$u^p - (\mu y)^{p-1} u = x \quad \text{and} \quad v^p - (\mu x)^{p-1} v = y, \tag{9.1}$$

we get $u^p, v^p \in \mathfrak{a}A$. Since the finite ring extension $R \subset A$ is flat of degree p^2, we must have $\mathfrak{a}A = (u^p, v^p)$. More precisely, substituting the equations (9.1) into each others one obtains

$$x \cdot \text{unit} = u^p - \mu^{p-1} v^{p(p-1)} u \quad \text{and} \quad y \cdot \text{unit} = v^p - \mu^{p-1} u^{p(p-1)} v,$$

showing explicitly that $(x, y)A \subseteq (u^p, v^p)$. Since $z = xu - yv$, we have $(u^p, v^p) = \mathfrak{a}A$.

The blow-up Z' of the ideal (u^p, v^p) in $\operatorname{Spec}(A)$ is covered by two charts. The u^p-chart has generators u, v, and v^p/u^p, so v/u satisfies an obvious integral equation, and we also have $v = v/u \cdot u$. It follows that on the normalization the chart becomes regular. The situation on the v^p-chart is similar, and we see that the scheme Y' is regular. □

Theorem 9.4. *The preimage of each y_i under the map $Y' \to Y$ consists of a single regular point $x_i \in Y'$, and $\mathscr{O}_{Y,y_i} = (\mathscr{O}_{Y',x_i})^G$. Thus y_i is a $\mathbb{Z}/p\mathbb{Z}$-quotient singularity whose resolution has dual graph A_{p-1} and associated discriminant group $\mathbb{Z}/p\mathbb{Z}$. The morphism $\operatorname{Spec} \mathscr{O}_{Y',x_i} \to \operatorname{Spec}(\mathscr{O}_{Y',x_i})^G$ is ramified in codimension 1 and the punctured spectrum of the rational double point y_i has trivial fundamental group.*

Proof. The G-action on the ring A induces a G-action on the normalized blow-up Y', which on the field of fractions of the u-chart is given by

$$u \mapsto u + \mu y \quad \text{and} \quad v/u \mapsto (v + \mu x)/(u + \mu y).$$

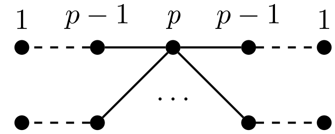
Since Y is normal, the induced morphism $Y' \to Y$ yields an identification $Y = Y'/G$.

Let E' denote the exceptional divisor of the blow-up $Y' \to \operatorname{Spec}(A)$ of the maximal ideal. Then the natural map $E' \to D_{\text{red}}$ induced by $Y' \to Y$ is purely inseparable of degree p, and, hence, the morphism $\operatorname{Spec} \mathscr{O}_{Y',x_i} \to \operatorname{Spec}(\mathscr{O}_{Y',x_i})^G$ is ramified at the codimension 1 point corresponding to E'. It follows from [3, Corollary 1.2] that the punctured spectrum of the rational double point y_i has trivial fundamental group. □

Remark 9.5. The occurrence of the A_{p-1}-singularities y_i on the quotient $Y = Y'/G$ is caused by points $x_i \in Y'$ where the ideal of the fixed scheme $Y'^G \subset Y'$ is not a Cartier divisor. Indeed, using Theorem 2 in [21], we find that when the action of σ on the local ring $A = k[[u, v]]$ is such that the ideal $(\sigma(u) - u, \sigma(v) - v)$ of the fixed scheme is principal, then the fixed ring $A^{\langle\sigma\rangle}$ is regular.

We leave the proof of the following proposition to the reader.

Proposition 9.6. *The multiplicities in the fundamental cycle $\mathbf{Z}$ of the resolution of $\operatorname{Spec} B_\mu$ are strictly decreasing along each terminal chain, as indicated below next to the corresponding vertex.*



The fundamental genus is $h^1(\mathscr{O}_{\mathbf{Z}}) = (p-2)(p+1)/2$, and $\mathbf{Z}^2 = -p$. Moreover, the canonical cycle is $K = -(p-2)\mathbf{Z}$.

10. Numerically Gorenstein intersection matrices

All wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularities resolved in this article are hypersurface singularities. We prove in this section that all wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities are hypersurface singularities. We then recall that the intersection matrix associated with a hypersurface singularity is always numerically Gorenstein. We show in Proposition 10.5 that any intersection matrix N whose discriminant group Φ_N is killed by 2 is automatically numerically Gorenstein. We exhibit in 10.7 an example when $p > 2$ of a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity which is not numerically Gorenstein.

Proposition 10.1. *Let $p = 2$. Let $A = k[[u, v]]$, endowed with a non-trivial action of $G = \mathbb{Z}/2\mathbb{Z}$. Then there exists a power series ring $R := k[[x, y]]$ such that A^G is k-isomorphic to $R[z]/(z^2 + sz + t)$, with $s, t \in R$.*

Proof. Let σ denote the generator of G. Proposition 2.9 in [31] allows us, if necessary, to replace the system of parameters (u, v) for A with a new system of parameters (again denoted by (u, v) below) with the following properties (use [31, Proposition 2.3]): let $x := u\sigma(u)$ and $y := v\sigma(v)$. Let $R := k[[x, y]]$ be the subring of A generated by k, x, and y. Then A is a free R-module of rank 4.

We have the inclusions $R \subset A^G \subset A$, and the fraction field of A^G is then of degree 2 over the fraction field of R. Since R is regular and A^G is Cohen-Macaulay because it is normal of dimension 2, we find that A^G is a free R-module of rank 2. Thus, R is a direct summand of A^G, with quotient A^G/R free of rank 1. We can therefore find an element $z \in A^G$ which generates the quotient A^G/R. It follows that the natural map $R[Z] \to A^G$ with $Z \mapsto z$ is surjective. Since $z \notin R$, it satisfies a quadratic equation $z^2 + sz + t = 0$, with $s, t \in R$ and $Z^2 + sZ + t$ irreducible in $R[Z]$. Since $R[Z]$ is a UFD, we find that $R[Z]/(Z^2 + sZ + t) \to A^G$ is an isomorphism. □

10.2. Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix. Let $H_0 \in \mathbb{Z}^n$ be the integer vector whose i-th coefficient is $h_i := -c_{ii} - 2$ for $i = 1, \ldots, n$. Since N is invertible, there exist a vector $K \in \mathbb{Q}^n$ such that $NK = H_0$. The vector K is called the *canonical cycle* of N. We say that N is *numerically Gorenstein* if $K \in \mathbb{Z}^n$.

When N is the intersection matrix associated with a collection of irreducible curves C_i, $i = 1, \ldots, n$ on a surface, each component C_i has an arithmetical genus $p_a(C_i)$. Our definition of numerically Gorenstein coincides with the usual one (see for instance [42], (2.5)) when all arithmetical genera are equal to 0.

Lemma 10.3. *Let k be a field of characteristic p. Let B denote a complete local ring of dimension 2, isomorphic to $k[[x, y, z]]/(f)$ for some $f \in (x, y, z)$, and formally smooth outside its closed point. Let $X \to \operatorname{Spec} B$ be a resolution of the singularity, with associated intersection matrix N. Assume that all the irreducible components in the exceptional locus of the resolution are smooth rational curves. Then N is numerically Gorenstein.*

Proof. We first use [1], 3.8, to find an algebraic scheme S over k and a point $s \in S$ such that the completion of $\mathcal{O}_{S,s}$ is isomorphic to B. The ring $\mathcal{O}_{S,s}$ is Gorenstein since its completion B is ([12], 21.18). Thus there exists an open set U of S, containing s, and such that U is everywhere Gorenstein ([15], 1.5). It follows that U has a canonical sheaf that is trivial. Consider a resolution $\pi : V \to U$ of the singularity $s \in U$. Then the canonical sheaf K_V on V is supported on the exceptional divisor of π. The adjunction formula for each irreducible component E_i shows that $(K_V \cdot E_i) + (E_i \cdot E_i) = 2p_a(E_i) - 2$. Since K_V is equal to a linear combination of the E_i, we find that the intersection matrix N of the exceptional locus is numerically Gorenstein. □

Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix with discriminant group Φ_N. As usual, denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$, and let p_i denote the order of the class of e_i in Φ_N. For each $i = 1, \ldots, n$, let $R_i \in \mathbb{Z}^n$ denote the unique positive vector such that $NR_i = -p_i e_i$. Let $(R_i)_j$ denote the j-th coefficient of R_i, and define

$$g_i := \sum_{j=1}^{n} (R_i)_j(|c_{jj}| - 2) = ({}^tR_i)H_0.$$

If the matrix N is such that $c_{jj} \leq -2$ for all $j = 1, \ldots, n$, then $g_i \geq 0$.

Lemma 10.4. *Let N be an intersection matrix. Then ${}^tK = (-g_1/p_1, \ldots, -g_n/p_n)$. In particular, the matrix N is numerically Gorenstein if and only if p_i divides g_i for each $i = 1, \ldots, n$.*

Proof. By hypothesis, we have $NK = H_0$ for some vector $K \in \mathbb{Q}^n$. It follows that ${}^tR_iNK = -p_iK_i = g_i$, and we find that $K_i = -g_i/p_i$. □

Proposition 10.5. *Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix with discriminant group Φ_N killed by 2. Then N is numerically Gorenstein.*

Proof. Our hypothesis implies that $p_i = 1$ or 2, for all $i = 1, \ldots, n$. We use the criterion given in 10.4: to show that N is numerically Gorenstein, it suffices to show, for each i, that the integer g_i is even when $p_i = 2$. Assume then that $p_i = 2$. Then by construction,

$${}^tR_iNR_i = -p_i(R_i)_i.$$

We now compute explicitly the term ${}^tR_iNR_i$ and obtain

$${}^tR_iNR_i = \sum_{j=1}^{n} c_{jj}(R_i)_j^2 + 2\sum_{j<k} c_{jk}(R_i)_j(R_i)_k.$$

Since p_i is even and $(R_i)_j^2 \equiv (R_i)_j \pmod 2$, we find that $\sum_{j=1}^{n} c_{jj}(R_i)_j$ is even, and so is g_i, as desired. □

Remark 10.6. Let $N = (c_{ij}) \in \mathrm{Mat}_n(\mathbb{Z})$ be an intersection matrix associated with the resolution of a hypersurface singularity, all of whose exceptional components are smooth rational curves. Assume that $c_{ii} \leq -2$ for all $i = 1, \ldots, n$. Laufer in [24], 3.7, provides additional constraints on the canonical vector K associated with such N, with an improvement by M. Tomari stated in the Addendum on page 496. A further improvement was found by Yau in [48], Theorems B and C, which show that for such N,

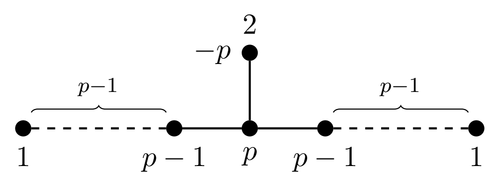
$$g_i/p_i \geq (|\mathbf{Z} \cdot \mathbf{Z}| - 2)z_i,$$

where ${}^t\mathbf{Z} = (z_1, \ldots, z_n)$ is the fundamental cycle of N. In other words, we have $-K \geq (|\mathbf{Z} \cdot \mathbf{Z}| - 2)\mathbf{Z}$. Note that the singularity in Proposition 9.6 satisfies $-K = (|\mathbf{Z} \cdot \mathbf{Z}| - 2)\mathbf{Z}$.

In the context of wild $\mathbb{Z}/2\mathbb{Z}$-quotient singularities treated in this article, the resolution of such a singularity has intersection matrix N with Φ_N killed by 2 and with $|\mathbf{Z} \cdot \mathbf{Z}| \leq 2$. Proposition 10.5 shows that any such N is always numerically Gorenstein, and since $|\mathbf{Z} \cdot \mathbf{Z}| \leq 2$ and $\mathbf{Z} > 0$, Laufer's constraints are also automatically satisfied.

Example 10.7. We exhibit below a wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity that is not numerically Gorenstein. For this, note first the following. Suppose that $N = (c_{ij})$ is an intersection matrix such that Φ_N is killed by p, and such that $c_{ii} = -2$ except for a unique vertex C with $2 < |(C \cdot C)|$ and $\gcd(p, |C \cdot C| - 2) = 1$. *Then N is numerically Gorenstein if and only if the class of e_C is trivial in Φ_N.* Indeed, if the class of e_C is trivial in Φ_N, then there exists an integer vector K_C with $NK_C = e_C$. It follows that $K := (|C \cdot C| - 2)K_C$ is such that $NK = H_0$. If the class of e_C is not trivial, then it must have order p, and so there exists an integer vector R such that $NR = pe_C$. If there also exists an integer vector K with $NK = H_0 = (|C \cdot C| - 2)e_C$, then the class of e_C in Φ_N has order dividing $|C \cdot C| - 2$, which is a contradiction since $\gcd(p, |C \cdot C| - 2) = 1$.

Let $p > 2$ be prime and consider now the wild $\mathbb{Z}/p\mathbb{Z}$-quotient singularity in [29], 6.8, with resolution graph with $r_1(i) = 1$. This resolution graph has a single vertex of self-intersection different from -2, namely the terminal vertex C with $r_1(i) = 1$ and self-intersection $-p$, represented as the top center vertex in the graph below. The class of e_C is not trivial in Φ_N, since the vector R whose multiplicities are indicated below is such that $NR = -pe_C$ and the greatest common divisor of the coefficients of R is equal to 1. Then the above argument shows that N cannot be numerically Gorenstein. In fact, the canonical vector K is $-(p-2)R/p$. The fundamental cycle is given in [30], 4.4, and it is shown in [30], 4.1, that this singularity is rational.



References

[1] M. Artin, *Algebraic approximation of structures over complete local rings*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 23–58.

[2] M. Artin, *Wildly ramified $Z/2$ actions in dimension two*, Proc. Amer. Math. Soc. **52** (1975), 60–64.

[3] M. Artin, *Coverings of the rational double points in characteristic p*, In: W. Baily, T. Shioda (eds.), Complex analysis and algebraic geometry, pp. 11–22. Iwanami Shoten, Tokyo, 1977.

[4] P. Berthelot, A. Grothendieck, and L. Illusie (eds.), *Théorie des intersections et théorème de Riemann–Roch* (SGA 6), Springer, Berlin, 1971.

[5] J. Bochnak and S. Łojasiewicz, *Remarks on finitely determined analytic germs*, 1971 Proceedings of Liverpool Singularities—Symposium, I (1969/70) pp. 262–270, Lecture Notes in Mathematics, Vol. 192, Springer, Berlin.

[6] W. Bosma, J. Cannon, and C. Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), 235–265. `http://magma.maths.usyd.edu.au/magma/`

[7] E. Campbell and D. Wehlau, *Modular invariant theory*, Encyclopaedia of Mathematical Sciences, 139. Invariant Theory and Algebraic Transformation Groups, 8. Springer-Verlag, Berlin, 2011.

[8] B. Conrad, B. Edixhoven, and W. Stein, $J_1(p)$ has connected fibers, Doc. Math. **8** (2003), 331–408.

[9] D. Cox, J. Little, and H. Schenck, *Toric varieties*, American Mathematical Society, Providence, RI, 2011.

[10] V. I. Danilov, *The geometry of toric varieties*, Russian Math. Surveys **33** (1978), no. 2, 97–154.

[11] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann, Singular 3-1-6 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de (2012).

[12] D. Eisenbud, *Commutative algebra. With a view toward algebraic geometry*, Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[13] R. Fossum, *The divisor class group of a Krull domain*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 74. Springer-Verlag, New York–Heidelberg, 1973.

[14] W. Fulton, *Intersection theory*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1998.

[15] S. Greco and M. Marinari, *Nagata's criterion and openness of loci for Gorenstein and complete intersection*, Math. Z. **160** (1978), no. 3, 207–216.

[16] G.-M. Greuel and H. Kröning, *Simple singularities in positive characteristic*, Math. Z. **203** (1990), 339–354.

[17] A. Grothendieck and J. Dieudonné, *Éléments de géométrie algébrique*, Publ. Math. IHÉS **4** (Chapter 0, 1–7, and I, 1–10), **8** (II, 1–8), **11** (Chapter 0, 8–13, and III, 1–5), **17** (III, 6–7), **20** (Chapter 0, 14–23, and IV, 1), **24** (IV, 2–7), **28** (IV, 8–15), and **32** (IV, 16–21), 1960–1967.

[18] F. Hirzebruch and K. Jänich, *Involutions and singularities*, in Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968) pp. 219–240, Oxford Univ. Press, London, 1969.

[19] H. Ito and S. Schröer, *Wild quotient surface singularities whose dual graphs are not star-shaped*, Asian J. Math. **19** (2015), 951–986.

[20] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, *Toroidal embeddings I*, Springer Lecture Notes 339, Berlin, 1973.

[21] F. Kiràly and W. Lüktebohmert, *Group actions of prime order on local normal rings*, Algebra Number Theory **7** (2013), no. 1, 63–74.

[22] S. Kleiman, *Toward a numerical theory of ampleness*, Ann. Math. **84** (1966), 293–344.

[23] H. Laufer, *On minimally elliptic singularities* Amer. J. Math. **99** (1977), no. 6, 1257–1295.

[24] H. Laufer, *The multiplicity of isolated two-dimensional hypersurface singularities*, Trans. Amer. Math. Soc. **302** (1987), no. 2, 489–496.

[25] J. Lipman, *Rational singularities, with applications to algebraic surfaces and unique factorization*, Inst. Hautes Études Sci. Publ. Math. **36** (1969), 195–279.

[26] J. Lipman, *Desingularization of two-dimensional schemes*, Ann. Math. (2) **107** (1978), no. 1, 151–207.

[27] D. Lorenzini, *Jacobians with potentially good ℓ-reduction*, J. reine angew. Math. **430** (1992), 151–177.

[28] D. Lorenzini, *Wild quotient singularities of surfaces*, Math. Z. **275** (2013), 211–232.

[29] D. Lorenzini, *Wild models of curves*, Algebra Number Theory **8** (2014), no. 2, 331–367.

[30] D. Lorenzini, *Wild quotients of products of curves*, Eur. J. Math. **4** (2018), no. 2, 525–554.

[31] D. Lorenzini and S. Schröer, *Moderately ramified actions in positive characteristic*, Math. Z. **295** (2020), no. 3-4, 1095–1142.

[32] A. Micali, *Sur les algèbres universelles*, Ann. Inst. Fourier (Grenoble) **14** (1964) fasc. 2, 33–87.

[33] K. Mitsui, *Quotient singularities of products of two curves*, Preprint, 2017.

[34] M. Miyanishi and P. Russell, *Purely inseparable coverings of exponent one of the affine plane*, J. Pure Appl. Algebra **28** (1983), no. 3, 279–317.

[35] J. Mott, *Eisenstein-type irreducibility criteria*, In: D. Anderson, E. Dobbs (eds.), Zero-dimensional commutative rings, pp. 307–329. Dekker, New York, 1995.

[36] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math., Inst. Hautes Étud. Sci. **9** (1961), 5–22.

[37] P. Orlik and P. Wagreich, *Isolated singularities of algebraic surfaces with $\mathbb{C}^*$ action*, Ann. of Math. (2) **93** (1971), 205–228.

[38] P. Orlik and P. Wagreich, *Singularities of algebraic surfaces with $\mathbb{C}^*$ action*, Math. Ann. **193** (1971), 121–135.

[39] P. Orlik and P. Wagreich, *Algebraic surfaces with k^*-action*, Acta Math. **138** (1977), 43–81.

[40] B. Peskin, *Quotient-singularities in characteristic p*, Thesis, Massachussetts Institute of Technology, 1980.

[41] B. Peskin, *Quotient-singularities and wild p-cyclic actions*, J. Algebra **81** (1983), 72–99.

[42] P. Popescu-Pampu and J. Seade, *A finiteness theorem for dual graphs of surface singularities*, Internat. J. Math. **20** (2009), no. 8, 1057–1068.

[43] M. Roczen, *Recognition of simple singularities in positive characteristic*, Math. Z. **210** (1992), no. 4, 641–653.

[44] P. Samuel, *Classes de diviseurs et dérivées logarithmiques*, Topology **3** (1964) suppl. 1, 81–96.

[45] S. Schröer, *The Hilbert scheme of points for supersingular abelian surfaces*, Arkiv Mat. **47** (2009), 143–181.

[46] S. Schröer, *A higher-dimensional generalization of Mumford's rational pullback for Weil divisors*, J. Singul. **19** (2019), 53–60.

[47] T. Tomaru, *On Gorenstein surface singularities with fundamental genus $p_f \geq 2$ which satisfy some minimality conditions*, Pacific J. Math. **170** (1995), no. 1, 271–295.

[48] S.-T. Yau, *The multiplicity of isolated two-dimensional hypersurface singularities: Zariski problem*, Amer. J. Math. **111** (1989), no. 5, 753–767.

Department of Mathematics, University of Georgia, Athens, GA 30602, USA
Email address: lorenzin@uga.edu

Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany
Email address: schroeer@math.uni-duesseldorf.de