ALGEBRAIC LOOP GROUPS

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Abstract. In this note we introduce algebraic loops, starting from the notion of an interval scheme, and define the algebraic loop group of a connected scheme with a geometric base point x_0 as the set of homotopy classes of algebraic loops based at x_0. The group structure is induced by concatenating algebraic loops. The main result is an isomorphism of this algebraic loop group to Grothendieck's algebraic fundamental group for proper connected schemes over a field.

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Introduction

The fundamental group $\pi_1(X, x_0)$ of a topological space X at a base point x_0 is an invariant of great significance, even more so as its definition is elementary and intuitive: the elements are loops up to homotopy, where a loop is a continuous morphism $I \to X$ from a closed interval I, such that the end points of I are mapped to the base point x_0 and two such loops are homotopic if one can be continuously deformed into the other keeping the base point x_0 fixed. Alternatively, the fundamental group can be described as the X-automorphisms of a universal covering space of X.

On the other hand, if X is a scheme the first construction above makes little sense. However, Grothendieck realized that the second description has a reasonable analog in algebraic geometry, see [6]. In fact he introduces the notion of a *Galois category* which is a category $\mathcal{C}$, the objects of which should be considered as abstract finite coverings, which admits a fiber functor $\Phi : \mathcal{C} \to (\text{FinSet})$ to the category of finite sets satisfying certain properties; these properties ensure that the automorphism group of Φ is equal to the opposite group of the automorphisms of an abstract pro-finite universal covering. The algebraic fundamental group $\pi_1^{\text{alg}}(X, x_0)$ of a connected scheme X with geometric point x_0 is then defined by applying this

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general construction to the category (FinEt/X) of finite étale coverings of X with fiber functor given by taking the fibers over the geometric point x_0. If X is connected of finite type over the complex numbers the group $\pi_1^{\mathrm{alg}}(X, x_0)$ is equal to the pro-finite completion of the fundamental group of the topological space $X(\mathbb{C})$ (with analytic topology). At the same time applying this construction to a point recovers the absolute Galois group of the field defining the point.

In this note we observe that the original construction of the fundamental group using loops has a meaningful analogy for schemes, once the notions of interval and loop are interpreted in an algebraic manner. More precisely, the crucial properties of the interval I in the construction above are: I is connected, quasi- compact, one dimensional, with no non-trivial finite covers, and I has two distinguished points. Translating these properties to algebraic geometry we define in Section 3 an *interval scheme* as a reduced, connected, affine, and one-dimensional scheme I which has no non-trivial finite étale covers and two distinguished closed points with separably closed residue fields. An *algebraic loop* on a scheme X based at a geometric point x_0 is a morphism of schemes $I \to X$, such that its restriction to the two closed points factors via x_0. The algebraic loops define monodromy transformations and two algebraic loops are homotopic if and only if the attached monodromies agree. The *algebraic loop group* $\pi_0\Omega^{\mathrm{alg}}(X, x_0)$ is defined as the set of homotopy classes of algebraic loops; the group structure is induced by concatenating algebraic loops, see Section 4.

Interval schemes are very often non-noetherian. One example of an interval scheme is the universal Galois cover (in the sense of Grothendieck) of a noetherian, connected, affine, reduced, and one dimensional scheme. Such universal Galois covers were systematically studied in [17], where Vakil and Wickelgren define the fundamental group scheme using universal covers, which are certain pro-finite étale maps. The notion of interval scheme introduced above is also inspired by their work. But there are also bigger examples of interval schemes which are more direct to obtain, for example, if R is an integral noetherian one-dimensional ring and A is its integral closure in the separable closure of the fraction field of R, then the choice of two geometric and closed points in $\operatorname{Spec} A$ turns it into an interval scheme.

By construction the monodromy induces an injective group homomorphism

$$(*) \qquad \pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0),$$

where "op" refers to the opposite group structure. The main result of this note is the following, see Theorem 4.4:

Theorem. *Let X be a proper and connected scheme over a field k with a geometric point $x_0 : \operatorname{Spec} k^{\mathrm{sep}} \to X$. Then the homomorphism $(*)$ is bijective.*

The main step in the proof of the above theorem is a Lefschetz type result saying that for a proper and connected k-scheme X we find a closed connected curve $C \subset X$ such that the algebraic fundamental group of C surjects to the one of X, see Proposition 5.5. This is well-known in the case where X is Cohen–Macaulay and projective over a field, see [7], Exposé XII. We reduce the general situation to this using a van-Kampen-like argument and Macaulayfication, which was in a special case constructed by Faltings and later more general by Kawasaki, see [11]. The proof of Theorem 4.4 is given in Section 6. For affine schemes the map $(*)$ will not be an isomorphism on the nose.

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1. Monodromy

Let Y be a scheme, and write (FinEt/Y) for the category of Y-schemes X whose structure morphism $f : X \to Y$ is finite and étale. Note that such an f is proper, affine, flat, of finite presentation, and for each point $b \in Y$ the fiber $f^{-1}(b)$ is the spectrum of some étale algebra over the residue field $k = \kappa(b)$. See, e.g., [13], Theorem 5.10 and Exercise 5.21, for the following result.

Proposition 1.1. *Suppose Y is connected, and $f : X \to Y$ finite and étale. Then there is a finite étale morphism $Y' \to Y$ such that $X' = X \times_Y Y'$ is isomorphic to the disjoint union $\coprod_{i=1}^r Y'$ for some integer $r \geqslant 0$.*

Corollary 1.2. *Suppose Y is connected, and $f : X \to Y$ finite and étale. Then X has only finitely many connected components $U \subset X$, each of which is open-and-closed. Moreover, the induced morphism $U \to X$ is finite and étale.*

Proof. Take $Y' \to Y$ as in Proposition 1.1. Since the projection $X' \to X$ is surjective the connected components of X are images of the connected components of X', hence there are only finitely many. This implies that the connected components U of X are open and closed and hence the composition $U \hookrightarrow X \to Y$ is étale and finite. □

The proposition tells us that the Y-scheme X is a *twisted form* of the disjoint union $\coprod_{i=1}^r Y$, with respect to the étale topology. It thus corresponds to a class in the non-abelian cohomology set $H^1(Y, S_r)$, with coefficients in the symmetric group S_r on $r \geqslant 0$ letters ([5], Chapter III, Section 2.3). To summarize:

Proposition 1.3. *If Y is connected, the following are equivalent:*

(i) Every finite étale Y-scheme is isomorphic to some $\coprod_{i=1}^r Y$, $r \geqslant 0$.
(ii) We have $H^1(Y, S_r) = \{*\}$ for all integers $r \geqslant 0$.
(iii) Each finite étale morphism $X \to Y$ from a connected scheme X is an isomorphism.

Let X be a Y-scheme, with structure morphism $f : X \to Y$, and $a : A \to X$ be some other morphism. To simplify notation, we write

$$X(A) = \mathrm{Hom}_Y(A, X) = \{a' : A \longrightarrow X \mid f \circ a' = a\}$$

of liftings of $a : A \to Y$ with respect to $f : X \to Y$.

Proposition 1.4. *Suppose that Y is connected, with $H^1(Y, S_r) = \{*\}$ for all $r \geqslant 0$, and $f : X \to Y$ is finite and étale. Let $a : A \to Y$ and $b : B \to Y$ be morphisms with connected domains. Then the sets $X(A)$ and $X(B)$ are finite, and for each $a' \in X(A)$ there is a unique $b' \in X(B)$ such that the images $a'(A)$ and $b'(B)$ lie in the same connected component of X.*

Proof. By Proposition 1.3, we may assume $X = \coprod_{i=1}^{r} Y$, for some $r \geqslant 0$. The set $X(A)$ is bijective to the set of sections of $\coprod_{i=1}^{r} A = X \times_Y A \to A$, whence is finite, and we see that every $a' \in X(A)$ corresponds uniquely to one of the r-many maps

$$A \hookrightarrow \coprod_i A \xrightarrow{\coprod a} \coprod_i Y = X.$$

This implies the statement. □

1.5. In the situation of Proposition 1.4 we obtain a mapping

$$\mu_X : X(A) \longrightarrow X(B), \quad a' \longmapsto b',$$

which is called the *monodromy*, and will play a crucial role throughout. We regard it as a transformation between functors $(\mathrm{FinEt}\,/Y) \to (\mathrm{FinSet})$ given by $X \mapsto X(A)$ and $X \mapsto X(B)$. By Proposition 1.4, the monodromy μ_X is a natural isomorphism and is given as the composition of the following natural bijections

$$\mu_X : X(A) \longrightarrow \pi_0(X \times_Y A) \longrightarrow \pi_0(X) \longrightarrow \pi_0(X \times_Y B) \longrightarrow X(B),$$

where $\pi_0(X)$ denotes the set of connected components of X.

2. Galois categories

In this section we recall the notion of Galois categories, which where introduced by Grothendieck to unify Galois theory from algebra and covering space theory from topology ([6], Exposé V).

2.1. Recall that a category $\mathcal{C}$ is a *Galois category* if there is a functor $\Phi : \mathcal{C} \to (\mathrm{FinSet})$ such that the following six axioms hold:

- (G1) Fiber products and final objects exist in $\mathcal{C}$.
- (G2) Finite sums and quotients by finite group actions exist in $\mathcal{C}$.
- (G3) Ever morphism $X' \to X$ in $\mathcal{C}$ factors into a strict epimorphism $X' \to U$ and the inclusion of a direct summand $U \subset X$.
- (G4) The functor Φ commutes with fiber products and final objects.
- (G5) It also commutes with finite direct sums and forming quotients by finite group actions, and transforms strict epimorphisms into surjections.
- (G6) If for a morphism $u : X' \to X$ in $\mathcal{C}$ the resulting map $\Phi(u)$ is bijective, then u is an isomorphism.

One calls Φ a *fundamental functor* for the Galois category $\mathcal{C}$, and denotes by $\pi = \mathrm{Aut}(\Phi)$ the group of natural isomorphisms of the fundamental functor to itself. In turn, we have an inclusion

$$\pi \subset \prod_{X \in \mathcal{C}} S_{\Phi(X)}$$

inside a product of symmetric groups $\mathrm{Aut}(\Phi(X)) = S_{\Phi(X)}$. These groups are finite. We endow them with the discrete topology, and the product with the product topology. The latter becomes a topological group that is compact and totally disconnected. Such topological groups are also called *pro-finite groups*. One easily checks that the subgroup π is closed, and thus inherits the structure of a pro-finite group. Up to uncanonical isomorphism, the pro-finite group π depends only on the Galois category $\mathcal{C}$, and not on the choice of fiber functor Φ.

Now write (π-FinSet) for the category of finite sets F endowed with a π-action from the left, where the kernel of the canonical homomorphism $\pi \to S_F$ is closed. In other words, the action $\pi \times F \to F$ is continuous, when the finite set F is endowed with the discrete topology. With respect to the natural π-action on the $\Phi(X)$, $X \in \mathcal{C}$, the fundamental functor becomes a functor

$$\Phi : \mathcal{C} \longrightarrow (\pi\text{-FinSet}),$$

and Grothendieck deduced with the axioms (G1)–(G6) that the above is an equivalence of categories. Conversely, if G is a pro-finite group, the category (G-FinSet) is a Galois category: The functor Φ that forgets the G-action is a fundamental functor, and the resulting $\pi = \mathrm{Aut}(\Phi)$ becomes identified with G.

One should see $\pi = \mathrm{Aut}(\Phi)$ as a common generalization of the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of a field F, and the pro-finite completion of the fundamental group $\widehat{\pi}_1(Y, y_0)$, say for connected and locally simply-connected topological spaces Y.

2.2. An exact functor between Galois categories is a covariant functor $H : \mathcal{C} \to \mathcal{C}'$, such that $\Phi' \circ H$ is a fiber functor for $\mathcal{C}$, whenever Φ' is a fiber functor for $\mathcal{C}'$. An exact functor H induces a morphism $h : \pi' \to \pi$ between the corresponding fundamental groups with reversed direction, which is well-defined up to conjugation. The following is equivalent (see [6, Exposé V, Proposition 6.9])

(1) $H : \mathcal{C} \to \mathcal{C}'$ is fully faithful.
(2) $h : \pi' \to \pi$ is surjective.
(3) For each connected object $X \in \mathcal{C}$, the object $H(X)$ is connected.

2.3. Let Y be a connected scheme. Then $\mathcal{C} = (\mathrm{FinEt}/Y)$ becomes a Galois category: For each morphism $y_0 : \mathrm{Spec}(K) \to Y$, where K is a separably closed field, we get a fiber functor

$$\Phi_{y_0} : (\mathrm{FinEt}/Y) \longrightarrow (\mathrm{FinSet}), \quad X \longmapsto X(K)$$

given by the set of morphisms $b' : \mathrm{Spec}(K) \to X$ lifting the given $b : \mathrm{Spec}(K) \to Y$. The resulting group

$$\pi_1^{\mathrm{alg}}(Y, y_0) = \pi = \mathrm{Aut}(\Phi_{y_0})$$

is called the *algebraic fundamental group* of the connected scheme Y with respect to y_0. In analogy to topology, the latter is called *base point*.

3. Interval schemes

In algebraic topology, the *standard interval* $I = [0, 1]$ plays a central role. From our perspective, the following are the crucial properties:

(i) The topological space I is connected and quasi-compact.
(ii) There are two distinguished points $0, 1 \in I$.
(iii) The universal covering $\tilde{I} \to I$ is a homeomorphism.
(iv) The interval I is one-dimensional.

In this section we introduce a class of schemes with analogous properties. We fix a separably closed field K.

Definition 3.1. An *interval scheme with K-valued endpoints* is a triple (I, a_0, a_1), where I is a reduced, connected, affine, and one-dimensional scheme such that $H^1(I, S_r) = \{*\}$ for all $r \geqslant 0$, and $a_i : \operatorname{Spec}(K) \to X$ are two closed embeddings.

By Proposition 1.3 the vanishing of the non-abelian cohomology groups means that each finite étale covering of I is trivial. The point a_0 is called the *left endpoint*, whereas a_1 is the *right endpoint*. By abuse of notation, we usually write I for the interval scheme (I, a_0, a_1). Note that interval schemes are very often non-noetherian. The following gives the most basic class of interval schemes:

Proposition 3.2. *Let A be a one-dimensional integral ring that is normal and whose field of fractions $F = \operatorname{Frac}(A)$ is separably closed, and suppose there are two surjections $\varphi_i : A \to K$. Then $I = \operatorname{Spec}(A)$ becomes an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.*

Proof. Let $X \to I$ be a connected finite étale cover. It follows that X is an affine connected normal scheme, and thus is integral. In particular the function field L of X is a finite separable field extension $L/\operatorname{Frac}(A)$. Since $\operatorname{Frac}(A)$ is separably closed we find $L \cong \operatorname{Frac}(A)$. Hence $X \to I$ is an isomorphism. Thus I has no non-trivial finite étale cover and therefore defines an interval scheme. □

Example 3.3. Rings as in Proposition 3.2 easily occur as follows: Suppose that R is a one-dimensional noetherian ring, endowed with two integral homomorphisms $\psi_i : R \to K$, i.e., each $\lambda \in K$ satisfies an algebraic equation over R. Choose a separable closure F^{sep} for the field of fractions $F = \operatorname{Frac}(R)$, and write $A = R^{\text{sep}}$ for the integral closure of $R \subset F^{\text{sep}}$. By construction A is integral and normal, with field of fractions F^{sep}, and the ring extension $R \subset A$ is integral. According to the Going-Up Theorem, the map $\operatorname{Spec}(A) \to \operatorname{Spec}(R)$ is surjective, thus the $\psi_i : R \to K$ extend to some homomorphisms $\varphi_i : R^{\text{sep}} \to K$. Proposition 3.2 yields that the scheme $I = \operatorname{Spec}(R^{\text{sep}})$ in an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.

Recall that a local ring R is called *strictly henselian* if each factorization $P \equiv P_1 P_2$ into coprime polynomials over the residue field $k = R/\mathfrak{m}_R$ of a monic polynomial $P \in R[T]$ is induced by a factorization over R, and moreover the residue field is separably closed. See [9], Proposition (18.8.1) for the next example of an interval scheme, for which the image points of the two end points agree.

Proposition 3.4. *Let A be a one-dimensional local ring that is strictly henselian, and whose residue field is isomorphic to K, and let $\varphi_i : A/\mathfrak{m}_A \to K$ be two isomorphisms. Then $I = \operatorname{Spec}(A)$ becomes an interval scheme, where the endpoints a_i correspond to the homomorphisms φ_i.*

3.5. Let I, J be two interval schemes, with K-valued endpoints a_0, a_1 and b_0, b_1, respectively. We write $I * J$ for the concatenation of I and J with respect to the right endpoint on I and the left endpoint on J. In other words, we have a cocartesian square in the category of schemes

$$\begin{array}{ccc} \operatorname{Spec}(K) & \xrightarrow{b_0} & J \\ {\scriptstyle a_1}\downarrow & & \downarrow \\ I & \longrightarrow & I * J. \end{array} \tag{3.1}$$

Note that the cocartesian square above exits in the category of schemes by [3], Théorème 5.4, and is in fact also cartesian. The scheme $I * J$ comes with closed embeddings of I and J, and we take $a_0 \in I \subset I \star J$ as new left endpoint, and $b_1 \in J \subset I * J$ as new right endpoint.

Lemma 3.6. *In the above situation, the concatenation $I * J$ is an interval scheme, with endpoints a_0 and b_1.*

Proof. Write $I = \operatorname{Spec} A$ and $J = \operatorname{Spec} B$. Then $I * J = \operatorname{Spec}(A \times_{a_1^*, K, b_0^*} B)$ is affine and reduced. By construction we have $I * J = I \cup J$ and $I \cap J = a_1(\operatorname{Spec} K) = b_0(\operatorname{Spec} K)$. Hence $I * J$ is also one-dimensional and connected. Let $X \to I * J$ be a finite étale cover. By base change we obtain finite étale covers $I_X := I \times_{I*J} X \to I$, $J_X := J \times_{I*J} X \to J$, and $\operatorname{Spec}(K)_X = \operatorname{Spec}(K) \times_{I*J} X \to \operatorname{Spec} K$. All these étale covers are in fact trivial, i.e., decompose as a disjoint union of r-many copies of the respective base, where $r = [X : I * J]$. We obtain

$$X = I_X \cup J_X = \Big(\coprod_{i=1}^{r} I\Big) \cup \Big(\coprod_{i=1}^{r} J\Big) \quad \text{and} \quad \Big(\coprod_{i=1}^{r} I\Big) \cap \Big(\coprod_{i=1}^{r} J\Big) = \coprod_{i=1}^{r} \operatorname{Spec} K. \tag{3.2}$$

Hence $X = \coprod_{i=1}^{r} I * J$. Thus any finite étale cover of $I * J$ is trivial and hence $I * J$ is an interval scheme. □

4. The algebraic loop group

Fix some separably closed field K and let Y be a scheme, endowed with two K-valued points $y_i : \operatorname{Spec}(K) \to Y$. We may regard this as an object in the category (K^2/Sch) of schemes endowed with two K-valued points.

Definition 4.1. An *algebraic path* in Y starting at y_0 and ending in y_1 is an interval scheme (I, a_0, a_1) with K-valued endpoints, together with a morphism

$$w : (I, a_0, a_1) \longrightarrow (Y, y_0, y_1)$$

of schemes endowed with two K-valued points. An algebraic path w is called *algebraic loop* if $y_0 = y_1$.

By abuse of notation, we often write $w : I \to Y$ for the algebraic path $w : (I, a_0, a_1) \to (Y, b_0, b_1)$. For each finite étale map $X \to Y$, the base change induces a finite étale map $X \times_Y I \to I$, which takes the form $\coprod_{i=1}^{r} I$, for some $r \geqslant 0$, and we have an identification $X(y_i) = (X \times_Y I)(a_i)$ of fiber sets. In turn, the monodromy gives a transformation

$$\mu_w : X(y_0) = (X \times_Y I)(a_0) \longrightarrow (X \times_Y I)(a_1) = X(y_1)$$

that is bijective, and natural in the objects $X \in (\mathrm{FinEt}/Y)$. In other words, the monodromy μ_w attached to the path $w : I \to Y$ from y_0 to y_1 is a bijective natural transformation between fiber functors

$$\Phi_{y_0}, \Phi_{y_1} : (\mathrm{FinEt}/Y) \longrightarrow (\mathrm{FinSet}).$$

We now use this monodromy to give an algebraic version of homotopy:

Definition 4.2. We say that two algebraic paths $w : I \to Y$ and $v : J \to Y$ from y_0 to y_1 are *homotopic* if $\mu_w = \mu_v$ as natural transformations from Φ_{y_0} to Φ_{y_1}.

4.3. We denote the set of algebraic loops in Y based at the geometric point y_0 by

$$\Omega^{\mathrm{alg}}(Y, y_0) = \{w : (I, a_0, a_1) \longrightarrow (Y, y_0, y_0)\}.$$

Let $w : (I, a_0, a_1) \to (Y, y_0, y_0)$ and $v : (J, b_0, b_1) \to (Y, y_0, y_0)$ be two loops based at y_0. It follows from the pushout diagram (3.1) and Lemma 3.6 that we can concatenate these loops to get a new loop

$$w * v : (I * J, a_0, b_1) \longrightarrow (Y, y_0, y_0).$$

The monodromy transformation corresponding to $w * v$ can be factored, for $X \in (\mathrm{FinEt}\,/Y)$ as

$$\begin{array}{ccc} X(y_0) & \xrightarrow{\quad\mu_{w*v}\quad} & X(y_0) \\ \uparrow\simeq & & \uparrow\simeq \\ (I*J)_X(a_0) & \xrightarrow{\quad\simeq\quad} & (I*J)_X(b_1) \\ \uparrow\simeq & & \uparrow\simeq \\ I_X(a_0) & \xrightarrow{\simeq} I_X(a_1) \cong (I*J)_X(a_1) = (I*J)_X(b_0) \cong J_X(b_0) \xrightarrow{\simeq} & J_X(b_1), \end{array}$$

where we use the notation $Z_X = X \times_Y Z$, the vertical maps in the upper square are induced by base change of $w * v$, and the isomorphisms $I_X(a_i) \cong (I * J)_X(a_i)$ and $J_X(b_i) \cong (I * J)_X(b_i)$, for $i = 0, 1$, are induced by the natural closed immersions $I \hookrightarrow I * J$ and $J \hookrightarrow I * J$. The upper square commutes by the definition of μ_{w*v}, the lower square commutes by the construction of the isomorphisms, see the proof of Proposition 1.4. Thus the whole square commutes and by definition of the monodromy we obtain the equality of functors $(\mathrm{FinEt}\,/Y) \to (\mathrm{FinSet})$

$$\mu_{w*v} = \mu_v \circ \mu_w. \tag{4.1}$$

Furthermore the universal property of pushout diagrams yields for three loops u, v, w based at y_0 a canonical isomorphism

$$(u * v) * w \cong u * (v * w) \quad \text{in } (\mathrm{Sch}/Y). \tag{4.2}$$

Clearly, homotopy between paths defines an equivalence relation $w \sim v$ and we denote by

$$\pi_0\Omega^{\mathrm{alg}}(Y, y_0)$$

the set of homotopy classes of algebraic loops based at y_0. We denote by $[w]$ the homotopy class of a loop $w : I \to Y$ at y_0. It follows from (4.1) that we obtain a well defined operation

$$* : \pi_0\Omega^{\mathrm{alg}}(Y, y_0) \times \pi_0\Omega^{\mathrm{alg}}(Y, y_0) \longrightarrow \pi_0\Omega^{\mathrm{alg}}(Y, y_0), \quad ([w], [v]) \mapsto [w] * [v].$$

This operation is associative by (4.2) and clearly any constant loop $I \to y_0 \to Y$ has the same homotopy class denoted by e, which defines a neutral element for $*$. Moreover, for a loop $w : (I, a_0, a_1) \to (Y, y_0, y_0)$ we define the loop $w' : (I, a_1, a_0) \to (Y, y_0, y_0)$ by switching the end points of I. Clearly $[w] * [w'] = e$. Hence concatenation of algebraic loops defines a group structure on $\pi_0\Omega^{\mathrm{alg}}(Y, y_0)$, which we therefore call the *algebraic loop group*.

By definition of the algebraic fundamental group, our homotopy relation, and the relation (4.1) the algebraic loop space $\Omega^{\mathrm{alg}}(Y, y_0)$ induces an injective homomorphism

$$\pi_0\Omega^{\mathrm{alg}}(Y, y_0)^{\mathrm{op}} \longrightarrow \pi_1^{\mathrm{alg}}(Y, y_0), \quad w \longmapsto \mu_w \tag{4.3}$$

of groups, where we use the opposite group structure on the left hand side. We regard this as an inclusion of groups.

The following is the main result of this note.

Theorem 4.4. *Let X be a proper and connected scheme over a field k with a geometric point $x_0 : \operatorname{Spec} k^{\mathrm{sep}} \to X$. Then the canonical homomorphism (4.3)*

$$\pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} \xrightarrow{\simeq} \pi_1^{\mathrm{alg}}(X, x_0)$$

is bijective.

The proof of Theorem 4.4 requires some preparations and will be given in Section 6. We remark that we do not expect (4.3) to be an isomorphism for non-proper schemes.

5. A Lefschetz type theorem

The *Lefschetz Hyperplane Theorem* gives a strong relation between the homology of a projective complex manifold X of dimension $n \geqslant 2$ and the homology of an ample divisor $D \subset X$. The original arguments appear in [12], Chapter V, Section III. Analogous statements for fundamental groups were first obtained by Bott [1]: The induced map $\pi_1(D, x_0) \to \pi_1(X, x_0)$ is bijective for $n \geqslant 3$, and at least surjective for $n \geqslant 2$. The latter statement extends to projective schemes X over arbitrary ground fields k: According to [7], Exposé XII, Corollary 3.5 the map $\pi_1^{\mathrm{alg}}(D, x_0) \to \pi_1^{\mathrm{alg}}(X, x_0)$ is surjective provided that $\operatorname{depth}(\mathscr{O}_{X,a}) \geqslant 2$ for each $a \in X$. If X is additionally *Cohen–Macaulay*, i.e., at every point $a \in X$ the equality $\dim(\mathscr{O}_{X,a}) = \operatorname{depth}(\mathscr{O}_{X,a})$ holds, the above can be iterated and one finds a surjection $\pi_1^{\mathrm{alg}}(C, x_0) \to \pi_1^{\mathrm{alg}}(X, x_0)$, where $C \subset X$ is a connected projective curve.

In this section we generalize the latter statement to arbitrary proper k-schemes.

5.1. Let X be a non-empty connected noetherian scheme. We consider the following property:

(C) For each closed subscheme $Z \subset X$ with $\dim(Z) \leqslant 0$, there is a connected closed subscheme $C \subset X$ with $0 \leqslant \dim(C) \leqslant 1$ and $Z \subset C$ such that, for each finite étale covering $U \to X$ with connected total space, the restriction $C_U = C \times_X U$ remains connected.

Remark 5.2. We remark that property (C) does not hold for certain affine schemes as the following example shows (cf. Lemma 5.4 in [2]): Let k be a field of characteristic $p > 0$. Let $C \subset \mathbb{A}^2_k$ be a closed 1-dimensional connected subscheme. Then there exists a connected finite étale cover $u : X \to \mathbb{A}^2_k$, with trivial base change $u^{-1}(C) \to C$.

Indeed we may assume C is reduced, hence Cohen–Macaulay. Thus it has no embedded components and we can write $C = V(f)$, for some $f \in k[x, y]$. Take

$h \in (f) \subset k[x,y]$ which is not equal to $g^p - g$, for some $g \in k[x,y]$. Via the isomorphism induced by the Artin-Schreier sequence

$$\frac{k[x,y]}{\{g^p - g \mid g \in k[x,y]\}} \xrightarrow{\simeq} H^1_{\mathrm{et}}(\mathbb{A}^2_k, \mathbb{Z}/p)$$

h corresponds to a non-trivial $\mathbb{Z}/p$-torsor $u_h : X \to \mathbb{A}^2_k$, in particular it is a connected finite étale cover of $\mathbb{A}^2_k$. On the other hand h maps to zero on $A/\{a^p - a \mid a \in A\}$, where $A = k[x,y]/(f)$, i.e., the pullback of u_h over C is trivial.

Lemma 5.3. *Let X be a non-empty connected noetherian scheme. Let $f : X' \to X$ be a proper and surjective morphism from a noetherian scheme X' and denote by X'_v the connected components of X'. If property (C) holds for all X'_v, then it also holds for X.*

The proof of this lemma is inspired by the Seifert–van Kampen Theorem from [16], but is more elementary. We first gather some basic material on graphs.

5.4. Let $f : X' \to X$ be as in the statement of Lemma 5.3. Set $X'' = X' \times_X X'$ and write $\mathrm{pr}_1, \mathrm{pr}_2 : X'' \to X'$ for the two projections. The schemes X' and X'' are noetherian, hence the sets of connected components $\pi_0(X')$ and $\pi_0(X'')$ are finite. Consider the induced maps

$$\mathrm{pr}_1 \times \mathrm{pr}_2 : \pi_0(X'') \longrightarrow \pi_0(X') \times \pi_0(X').$$

This defines an *oriented graph* $\Gamma = (E, V, \mathrm{pr}_1 \times \mathrm{pr}_2)$: in the sense of Serre [14], Section 2.1: The set of *vertices* is $V = \pi_0(X')$, and the set of *oriented edges* is $E = \pi_0(X'')$. The endpoints of an edge $e \in E$ are the images $v_i = \mathrm{pr}_i(e)$. The orientation is given by declaring v_1 as the initial vertex, and v_2 as the terminal vertex. Note that there might be edges with the same initial and terminal vertex, and also several edges with the same initial and the same terminal vertices. By abuse of notation, we also write $v \in \Gamma$ and $e \in \Gamma$ to denote vertices and edges of the graph, if there is no risk of confusion. A morphism $f : \Gamma \to \Gamma'$ between oriented graphs comprises compatible maps $V \to V'$ and $E \to E'$. We simply say that f is a *map of oriented graphs.*

Note that the graph Γ constructed above is connected, since the scheme X is connected.

Proof of Lemma 5.3. Since $f : X' \to X$ is proper, the image $f(Z)$ of a closed subscheme $Z \subset X'$ is closed and satisfies $\dim f(Z) \leqslant \dim(Z)$. In particular, closed points are mapped to closed points. The assertion is trivial for $\dim(X) \leqslant 1$ We now assume $\dim X \geqslant 2$. We use the notation from 5.4. Let $Z \subset X$ be a zero-dimensional closed subscheme. For each edge $e \in \Gamma$ choose a closed point $x_e \in X''_e$. Set

$$x_{e,i} := \mathrm{pr}_i(x_e) \in X'_v, \quad \text{where } v = \mathrm{pr}_i(e) \in \Gamma,\ i = 1, 2.$$

Since Γ is finite we find for each vertex $v \in \Gamma$ a 0-dimensional closed subset $Z'_v \subset X'_v$, such that

(a) $Z \cap f(X'_v) \subset f(Z'_v)$ and
(b) $x_{e,i} \in Z'_v$, for all edges $e \in \Gamma$ with $v = \mathrm{pr}_i(e)$ for $i = 1$ or 2.

Condition (b) is immediate, and one can achieve (a) by picking a closed point in each of the finitely many schemes $f^{-1}(z) \cap X'_v$, with $z \in Z \cap f(X'_v)$. By the surjectivity of f we have

$$Z \subset \bigcup_v f(Z'_v). \tag{5.1}$$

Applying (C) to X'_v and Z'_v we find an at most 1-dimensional closed subscheme $C'_v \subset X'_v$ containing Z'_v, such that the pullback of any connected finite étale cover of X'_v to C'_v stays connected. Then $C = \bigcup_v f(C'_v)$ is closed, at most 1-dimensional, and contains Z, by (5.1). It remains to show that for each connected finite étale cover $U \to X$ the pullback $U \times_X C$ remains connected (then C is connected as well).

To this end fix such a finite étale cover $u : U \to X$. Denote by Γ_U the graph defined by $\mathrm{pr}_1 \times \mathrm{pr}_2 : \pi_0(U'') \to \pi_0(U') \times \pi_0(U')$, where $U' = U \times_X X'$ and $U'' = U \times_X X'' = U' \times_U U'$. We obtain a surjection of graphs $u : \Gamma_U \to \Gamma$ and for each edge $\epsilon \in \Gamma_U$ we obtain a finite and étale morphism between connected schemes $U''_\epsilon \to X''_{u(\epsilon)}$ which therefore is surjective. Thus for each edge $e \in \Gamma$ and edge $\epsilon \in \Gamma_U$ mapping to e we can choose a closed point $x_{U,\epsilon} \in U_\epsilon$ with $u(x_{U,\epsilon}) = x_e$.

For a vertex $w \in \Gamma_U$ mapping to $v \in \Gamma$ denote by I_w the image of $U'_w \times_{X'_v} C'_v$ under the map

$$U' \times_{X'} C'_v = U \times_X C'_v \longrightarrow U \times_X C_v.$$

The map is induced by the base change with the composition $C'_v \hookrightarrow X'_v \hookrightarrow X' \xrightarrow{f} X$, and therefore is closed and surjective. Hence

$$U \times_X C_v = \bigcup_{w \in u^{-1}(v)} I_w \quad \text{and} \quad U \times_X C = \bigcup_{w \text{ edge in } \Gamma_U} I_w, \tag{5.2}$$

where each I_w is closed. By our choice of C'_v the pullback of the connected étale cover $U'_w \to X'_v$ over C'_v remains connected. Thus I_w is the image of a connected scheme and is hence connected. Let w_1 and w_2 be the initial and the terminal vertices of an edge $\epsilon \in \Gamma_U$, then $x_{U,\epsilon} \in U''_\epsilon$ maps via the ith projection to points $\mathrm{pr}_i(x_{U,\epsilon})$ in $U'_{w_i} \times_{X'_{u(w_i)}} C'_{u(w_i)}$, $i = 1, 2$, and these points map to same point in U. Thus the intersection $I_{w_1} \cap I_{w_2}$ is non-empty, if w_1 and w_2 are linked by an edge in Γ_U. Since the graph Γ_U is connected so is the scheme $U \times_X C$. This completes the proof. □

Proposition 5.5. *Let X be a connected scheme that is proper over a field k. Then X has property (C).*

Proof. We proceed by induction on $n = \dim X$. There is nothing to prove for $n \leqslant 1$. Assume $n \geqslant 2$ and that (C) holds for all connected schemes that are proper over k and have dimension $\leqslant n-1$. Using Lemma 5.3 we can make the following reductions:

(i) X reduced (by considering the proper morphism $X_{\mathrm{red}} \to X$);
(ii) X projective over k (Chow's Lemma);
(iii) X integral (by considering the proper and surjective morphism $\coprod X_i \to X$, with X_i the irreducible components of X).

According to Kawasaki's result ([11], Theorem 1.1) there is a proper birational $X' \to X$ such that the scheme X' is Cohen–Macaulay. Moreover, this Macaulayfication arises as a sequence of blowing-ups. Hence applying Lemma 5.3 one more time, we are reduced to consider the case that X is projective, integral, and Cohen–Macaulay over a field k. Let $Z \subset X$ be a closed subscheme with $\dim(Z) \leqslant 0$. By, e.g., [4], Theorem 5.1, we find an effective ample divisor $D \subset X$ containing Z. By induction the following claim implies that X satisfies (C):

Claim 5.6. Let $X' \to X$ be an étale covering whose total space is connected. Then the restriction $D' = X' \times_X D$ remains connected.

The argument to prove the claim is similar to [10], Chapter II, Corollary 6.2. Let us recall it for the sake of completeness: Consider the ample invertible sheaf $\mathscr{L} = \mathscr{O}_X(D)$. Since $X' \to X$ is finite and surjective, the inclusion $D' \subset X'$ remains an effective Cartier divisor, and the corresponding invertible sheaf is the pullback $\mathscr{L}' = \mathscr{L}|X'$, which is still ample. Since X is projective and Cohen–Macaulay, so is X'. Let $\omega_{X'}$ be the dualizing sheaf over k. Then $h^1(\mathscr{L}'^{\otimes -t}) = h^{n-1}(\omega_{X'} \otimes \mathscr{L}'^{\otimes t})$ for every integer t. The right hand side vanishes for t sufficiently large, because $\mathscr{L}'$ is ample and $n-1 \geqslant 1$. Replacing D by tD, we may assume $H^1(X', \mathscr{L}'^{\otimes -1}) = 0$. The short exact sequence $0 \to \mathscr{L}^{\otimes -1} \to \mathscr{O}_{X'} \to \mathscr{O}_{D'} \to 0$ thus gives a surjection of rings $H^0(X', \mathscr{O}_{X'}) \to H^0(D', \mathscr{O}_{D'})$. The term on the left is a finite extension of the ground field k because X' is integral and proper. Hence the above map is bijective, and D' must be connected. ☐

In view of 2.2 we obtain the following corollary.

Corollary 5.7. *Let k be a field and set $K = k^{\mathrm{sep}}$. Let X be a connected scheme which is proper over k and let $x_0 : \operatorname{Spec} K \to X$ be a geometric point. Then there exists a connected, reduced, affine, and 1-dimensional scheme C of finite type over k and a k-morphism $C \to X$, such that x_0 factors via C and the induced map*

$$\pi_1^{\mathrm{alg}}(C, x_0) \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0)$$

is surjective.

Proof. By Proposition 5.5 and 2.2 we find a connected closed subscheme $C_1 \subset X$ of dimension at most 1, such that x_0 factors via C_1 and the induced map

$$\pi_1^{\mathrm{alg}}(C_1, x_0) \longrightarrow \pi_1^{\mathrm{alg}}(X, x_0)$$

is surjective. Since passing to the reduced subscheme does not change the fundamental group, we may assume C_1 reduced. If $\dim(C_1) = 0$, then $C_1 = \operatorname{Spec} L$ with L a subfield of K. In this case we can take $C := \mathbb{A}^1_L$ with map $\mathbb{A}^1_L \to \operatorname{Spec} L = C_1 \to X$ and factorization of x_0 given by the composition of $\operatorname{Spec} K \to \operatorname{Spec} L$ with the inclusion of the zero section $\operatorname{Spec} L \hookrightarrow \mathbb{A}^1_L$.

Assume $\dim(C_1) = 1$. Note that C_1 is quasi-projective, hence we find an affine open $U \subset C_1$ which is connected and contains the singular locus of C_1 and the image point of x_0. In particular we remove from C_1 only finitely many regular closed points, whose local rings are therefore discrete valuation rings. Hence it follows from [15, Tag 0BSC] that $\pi_1^{\mathrm{alg}}(U, x_0) \to \pi_1^{\mathrm{alg}}(C_1, x_0)$ is surjective and we can take $C := U$. ☐

6. Proof of Theorem the main theorem

We prove Theorem 4.4, i.e., for a proper and connected scheme X over a field k with geometric point $x_0 : \operatorname{Spec} k \to X$ we want to show that the natural injective group homomorphism $\pi_0\Omega^{\mathrm{alg}}(X, x_0) \to \pi_1^{\mathrm{alg}}(X, x_0)$ is surjective as well.

Set $K := k^{\mathrm{sep}}$. By Corollary 5.7 we find a connected, affine, reduced and 1-dimensional k-scheme of finite type C with a morphism $C \to X$ such that x_0 factors via C and the natural $\pi_1^{\mathrm{alg}}(C, x_0) \twoheadrightarrow \pi_1^{\mathrm{alg}}(X, x_0)$ is surjective. We obtain a commutative diagram

$$\begin{array}{ccc} \pi_0\Omega^{\mathrm{alg}}(C, x_0)^{\mathrm{op}} & \longrightarrow & \pi_1^{\mathrm{alg}}(C, x_0) \\ \downarrow & & \downarrow \\ \pi_0\Omega^{\mathrm{alg}}(X, x_0)^{\mathrm{op}} & \longrightarrow & \pi_1^{\mathrm{alg}}(X, x_0), \end{array}$$

in which the vertical arrow on the right is surjective. It therefore remains to show that the top horizontal arrow is surjective. To this end we choose some pro-object $(C_i)_{i\in I}$ in (FinEt/C) representing Φ_{x_0}, see [6], Exposé V, 4. We write $\mathrm{Pro}(\mathrm{FinEt}/C)$ for pro-objects in (FinEt/C). Since I is a filtered set and the transition maps $C_i \to C_j$ are finite étale, and therefore affine, we may form the projective limit $\widetilde{C} = \varprojlim_{i\in I} C_i$ in the category of C-schemes, see [8], Proposition (8.2.3). For a C-scheme T we obtain a functorial isomorphism

$$\mathrm{Hom}_C(T, \widetilde{C}) = \varprojlim_i \mathrm{Hom}_C(T, C_i).$$

In particular an element $a_0 \in \varprojlim \Phi_{x_0}(C_i)$ is a morphism $a_0 : \mathrm{Spec}\, K \to \widetilde{C}$ over x_0. Furthermore, by [6], Exposé V, 4., h), we have

$$\begin{aligned} \mathrm{Aut}_{\mathrm{Pro}(\mathrm{FinEt}/C)}((C_i)_i) &= \mathrm{Hom}_{\mathrm{Pro}(\mathrm{FinEt}/C)}((C_i)_i, (C_j)_j) \\ &= \varprojlim_j \mathrm{Hom}_{\mathrm{Pro}(\mathrm{FinEt}/C)}((C_i)_i, C_j) \\ &= \varprojlim_j \mathrm{Hom}_C(C_j, C_j) = \varprojlim_j \mathrm{Hom}_C(\widetilde{C}, C_j) \\ &= \mathrm{Hom}_C(\widetilde{C}, \widetilde{C}) = \mathrm{Aut}_C(\widetilde{C}). \end{aligned}$$

By *loc. cit.*, the choice of $a_0 : \mathrm{Spec}\, K \to \widetilde{C}$ over x_0 yields a functorial isomorphism

$$\mathrm{Hom}_C(\widetilde{C}, U) \xrightarrow{\simeq} \Phi_{x_0}(U), \quad f \longmapsto f \circ a_0, \qquad U \in (\mathrm{FinEt}/C).$$

We obtain an isomorphism

$$\theta : \mathrm{Aut}_C(\widetilde{C})^{\mathrm{op}} \xrightarrow{\simeq} \mathrm{Aut}(\Phi_{x_0}) = \pi_1^{\mathrm{alg}}(C, x_0),$$

which sends a C-automorphism $\sigma : \widetilde{C} \to \widetilde{C}$ to the automorphism $\theta(\sigma)$ of Φ_{x_0}, which on $U \in (\mathrm{FinEt}/C)$ is given by

$$\Phi_{x_0}(U) \ni f \circ a_0 \longmapsto f \circ (\sigma \circ a_0) \in \Phi_{x_0}(U), \quad f \in \mathrm{Hom}_C(\widetilde{C}, U).$$

We claim that for any σ the automorphism $\theta(\sigma)$ is in the image of $\pi_0\Omega^{\mathrm{alg}}(C, x_0)^{\mathrm{op}}$. Indeed, by construction $\widetilde{C}$ is affine, reduced, connected and 1-dimensional and has only trivial S_r-torsors for all $r \geqslant 1$. We have the K-rational point $a_0 : \mathrm{Spec}\, K \to \widetilde{C}$ over $x_0 : \mathrm{Spec}\, K \to C$. We note that a_0 is a closed immersion. Indeed, for any finite separable field extension L/k a connected component C_0 of $C \otimes_k L$ is a finite étale covering of C; hence we have a map $\widetilde{C} \to C_0$. It follows that the algebraic closure of k in $H^0(\widetilde{C}, \mathscr{O}_{\widetilde{C}})$ is equal to $K = k^{\mathrm{sep}}$, which implies that a_0 is a closed immersion. Thus $(\widetilde{C}, a_0, \sigma \circ a_0)$ is an interval scheme in the sense of Definition 3.1 and the map

$\widetilde{C} \to C$ induces an algebraic loop $w : (\widetilde{C}, a_0, \sigma \circ a_0) \to (C, x_0, x_0)$. The map (4.3) sends the loop w to the monodromy μ_w which by construction is equal to $\theta(\sigma)$. This completes the proof of the theorem. □

References

[1] R. Bott: On a theorem of Lefschetz. Michigan Math. J. 6 (1959), 211–216.
[2] H. Esnault: Survey on some aspects of Lefschetz theorems in algebraic geometry. Rev. Mat. Complut. 30 (2017), 217–232.
[3] D. Ferrand: Conducteur, descente et pincement. Bull. Soc. Math. France 131 (2003), No. 4, 553 – 585
[4] O. Gabber, Q. Liu, and D. Lorenzini: Hypersurfaces in projective schemes and a moving lemma. Duke Math J. 164 (2015), No. 7, 1187 – 1270
[5] J. Giraud: Cohomologie non abélienne. Springer, Berlin, 1971.
[6] A. Grothendieck: Revêtements étales et groupe fondamental (SGA 1). Springer, Berlin, 1971.
[7] A. Grothendieck: Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA 2). North-Holland Publishing Company, Amsterdam, 1968.
[8] A. Grothendieck: Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. Pub. Math. IHÉS, Tome 28 (1966) , pp. 5-255.
[9] A. Grothendieck: Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie. Pub. Math. IHÉS, Tome 32 (1967) , pp. 5-361.
[10] R. Hartshorne: Ample subvarieties of algebraic varieties. Springer, Berlin, 1970.
[11] T. Kawasaki: On Macaulayfication of Noetherian Schemes. Trans. Amer. Math. Soc. 352 (2000), No. 6, 2517–2552.
[12] S. Lefschetz: L'analysis situs et la géométrie algébrique. Gauthier-Villars, Paris, 1924.
[13] H. Lenstra: Galois theory for schemes. Lecture notes, Universiteit Leiden, 1985. http://www.math.leidenuniv.nl/~hwl/PUBLICATIONS/pub.html.
[14] J.-P. Serre:Trees. Springer, Berlin, 1980.
[15] The Stacks project authors: The Stacks project. https://stacks.math.columbia.edu (2021).
[16] J. Stix: A general Seifert-Van Kampen theorem for algebraic fundamental groups. Publ. Res. Inst. Math. Sci. 42 (2006), no. 3, 763–786.
[17] R. Vakil, K. Wickelgren: Universal covering spaces and fundamental groups in algebraic geometry as schemes. J. Théor. Nombres Bordeaux 23 (2011), no. 2, 489–526.

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