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ALBANESE MAPS FOR OPEN ALGEBRAIC SPACES

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Abstract. We show that for each algebraic space that is separated and of finite type over a field, and whose affinization is connected and reduced, there is a universal morphism to a para-abelian variety. The latter are schemes that acquire the structure of an abelian variety after some ground field extension. This generalizes classical results of Serre on universal morphisms from algebraic varieties to abelian varieties. Our proof relies on corresponding facts for the proper case, together with the theory of Macaulayfication, removal of singularities by alterations, pseudo-rational singularities, ind-objects, and Bockstein maps. It turns out that the formation of the Albanese variety commutes with base-change up to universal homeomorphisms. We also give a detailed analysis of Albanese maps for algebraic curves and algebraic groups, with special emphasis on imperfect ground fields.

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Introduction

The *Albanese variety* and the *Albanese map* are fundamental objects in algebraic geometry. Originally, the construction was purely transcendental, depending on path integrals over closed holomorphic one-forms. In the form given by Blanchard [7], it the universal holomorphic map $f : X \to \mathrm{Alb}(X)$ from a compact connected complex space X endowed with a base-point x_0 to a complex torus $A = \mathrm{Alb}(X)$, where the image of $x_0 \in X$ is the zero element $0 \in A$.

Over arbitrary ground field k, Albanese maps for proper varieties and schemes where constructed by Matsusaka [34] and Grothendieck [27], by using the Picard scheme. In the absence of rational points $x_0 \in X$, however, notorious complications

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arise. These problems are particular severe over imperfect fields k of characteristic $p > 0$, when X may become geometrically non-reduced.

To circumvent such issues one has to replace abelian varieties by the so-called *para-abelian varieties*. The latter are schemes P such that for some field extension $k \subset k'$, the base-change $P' = P \otimes_k k'$ admits the structure of an abelian variety. Roughly speaking, these schemes are like abelian varieties, but may lack rational points and group laws. This notion was introduced and analyzed in [32], where we constructed for any proper algebraic space X with $h^0(\mathscr{O}_X) = 1$ a universal morphism $f : X \to \mathrm{Alb}_{X/k}$ to a para-abelian variety. The defining property of this *Albanese map* is that it induces an isomorphism between *maximal abelian subvarieties* inside the Picard schemes. These Albanese maps have excellent properties: They are functorial in X, commute with ground field extensions $k \subset k'$, and are equivariant with respect to the action of the group scheme $\mathrm{Aut}_{X/k}$.

The goal of this paper is to *extend the theory of Albanese maps by removing the assumption that X is universally closed*. In other words, given an algebraic space U that is separated and of finite type, we seek a universal morphism to a para-abelian variety. The first main result of this paper gives a rather comprehensive answer:

Theorem. *(See Thm. 5.4) Let k be a ground field of characteristic $p \geq 0$, and U be an algebraic space that is separated and of finite type. Suppose the affinization $U^{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(U, \mathscr{O}_U)$ is connected and reduced, and that k coincides with the essential field of constants for U. Then there is a universal morphism $f : U \to \mathrm{Alb}_{U/k}$ to a para-abelian variety, which is functorial in U.*

This extends results of Serre [46] on universal maps from algebraic varieties to abelian varieties, obtained in classical language. It also generalizes more recent results of Wittenberg ([49], Appendix) and Achter, Casalaina-Martin and Vial ([1], Appendix), where geometrically reduced schemes were treated. Our assumption on the *essential fields of constants*, a technical concept introduced in Section 5, ensures that for all compactifications $U \subset X$ we have $h^0(\mathscr{O}_X) = 1$. Note that this automatically holds after passing to to a finite field extension inside the integral domain $\Gamma(U, \mathscr{O}_U)$.

The main idea for the proof of the theorem is conceptual and direct: We consider *compactifications* $i_\lambda : U \to X_\lambda$ and the resulting *maximal abelian subvarieties* A_λ inside the Picard schemes $\mathrm{Pic}_{X_\lambda/k}$. The existence of compactifications goes back to Nagata [39]; for algebraic spaces this was more recently established by Conrad, Lieblich and Olsson [16]. The collection $(X_\lambda, i_\lambda)_{\lambda \in L}$ of all compactifications is essentially a filtered ordered set, so we get an ind-object of abelian varieties $(A_\lambda)_{\lambda \in L}$. Note that the concept of *ind-objects* plays a crucial role in several category-theoretic constructions of Grothendieck. We then use results of de Jong [18] on removal of singularities by alterations, together with results of Kawasaki [29] on the existence of Cohen–Macaulayfication, and Kovács [31] on the behavior of direct image sheaves $R^if_*(\mathscr{O}_X)$ over pseudo-rational singularities, to show that $(A_\lambda)_{\lambda \in L}$ is *essentially constant*. It follows that for sufficiently large λ, the Albanese varieties $\mathrm{Alb}_{X_\lambda/k}$ do not depend on the index. These give the desired Albanese variety $\mathrm{Alb}_{U/k}$, whose universal property easily follows from the theory of rational maps.

Recall that in differential topology, an *open manifold* is a manifold without boundary whose connected components are non-compact. In analogy, one may call an algebraic spaces that is separated and of finite type and whose irreducible components are non-proper an "open algebraic spaces", as in the title.

There is a crucial difference between the proper and the open situation: In the latter case, the category $\operatorname{Cpt}(U)$ of all compactifications usually changes under ground field extension $k \subset k'$ in a significant way, for lack of initial object. It is therefore a priori unclear how the Albanese map behaves under ground field extension. Our second main result clarifies this:

Theorem. *(See Thm. 6.1) The comparison map $\operatorname{Alb}_{U\otimes k'/k'} \to \operatorname{Alb}_{U/k} \otimes k'$ is a finite universal homeomorphisms. For separable extensions $k \subset k'$, it is an isomorphism.*

We shall see that over imperfect fields k, the Albanese variety may indeed change upon inseparable extensions. This phenomenon already appears for algebraic curves C, if the *canonical compactification* $\bar{C}$ by regular points at infinity is not smooth at infinity, as we show in Theorem 7.5.

Our theory of Albanese maps also has consequences for *algebraic groups*, that is, group schemes G of finite type. For smooth G the existence of an Albanese map is a classical result, obtained in various degrees of generality by Barsotti [5], Rosenlicht [43] and Chevalley [13]. Modern accounts are given by Conrad [14] and Brion [10], but the general case was apparently not covered so far. Each algebraic group G sits in a central extension $0 \to N \to G \to G^{\text{aff}} \to 1$, where the kernel N of the affinization map is *anti-affine*, a notion introduced and analyzed by Brion [9]. Suppose now that G^{aff} and equivalently G are reduced and connected, such that our theory of Albanese varieties applies.

Theorem. *(See Thm. 8.5) We have $\operatorname{Alb}_{G/k} = N/N'$, where $N' \subset N$ is the smallest subgroup scheme such that N/N' is proper and the induced projection $G/N' \to G^{\text{aff}}$ admits a section.*

Note that the section, if it exists, does not necessarily respect the group laws. So the group scheme G/N' has as underlying scheme $N/N' \times G^{\text{aff}}$, and the group law arises from the product group law by modifying it with a Hochschild cocycle from $Z^2(G^{\text{aff}}, G/N')$. The result seems to be relevant in connection with the *pseudo-abelian varieties*. This are certain extensions of smooth connected unipotent group schemes by abelian varieties, introduced and studied by Totaro [48]. Using reduced connected unipotent group schemes U, supersingular abelian varieties N, and Hochschild cohomology we construct in Proposition 8.7 algebraic groups whose Albanese map does not respect the group law.

The paper is structured as follows: In Section 1 we recall generalities on compactifications, Macaulayfications, and resolution of singularities by alterations, as well as the theory of para-abelian varieties. Section 2 contains an analysis of the map $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X)$ for proper modifications $f : X \to Y$, in particular if Y is regular, and discusses consequences for the Picard group. This is used in Section 3, to understand the effect on the abelian part of the Picard scheme. In Section 4 we briefly recall the notion of ind-objects, and give a characterization of ind-objects of abelian varieties that are essentially constant. Using this, we construct in Section 5 the Albanese map for an algebraic space U that is separated and of finite type, in terms of the system of its compactifications (X_λ, i_λ). In Section 6 we study the behavior of the Albanese variety under ground field extensions. The last two sections study algebraic curves and algebraic groups, respectively.

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1. Recollections and generalities

Let S be a base scheme, and write (Aff/S) for the category of affine S-schemes. Recall that an *algebraic space* is a contravariant functor $X : (\mathrm{Aff}/S) \to (\mathrm{Set})$ that satisfies the sheaf axiom with respect to the étale topology, for which the diagonal monomorphism $X \to X \times X$ is relatively representable by schemes, and such that there is an étale surjection $U \to X$ from some scheme U. Algebraic spaces are important generalizations of schemes that in many situations allow more freely the formation of quotients. Those that are representable by schemes are called *schematic*. We refer to the monographs of Artin [3], Knutson [30], Olsson [40] and the stacks project [47] for a comprehensive treatment.

Let X be an algebraic space. Although not immediate from the definition, it comes with a topological space, and I want to discuss this matter first: A *point* is an equivalence class of some morphism $a : \mathrm{Spec}(K) \to X$, where K is a field, and the equivalence relation is generated by the factorization relation. The set of all points is denoted by $|X|$. It is endowed with the *Zariski topology*, which is the finest topology that renders all maps $|U| \to |X|$ continuous, where $U \to X$ runs through the étale maps from a scheme U. If our algebraic space is schematic, each $a \in |X|$ has via the image $x \in X$ a canonical representation by $\mathrm{Spec}\,\kappa(x) \to X$, so the above $|X|$ can be identified with the usual underlying topological space of X.

A *geometric point* is a morphism $\bar{a} : \mathrm{Spec}(\Omega) \to X$ for some algebraically closed field Ω. Note that here we do not pass to equivalence classes. Such morphisms lift to every étale surjection $U \to X$ from a scheme U. Any such lift gives a point $u \in U$, and the relative separable closure for the inclusion $\kappa(u) \subset \Omega$ depends only on the point $a \in |X|$ represented by the geometric point $\bar{a}$. We denote this by $\kappa(a)^{\mathrm{sep}}$. Likewise, we write $\mathscr{O}_{X,a}^{\mathrm{sh}}$ for the resulting strictly henselian local ring, where the residue field $\kappa(a)^{\mathrm{sep}}$ is separably closed.

Let Y be a noetherian algebraic space. Then there is a dense open subspace $U \subset X$ that is isomorphic to a scheme ([40], Theorem 6.4.1). As explained above, we may regard the generic points $\eta \in |X|$ as generic points $\eta \in X$. A *modification* is a proper morphism $f : X \to Y$ such that $f^{-1}(V) \to V$ is an isomorphism for some dense open set $V \subset Y$, and that f induces a bijection between the sets of generic points. These are just the proper birational morphism, in case Y is integral. An *alteration* is a proper morphism $f : X \to Y$ such that $f^{-1}(V) \to V$ is a finite surjection for some dense open set $V \subset Y$, and that f induces a bijection between the sets of generic points. Let us now recall and collect three deep and fundamental results:

Theorem 1.1. *Suppose that $S = \operatorname{Spec}(R)$ is the spectrum of a noetherian ring. Let Y be an algebraic space that is separated and of finite type.*

(i) *There is an open embedding $Y \subset \bar{Y}$ with $\bar{Y}$ proper.*
(ii) *If the ring R admits a dualizing complex, there is a modification $f : X \to Y$ with X Cohen–Macaulay.*
(iii) *If R is excellent of dimension ≤ 2 and Y is reduced, there is an alteration $f : X \to Y$ with X regular.*

In the above, one moreover may choose X with an ample invertible sheaf.

The embedding in statement (i) is usually called *Nagata compactification*. The above general form is due to Conrad, Lieblich and Olsson ([16], Theorem 1.2.1). Note that the case of schemes was already treated by Lütkebohmert [33]. The other two statements are reduced to the case where Y is a scheme with Chow's Lemma, which was established by Rydh for algebraic spaces ([44], Theorem 8.8). Morphisms $f : X \to Y$ as in (ii) are called *Macaulayfications*. Under certain assumptions, such maps where first constructed by Faltings [21]. The above general form was established by Kawasaki [29]. Further generalizations, without assumptions on the dualizing complex of R, were recently obtained by Česnavičius [11]. Result (iii) is due to de Jong ([18], Corollary 5.15). The case of ground fields was already established earlier ([17], Theorem 4.1). Throughout the paper, we will freely use the above facts.

A scheme P over some field k is called a *para-abelian variety* if some base-change $P' = P \otimes_k k'$ admits the structure of an abelian variety. This notion seems to go back to Grothendieck ([27], Theorem 3.3), in somewhat different but equivalent form, and was thoroughly studied in [32]. Note that such P are projective, smooth, and connected, but may lack group laws and rational points. In fact, the group laws on P' correspond to the elements $e' \in P(k')$, by loc. cit. Proposition 4.3.

Let X be an algebraic space over some base scheme S whose structure morphism $X \to S$ is proper, flat, of finite presentation and cohomologically flat in degree $d = 0$. The latter means that $f_*(\mathscr{O}_X)$ is locally free of finite rank, and that its formation commutes with base-change. Then the sheafification of the functor $R \mapsto \operatorname{Pic}(X \otimes R)$ with respect to the fppf topology is representable by an algebraic space $\operatorname{Pic}_{X/S}$, which is locally of finite presentation ([2], Theorem 7.3). Moreover, the subsheaf $\operatorname{Pic}^\tau_{X/S}$ stemming from fiberwise numerically trivial sheaves is representable by an algebraic space that is of finite presentation, and the inclusion is an open embedding ([32], Theorem 2.1).

A *family of para-abelian varieties* is a proper, flat morphism $P \to S$ of finite finite presentation whose fibers are para-abelian varieties. Then the subgroup scheme $G \subset \operatorname{Aut}_{P/S}$ that fixes $\operatorname{Pic}^\tau_{P/S}$ is a family of abelian varieties, its action on P is free and transitive, and we have an identification $\operatorname{Pic}^\tau_{P/S} = \operatorname{Pic}^\tau_{G/S}$ ([32], Section 5). A morphism $f : X \to P$ to a family of para-abelian varieties is called an *Albanese map* if the resulting $f^* : \operatorname{Pic}^\tau_{P/S} \to \operatorname{Pic}^\tau_{X/S}$ is a monomorphism and identifies the abelian varieties $A_s = \operatorname{Pic}^\tau_{P/S} \otimes \kappa(s)$ with the *maximal abelian subvarieties* inside the group schemes $G_s = \operatorname{Pic}^\tau_{X/S} \otimes \kappa(s)$, for all points $s \in S$. If it exists, it is universal for morphism into families of para-abelian varieties. We then set $\operatorname{Alb}_{X/S} = P$ and call it the *family of Albanese varieties*. Over ground fields, the existence is automatic ([32], Corollary 10.5):

Theorem 1.2. *If $S = \operatorname{Spec}(k)$ is the spectrum of a field, then every proper algebraic space X with $h^0(\mathscr{O}_X) = 1$ has an Albanese map $X \to \operatorname{Alb}_{X/k}$. Moreover, the formation of the Albanese variety $\operatorname{Alb}_{X/k}$ is functorial in X, equivariant with respect to the action of the group scheme $\operatorname{Aut}_{X/k}$, and commutes with ground field extensions.*

Let $h : P_1 \to P_2$ be a morphism between para-abelian varieties over a field k. Write $G_i \subset \operatorname{Aut}_{P_i/k}$ for the subgroup schemes that fix $\operatorname{Pic}^\tau_{P_i/k}$. Then there is a unique homomorphism $h_* : G_1 \to G_2$ between these abelian varieties that makes h equivariant with respect to the resulting G_1-actions ([32], Proposition 5.4). Let $A_i = \operatorname{Pic}^\tau_{G_i/k}$ be the *dual abelian varieties*, and $h^* : A_2 \to A_1$ be the induced homomorphism. These maps are related as follows:

Proposition 1.3. *In the above situation, the following equivalences holds:*

(i) *h is surjective $\Leftrightarrow$ h_* is surjective $\Leftrightarrow$ h^* is finite.*
(ii) *h is finite $\Leftrightarrow$ h_* is finite $\Leftrightarrow$ h^* is surjective.*
(iii) *h has geometrically connected fibers $\Leftrightarrow$ $\operatorname{Ker}(h_*)^{\mathrm{aff}}$ is local $\Leftrightarrow$ $\operatorname{Ker}(h^*)^{\mathrm{aff}}$ is unipotent.*

Proof. It suffices to treat the case that k is algebraically closed. Then there is a rational point $a_1 \in P_1$ giving identifications $P_i = G_i$. So $h_* = h$, and we thus may start with a homomorphism $h : G_1 \to G_2$ of abelian varieties. Moreover, $\operatorname{Im}(h)$ and $\operatorname{Coker}(h)$ are abelian varieties, and $N = \operatorname{Ker}(h)$ is proper. These are related by three short exact sequences:

$$\begin{gathered} 0 \longrightarrow G \longrightarrow N \longrightarrow N/G \longrightarrow 0, \\ 0 \longrightarrow \operatorname{Im}(h) \longrightarrow G_2 \longrightarrow \operatorname{Coker}(h) \longrightarrow 0, \\ 0 \longrightarrow N \longrightarrow G_1 \longrightarrow \operatorname{Im}(h) \longrightarrow 0. \end{gathered}$$

The kernel G of the affinization map $N \to N^{\mathrm{aff}}$ connected and smooth ([19], Chapter III, 8.2) and here also proper. Thus N is an extension of some finite group scheme N/G by some abelian variety G. Now recall that for any abelian variety B, the dual abelian variety $B^* = \operatorname{Pic}^\tau_{B/k}$ represents the sheaf $\underline{\operatorname{Ext}}^1(B, \mathbb{G}_m)$, as explained in [36], Appendix, and that $\underline{\operatorname{Hom}}(B, \mathbb{G}_m) = 0$. Moreover, for any finite group scheme H, the *Cartier dual* $H^\vee$ represents the sheaf $\underline{\operatorname{Hom}}(H, \mathbb{G}_m)$.

Applying the contravariant functor $\underline{\operatorname{Hom}}(\cdot, \mathbb{G}_m)$ to the above short exact sequences and using Lemma 1.4 below, we obtain identifications $\underline{\operatorname{Ext}}^1(N, \mathbb{G}_m) = G^*$ and $\underline{\operatorname{Hom}}(N, \mathbb{G}_m) = (N/G)^\vee$, together with exact sequences $0 \leftarrow \operatorname{Im}(h)^* \leftarrow G_2^* \leftarrow \operatorname{Coker}(h)^* \leftarrow 0$ and $0 \leftarrow \underline{\operatorname{Ext}}^1(N, \mathbb{G}_m) \leftarrow G_1^* \leftarrow \operatorname{Im}(h)^* \leftarrow \underline{\operatorname{Hom}}(N, \mathbb{G}_m) \leftarrow 0$. In turn, we get a commutative diagram where the sequences with kinks are exact:

$$\begin{tikzcd} 0 & G^* \arrow[l] & G_1^* \arrow[l] & & G_2^* \arrow[ll, dashed, "h^*"'] \arrow[dl] & \operatorname{Coker}(h)^* \arrow[l] & 0 \arrow[l] \\ & & & \operatorname{Im}(h)^* \arrow[ul] \arrow[dl] & & & \\ & & 0 & & (N/G)^\vee \arrow[ul] & 0 \arrow[l] & \end{tikzcd}$$

Using the Snake Lemma, we see that $\mathrm{Ker}(h^*)$ is an extension of the abelian variety $\mathrm{Coker}(h)^*$ by the finite group scheme $(N/G)^\vee$. Consequently h is surjective if and only if $\mathrm{Ker}(h^*)$ is finite, which gives (i). Assertion (ii) follows from biduality ([38], Corollary on page 132). Equivalently, one may argue that h is finite if and only if $G^* = 0$, which means that h^* is surjective.

It remains to check (iii). Clearly, $h : G_1 \to G_2$ has geometrically connected fibers if and only if the finite group scheme $H = N/G$ is connected. Now decompose $H = H_u \oplus H_m$ into its unipotent and multiplicative part, and furthermore

$$H_u = H_u^0 \oplus H_u^e \quad \text{and} \quad H_m = H_m^0 \oplus H_m^e$$

into connected and étale parts. Such decompositions indeed exists since k is algebraically closed ([19], Chapter IV, §3, Theorem 1.1). The Cartier duals of the connected group schemes H_u^0 and H_m^0 are unipotent, whereas the étale parts H_u^e and H_m^e have multiplicative Cartier duals. This gives (iii). $\square$

In the above proof, we have used the following fact:

Lemma 1.4. *For each inclusion of abelian varieties $A \subset B$, the induced map $\underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) \to \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m)$ is surjective. For each finite group scheme H, the sheaf $\underline{\mathrm{Ext}}^1(H, \mathbb{G}_m)$ vanishes.*

Proof. Note that for both statements one may replace the ground field k by any finite field extension. So by Poincare's Complete Reducibility Theorem ([38], Theorem 1 on page 173), we may assume that there is another abelian subvariety A' such that $A \oplus A' \to B$ is surjective, with finite kernel N. Choose an integer $n \geq 1$ that annihilates the group scheme N. In the long exact sequence

$$\ldots \longrightarrow \underline{\mathrm{Ext}}^1(B, \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(A, \mathbb{G}_m) \oplus \underline{\mathrm{Ext}}^1(A', \mathbb{G}_m) \longrightarrow \underline{\mathrm{Ext}}^1(N, \mathbb{G}_m) \longrightarrow \ldots,$$

the term on the right is annihilated by n, whereas the cokernel for the map on the left is an abelian variety. It follows that the map on the right is zero, thus the map on the left is surjective.

Concerning H, we have to show that each $0 \to \mathbb{G}_{m,R} \to E \to H_R \to 0$ over some ring R splits after base-change with respect to an fppf extension $R \subset R'$. According to [19], Chapter IV, §3, Theorem 1.1 there is a short exact sequence $0 \to H' \to H \to H'' \to 0$ where H' is unipotent and H'' is multiplicative. The arguments for [32], Proposition 6.1 show that $\underline{\mathrm{Ext}}^1(H'', \mathbb{G}_m) = 0$. It remains to treat the case that H is unipotent. In characteristic zero the finite group scheme H must be trivial, so we assume $p > 0$. After enlarging the ground field k, we may assume that H admits a composition series whose quotients are isomorphic to $(\mathbb{Z}/p\mathbb{Z})_k$ or α_p, which reduces the problem to these two particular cases. In both cases, the Kummer sequence yields a long exact sequence

$$\ldots \longrightarrow \underline{\mathrm{Ext}}^1(H, \mu_p) \longrightarrow \underline{\mathrm{Ext}}^1(H, \mathbb{G}_m) \xrightarrow{p} \underline{\mathrm{Ext}}^1(H, \mathbb{G}_m) \longrightarrow \ldots,$$

where the map on the right is zero. It thus suffices to verify that each extension $0 \to \mu_{p,R} \to E \to H_R \to 0$ spits over some fppf extension $R \subset R'$. For $H = (\mathbb{Z}/p\mathbb{Z})_k$, this happens if we trivialize the fiber of $E \to H_R$ over the 1-section, which is a $\mu_{p,R}$-torsor. For $H = \alpha_p$, we pass in our extension to Cartier duals, and argue in the same way. $\square$

2. Modifications and regularity

Let k be a ground field of arbitrary characteristic $p \geq 0$, and $f : X \to Y$ be a proper morphism between algebraic spaces that are separated and of finite type. The resulting Leray–Serre spectral sequence is

$$E_2^{rs} = H^r(Y, R^s f_*(\mathscr{O}_X)) \Longrightarrow H^{r+s}(X, \mathscr{O}_X). \tag{1}$$

If $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is bijective, the five-term exact sequence takes the form

$$0 \to H^1(\mathscr{O}_Y) \to H^1(\mathscr{O}_X) \to H^0(Y, R^1 f_*(\mathscr{O}_X)) \to H^2(\mathscr{O}_Y) \to H^2(\mathscr{O}_X). \tag{2}$$

By Zariski's Main Theorem, $\mathscr{O}_Y = f_*(\mathscr{O}_X)$ automatically holds if f is a modification and Y is normal. In this section we are particularly interested in the case that Y is *regular*. In other words, the Krull dimension of $\mathscr{O}_{Y,b}$ coincides with the dimension of the cotangent space $\mathfrak{m}_b/\mathfrak{m}_b^2$, for all points $b \in Y$. But note that Y may fail to be geometrically regular, and it actually may be geometrically non-reduced.

Proposition 2.1. *Suppose that Y is regular, and that for each generic point $\eta \in Y$ the fiber $f^{-1}(\eta)$ contains some $\kappa(\eta)$-valued point. Then the canonical maps $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X)$ are injective for all integers $i \geq 0$.*

Proof. It suffices to treat the case that Y is connected and hence integral. By assumption, there is an integral closed subspace $Z \subset X$ such that the induced map $f : Z \to Y$ is a modification. Let $Z' \to Z$ be a Macaulayfication where Z' admits an ample sheaf. The modification $Z' \to Y$ induces the composite map $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X) \to H^i(Z', \mathscr{O}_{Z'})$. Replacing X by Z', we may assume that $f : X \to Y$ is a modification with X Cohen–Macaulay and having an ample sheaf. Now the assertion follows from the next result. □

Proposition 2.2. *Suppose that Y is regular, and that $f : X \to Y$ is a modification with X Cohen–Macaulay. Then the canonical maps $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X)$ are bijective for all integers $i \geq 0$.*

Proof. Suppose first that f is projective, which is sufficient to establish Proposition 2.1. Let $V \to Y$ be an étale surjection from an affine scheme V. This scheme is regular, and by [31], Corollary 9.11 has only *pseudo-rational singularities*. In our setting this means that for each projective modification $g : U \to V$ with U Cohen–Macaulay, the canonical map $g_*(\omega_U) \to \omega_V$ is bijective. Here ω_U and ω_V are the coherent sheaves that yield, with appropriate shift, the dualizing complexes. Such morphisms g are called *pseudo-rational modifications*; according to loc. cit. Theorem 8.6 we have $R^s g_*(\mathscr{O}_U) = 0$ for all $s \geq 1$. This applies in particular for the base-change $U = X_V$. It follows that the higher direct images $R^s f_*(\mathscr{O}_X)$ vanish for all $s \geq 1$. In turn, the Leray–Serre spectral sequence (1) collapses, and we get the desired identification $H^i(Y, \mathscr{O}_Y) = H^i(X, \mathscr{O}_X)$. Note that Chatzistamatiou and Rülling [12] showed earlier that the higher direct images $R^s f_*(\mathscr{O}_X)$ vanish provided that Y and X are excellent regular schemes.

Suppose now that f is merely proper. Choose a Macaulayfication $X' \to X$ such that X' admits an ample sheaf. The maps in $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X) \to$

$H^i(X', \mathscr{O}_{X'})$ are injective, in light of Proposition 2.1, and the composition is bijective, by the previous paragraph. In turn, the inclusion $H^i(Y, \mathscr{O}_Y) \subset H^i(X, \mathscr{O}_X)$ must be an equality. □

In degree $i \leq 2$ we can weaken the assumptions further:

Proposition 2.3. *Suppose that Y is regular, and that $f : X \to Y$ is a modification. Then the canonical maps $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X)$ are bijective for $i = 1$, and injective for $i = 2$.*

Proof. Let $g : X' \to X$ be some Macaulayfication such that X' admits an ample sheaf. The Leray–Serre spectral sequence for the composite map gives an inclusion $R^1 f_*(\mathscr{O}_X) \subset R^1(f \circ g)_*(\mathscr{O}_{X'})$. We saw in the previous proof that the term on the right vanishes, thus $R^1 f_*(\mathscr{O}_X) = 0$. Now the exact sequence (2) gives the assertion. □

If Y is proper, the part of the Picard scheme that comprises the numerically trivial sheaves is called $\mathrm{Pic}^\tau_{Y/k}$. It contains the connected component $\mathrm{Pic}^0_{Y/k}$ of the origin as a subgroup scheme of finite index. Under suitable assumptions, these parts change only change little under modifications:

Proposition 2.4. *Suppose that Y is proper. Let $f : X \to Y$ be a proper morphisms with $\mathscr{O}_Y = f_*(\mathscr{O}_X)$ and $H^0(Y, R^1 f_*(\mathscr{O}_X)) = 0$. Then $f^* : \mathrm{Pic}^\tau_{Y/k} \to \mathrm{Pic}^\tau_{X/k}$ is a monomorphism, with finite cokernel.*

Proof. The formation of $f_*(\mathscr{O}_X)$ commutes with flat base-change. It follows that for each k-algebra R, the map $\mathscr{O}_{Y \otimes R} \to (f \otimes \mathrm{id}_R)_*(\mathscr{O}_{X \otimes R})$ is bijective as well, and the same holds for the multiplicative sheaf of units. In turn, the map on Picard groups $H^1(Y \otimes R, \mathscr{O}^\times_{Y \otimes R}) \to H^1(X \otimes R, \mathscr{O}^\times_{X \otimes R})$ is injective. Consequently, the map $f^* : \mathrm{Pic}_{Y/k} \to \mathrm{Pic}_{X/k}$ between sheafifications is a monomorphism.

Set $H = \mathrm{Pic}^0_{Y/k}$ and $G = \mathrm{Pic}^0_{X/k}$. We have an inclusion $H \subset G$, and our task is to show that $\dim(G) = \dim(H)$. The Lie algebras $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{g} = \mathrm{Lie}(G)$ are given by the cohomology groups $H^1(Y, \mathscr{O}_Y)$ and $H^1(X, \mathscr{O}_X)$, respectively. By (2), the morphism $f : X \to Y$ induces a bijection between these Lie algebras. In characteristic zero, we then have $h^1(\mathscr{O}_Y) = \dim(H)$, and likewise for X, so the result follows.

Suppose $p > 0$. Here we need additional arguments, which rely on Mumford's theory of Bockstein operations ([37], Lecture 27). First recall that the canonical map

$$H^i(Y, \mathscr{O}_Y) \longrightarrow H^i(X, \mathscr{O}_X) \tag{3}$$

is bijective for $i = 1$ and injective for $i = 2$, by the five-term sequence (2). To proceed, we replace k by its algebraic closure. What we have achieved is that the reduced part $G_{\mathrm{red}} \subset G$ is a subgroup schemes, which must be smooth, and $\dim(G)$ coincides with the vector space dimension of $\mathrm{Lie}(G_{\mathrm{red}})$.

Write $W_n = W_n(\mathscr{O}_Y)$ for the sheaf of *Witt vectors* of length n. This sheaf of rings comes with an additive map $V : W_n \to W_n$ called *Verschiebung*. The image of its m-fold iteration is denoted by $V^m_n \subset W_n$. As explained in [23], Section 2 the combination of the short exact sequences $0 \to V^r_{r+1} \to W_{r+1} \to W_r \to 0$ and

$0 \to V_r^1 \to W_r \to \mathscr{O}_Y \to 0$ yields $W_r(k)$-linear maps

$$\mathrm{Im}(H^i(W_r) \overset{\mathrm{can}}{\to} H^i(\mathscr{O}_Y)) \xrightarrow{\beta_r} \mathrm{Coker}(H^i(V_r^1) \overset{\mathrm{can}}{\to} H^{i+1}(V_{r+1}^r)), \tag{4}$$

where the image on the left if formed with respect to the surjection $W_r \to W_1 = \mathscr{O}_Y$, and the cokernel on the right comes from a canonical composite map. The above β_r are called *Bockstein operators*. The kernel of β_r comprises those cohomology classes in $H^i(Y, \mathscr{O}_Y)$ that extend to $H^i(Y, W_{r+1})$, and we write $H^i(Y, \mathscr{O}_Y)[\beta]$ for their intersection. This vector subspace has an important geometric meaning: According to [37], Theorem on page 196 we have $\mathrm{Lie}(G_{\mathrm{red}}) = H^1(Y, \mathscr{O}_Y)[\beta]$. It can also be seen as the intersection of the images for $H^1(Y, W_r) \to H^1(Y, \mathscr{O}_Y)$, $r \geq 1$.

Our remaining task is to verify that the latter images in the cohomology groups $H^1(Y, \mathscr{O}_Y) = H^1(X, \mathscr{O}_X)$ are the same, whether computed on Y or X. For this, it suffices to check that the canonical maps $H^1(Y, W_r) \to H^1(X, W_r)$, $r \geq 1$ are bijective. We proceed by induction on $r \geq 1$. The case $r = 1$ is trivial. Suppose now $r \geq 2$, and that the assertion holds for $r-1$. Consider the short exact sequence $0 \to V_r^{r-1} \to W_r \to W_{r-1} \to 0$ on Y. First note that $H^0(Y, W_r) \to H^0(Y, W_{r-1})$ is surjective, because the map of set-valued sheaves $W_r \to W_{r-1}$ admits a section, and the same holds on X. In turn, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1(Y, V_r^{r-1}) \arrow[r] \arrow[d] & H^1(Y, W_r) \arrow[r] \arrow[d] & H^1(Y, W_{r-1}) \arrow[r] \arrow[d] & H^2(Y, V_r^{r-1}) \arrow[d] \\
0 \arrow[r] & H^1(X, V_r^{r-1}) \arrow[r] & H^1(X, W_r) \arrow[r] & H^1(X, W_{r-1}) \arrow[r] & H^2(X, V_r^{r-1})
\end{tikzcd}$$

with exact rows. The Verschiebung $V : W_n \to W_n$ is additive, and its $(r-1)$-fold composition induces identifications $\mathscr{O} = W_r/V_r^1 \to V_r^{r-1}/V_r^r = V_r^{r-1}$ on both Y and X. Consequently, the vertical map on the left is bijective, and the vertical map on the right is injective. By induction, $H^1(Y, W_{r-1}) \to H^1(X, W_{r-1})$ is bijective, and we conclude with the Five Lemma ([35], Chapter I, Proposition 21.1). $\square$

We now combine our observations:

Theorem 2.5. *Suppose that Y is proper and regular, and $f : X \to Y$ is a modification. Then the induced homomorphism $f^* : \mathrm{Pic}^\tau_{Y/k} \to \mathrm{Pic}^\tau_{X/k}$ is a monomorphism, the induced map on Lie algebras is bijective, and the cokernel is finite.*

Proof. By Zariski's Main Theorem, the canonical map $\mathscr{O}_Y \to f_*(\mathscr{O}_X)$ is bijective. In light of the 5-term exact sequence, the condition $H^0(Y, R^1 f_*(\mathscr{O}_X)) = 0$ is equivalent to the condition that the canonical maps $H^i(Y, \mathscr{O}_Y) \to H^i(X, \mathscr{O}_X)$ are bijective for $i = 1$, and injective for $i = 2$. This indeed holds by Proposition 2.3. In particular, $\mathrm{Lie}(f^*)$ is bijective, and the assertion now follows from Proposition 2.4. $\square$

In many situations, f^* is actually an isomorphism, but I do not know if this holds in general.

3. The abelian part of the Picard scheme

Let k be a ground field of characteristic $p \geq 0$. As explained in [32], Section 7 any group scheme G of finite type contains a *maximal abelian subvariety* $A \subset G$, which is functorial in G and compatible with ground field extensions $k \subset k'$. Let

Y be a proper algebraic space. Recall that $\mathrm{Pic}^\tau_{Y/k}$ is a group scheme of finite type. The following analogous notation seems useful:

Definition 3.1. We write $\mathrm{Pic}^\alpha_{Y/k}$ for the maximal abelian subvariety inside $\mathrm{Pic}^\tau_{Y/k}$, and call it the *abelian part* of the Picard scheme.

Each proper morphism $f : X \to Y$ induces a homomorphism $f^* : \mathrm{Pic}^\alpha_{Y/k} \to \mathrm{Pic}^\alpha_{X/k}$ of abelian varieties. We need the following:

Proposition 3.2. *Suppose Y is regular and that $f : X \to Y$ is a modification. Then the homomorphism $f^* : \mathrm{Pic}^\alpha_{Y/k} \to \mathrm{Pic}^\alpha_{X/k}$ of abelian varieties is an isomorphism.*

Proof. According to Proposition 2.5 we may regard $P' = \mathrm{Pic}^\tau_{Y/k}$ as a subgroup scheme of $P = \mathrm{Pic}^\tau_{X/k}$, giving an inclusion $\mathrm{Pic}^\alpha_{Y/k} \subset \mathrm{Pic}^\alpha_{X/k}$. Moreover, the quotient P/P' is finite. Set $A = \mathrm{Pic}^\alpha_{X/k}$ and $A' = A \cap P'$. The subgroup scheme $A/A' \subset P/P'$ is finite. Being the quotient of an abelian variety, it has $h^0(\mathscr{O}_{A/A'}) = 1$. Thus A/A' is trivial, hence $A \subset P'$. The maximality of $\mathrm{Pic}^\alpha_{Y/k}$ yields $\mathrm{Pic}^\tau_{X/k} = A \subset \mathrm{Pic}^\alpha_{Y/k}$ inside P', and the assertion follows. $\square$

The main observation in this section is the following boundedness result on the abelian part for modifications:

Proposition 3.3. *There are constants $d \geq 0$ and $l \geq 1$ depending on our proper algebraic space Y such that for every modification $f : X \to Y$, the following holds:*

(i) *The abelian variety $\mathrm{Pic}^\alpha_{X/k}$ has dimension $\leq d$.*
(ii) *The kernel for the induced map $f^* : \mathrm{Pic}^\alpha_{Y/k} \to \mathrm{Pic}^\alpha_{X/k}$ has order $\leq l$.*

Proof. The idea is to reduce the problem to the case that Y is regular. Let Y' be an alteration of Y_{red} such that the Y' is regular, and let X' be the reduction for $X \times_Y Y'$. This gives a commutative diagram

$$\begin{array}{ccccc} X & \longleftarrow & Y_{\mathrm{red}} & \longleftarrow & X' \\ {\scriptstyle f}\downarrow & & \downarrow & & \downarrow \\ Y & \longleftarrow & Y_{\mathrm{red}} & \longleftarrow & Y' \end{array}$$

where the vertical maps are modifications. According to Proposition 3.2, the induced map $\mathrm{Pic}^\alpha_{Y'/k} \to \mathrm{Pic}^\alpha_{X'/k}$ is an isomorphism. Since $X' \to X$ is surjective, the kernel K for $\mathrm{Pic}_{X/k} \to \mathrm{Pic}_{X'/k}$ is affine, according to [6], Exposé XII, Corollary 1.5. So the same holds for the induced map $A \to A'$ on maximal abelian subvarieties. Its kernel $K \cap A$ is proper and affine, hence finite. Consequently $\dim(A) \leq \dim(A')$. Thus the dimension $d \geq 0$ for the abelian parts $\mathrm{Pic}^\alpha_{X'/k} = \mathrm{Pic}^\alpha_{Y'/k}$ is the desired bound.

Now set $B = \mathrm{Pic}^\alpha_{Y/k}$ and $B' = \mathrm{Pic}^\alpha_{Y'/k}$. As in the preceding paragraph, the kernel N for the induced map $B \to B'$ is finite. In light of the above commutative diagram and the identification $\mathrm{Pic}^\alpha_{Y'/k} = \mathrm{Pic}^\alpha_{X'/k}$, we see that N contains the kernel for $\mathrm{Pic}^\alpha_{Y/k} \to \mathrm{Pic}^\alpha_{X/k}$. Thus $l = |N|$ is the desired bound for the kernel orders. $\square$

4. Ind-objects of abelian varieties

Throughout this section $\mathcal{C}$ denotes some category. Let us briefly discuss the notation of *ind-objects*, which was introduced by Grothendieck ([4], Exposé 1, Section 8, compare also [28], Section 6.1). These are nothing but covariant functors $A : L \to \mathcal{C}$ defined on some filtered category L. Recall that if the category L is just an ordered set, *filtered* means that for all $\lambda, \lambda' \in L$ there is some $\mu \in L$ with $\lambda, \lambda' \leq \mu$. For simplicity one often writes ind-object as $(A_\lambda)_{\lambda \in L}$, and calls L the *index category*. The morphisms $t_{\lambda\mu} : A_\lambda \to A_\mu$ for $\lambda \to \mu$ are usually called *transition maps*.

Each ind object defines a presheaf "$\varinjlim$"A_λ on the category $\mathcal{C}$, via the formula $\mathrm{Hom}(X, \text{"}\varinjlim\text{"} A_\lambda) = \varinjlim \mathrm{Hom}(X, A_\lambda)$. Hence a morphism "$\varinjlim$"$A_\lambda \to$ "$\varinjlim$"B_μ is a natural transformation

$$\varinjlim_\lambda \mathrm{Hom}(X, A_\lambda) \xrightarrow{\Phi_X} \varinjlim_\mu \mathrm{Hom}(X, B_\mu).$$

By the universal property of direct limits and the Yoneda Lemma, this is a compatible collection of morphisms $f_\mu : \varinjlim_\mu \mathrm{Hom}(A_\lambda, B_\mu)$. This shows

$$\mathrm{Hom}(\text{"}\varinjlim_\lambda\text{"} A_\lambda, \text{"}\varinjlim_\mu\text{"} B_\mu) = \varprojlim_\lambda \varinjlim_\mu \mathrm{Hom}(A_\lambda, B_\mu). \tag{5}$$

The collection of all ind-object, together with the above Hom sets, form the category $\mathrm{Ind}(\mathcal{C})$. Each $B \in \mathcal{C}$ can be seen as an ind-object, with a singleton as index category. By the above a morphism $(A_\lambda)_{\lambda \in L} \to B$ is a compatible collection of morphisms $f_\lambda : A_\lambda \to B$. The morphism is an isomorphism if for some index λ_0, there is a morphism $g : B \to A_{\lambda_0}$ so that for all arrows $\lambda_0 \to \lambda$, the composition

$$B \xrightarrow{g} A_{\lambda_0} \xrightarrow{t_{\lambda_0,\lambda}} A_\lambda \xrightarrow{f_\lambda} B$$

coincides with the identity map for B. An ind-object $(A_\lambda)_{\lambda \in L}$ is called *constant* if all transition maps are isomorphisms. More generally, it is called *essentially constant* if there is an index λ_0 such that for all $\lambda_0 \to \lambda$ the transition maps $t_{\lambda_0,\lambda} : A_{\lambda_0} \to A_\lambda$ are isomorphisms. By abuse of notation we say that $B = A_{\lambda_0}$ is the *essential value*. Choosing for each λ' some diagram $\lambda' \to \lambda \leftarrow \lambda_0$, the compositions $f_\lambda = t_{\lambda_0,\lambda}^{-1} \circ t_{\lambda',\lambda_0}$ define compatible morphisms $f_\lambda : A_\lambda \to B$ that do not depend on the choices, and yield an isomorphism $f :$ "$\varinjlim$"$A_\lambda \to B$. Thus each essentially constant ind-object belongs to the essential image of the fully faithful inclusion $\mathcal{C} \subset \mathrm{Ind}(\mathcal{C})$.

We need the following criterion for abelian varieties:

Lemma 4.1. *Let $(A_\lambda)_{\lambda \in L}$ be an ind-object of abelian varieties over some ground field k. Suppose for each index λ, there are constants $d \geq 0$ and $l \geq 1$ such that for each arrow $\lambda \to \mu$ we have*

$$\dim(A_\mu) \leq d \quad \textit{and} \quad |\mathrm{Ker}(A_\lambda \to A_\mu)| \leq l.$$

Then the ind-object $(A_\lambda)_{\lambda \in L}$ is essentially constant.

Proof. We may replace L by any cofinal subcategory. It thus suffices to treat the case that the filtered category L is just a directed ordered set, having some smallest element λ. Since the dimensions of the A_μ are bounded, there is some $\lambda' \in L$ where $\dim(A_{\lambda'})$ takes the largest value. Replacing λ by λ', we may assume that this already happens for A_λ. Given $\lambda \leq \mu$, the transition map $A_\lambda \to A_\mu$ has finite kernel, and hence $\dim(A_\lambda) = \dim(A_\mu)$. Thus the dimensions are constant. In turn, the transition maps $A_\lambda \to A_\mu$ are surjective.

Set $A = A_\lambda$, and consider the kernels $K_\mu = \mathrm{Ker}(A \to A_\mu)$. The Isomorphism Theorem gives $A_\mu = A/K_\mu$. The orders $l_\mu = |K_\mu|$ satisfy $l_\mu \leq l_\eta$ whenever $\mu \leq \eta$. On the other hand we have $l_\mu \leq l$. Passing to some cofinal subset, we may assume that the orders l_μ are constant for all $\lambda < \mu$, so the inclusions $K_\mu \subset K_\eta$ are equalities. Hence the transition maps $A_\mu = A/K_\mu \to A/K_\eta = A_\eta$ are isomorphisms. $\square$

5. Compactifications and Albanese maps

Let k be a ground field of arbitrary characteristic $p \geq 0$, and U be an algebraic space that is separated and of finite type. We also assume that the ring of global sections $H^0(U, \mathscr{O}_U)$ is indecomposable and reduced. Equivalently, the affine hull $U^{\mathrm{aff}} = \mathrm{Spec}\, H^0(U, \mathscr{O}_U)$ is connected and reduced. Recall that if U is proper with $h^0(\mathscr{O}_U) = 1$, there is a universal map to a para-abelian variety ([32], Corollary 10.5). The ultimate goal of this section is to remove the properness assumption.

Recall that a *compactification* is a pair (X, i) where X is a proper algebraic space, and $i : U \to X$ is an open embedding such that X is the smallest closed subspace over which i factors. If X is a scheme, this means that $U \subset X$ is dense and contains the finite set $\mathrm{Ass}(\mathscr{O}_X)$. One also says that $U \subset X$ is *schematically dense*. We will apply the same locution for algebraic spaces.

The compactifications of U form a non-empty category $\mathrm{Cpt}(U)$, where an arrow $(X, i) \to (Y, j)$ is a morphism $f : X \to Y$ with $f \circ i = j$. We then say that X *dominates* Y. The schematic density of U ensures that the Hom sets are empty or singletons, in other words, the category $\mathrm{Cpt}(U)$ is equivalent to an ordered set.

Given compactifications (X_1, i_1) and (X_2, i_2), the smallest closed subspace over which the diagonal $(i_1, i_2) : U \to X_1 \times X_2$ factors defines another compactification, and we see that the opposite category $\mathrm{Cpt}(V)^{\mathrm{op}}$ is filtered. For each cofinal $L \subset \mathrm{Cpt}(V)^{\mathrm{op}}$, we thus get ind-objects

$$(H^j(X_\lambda, \mathscr{O}_{X_\lambda}))_{\lambda \in L} \quad \text{and} \quad (\mathrm{Pic}^\tau_{X_\lambda/k})_{\lambda \in L} \quad \text{and} \quad (\mathrm{Pic}^\alpha_{X_\lambda/k})_{\lambda \in L}$$

taking respective values in finite-dimensional vector spaces, group schemes of finite type, and abelian varieties. These ind-objects should be seen as invariants of interest for the algebraic space U. Before we proceed we have to address the problem of so-called *constant field extensions.*

Lemma 5.1. *For each compactification (Y, j) the finite k-algebra $\Gamma(Y, \mathscr{O}_Y)$ is a field. Moreover, the ind-object $(\Gamma(X_\lambda, \mathscr{O}_{X_\lambda}))_{\lambda \in L}$ of fields is essentially constant.*

Proof. The morphism $j : U \to Y$ induces an inclusion $\Gamma(Y, \mathscr{O}_Y) \subset \Gamma(U, \mathscr{O}_U)$. Since Y is proper, the k-algebra $F = \Gamma(Y, \mathscr{O}_Y)$ is finite. Hence each point in $\mathrm{Spec}(F)$ is generic, so the dominant map $Y \to \mathrm{Spec}(F)$ is surjective. Since the ring $\Gamma(U, \mathscr{O}_U)$ is reduced, the same holds for the subring F. Using that U is connected, and also dense in Y, we infer that Y and hence its image $\mathrm{Spec}(F)$ is connected. This proves the first assertion.

Before we come to the second assertion, we make a little observation: Let $Y' \to Y$ be the normalization of Y_{red}. Consider the finite k-algebra $R = \Gamma(Y', \mathscr{O}_{Y'})$, which is a finite product of finite field extensions of k. Let $f : (X, i) \to (Y, j)$ be a morphism from another compactification, and set $X' = (Y' \times_Y X)_{\mathrm{red}}$. We obtain a commutative diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \Big\downarrow & & \Big\downarrow f \\ Y' & \longrightarrow & Y. \end{array}$$

The first projection $X' \to Y'$ is a modification of the normal algebraic space Y', and Zariski's Main theorem implies that $H^0(Y', \mathscr{O}_{Y'}) = H^0(X', \mathscr{O}_{X'})$. The second projection $X' \to X$ induces a homomorphism $H^0(X, \mathscr{O}_X) \to H^0(X', \mathscr{O}_{X'})$, which must be injective because $F = H^0(X, \mathscr{O}_X)$ is a field. It follows that $h^0(\mathscr{O}_X) \leq h^0(\mathscr{O}_{Y'})$. Summing up, the integers $h^0(\mathscr{O}_X)$ are bounded above by some number that depends only on Y.

This easily gives the second assertion: By passing to a cofinal set, we may assume that L has a smallest member (Y, j). According to the preceding paragraph, the numbers $h^0(\mathscr{O}_{X_\lambda})$ are bounded. It follows that the ind-objects $(\Gamma(X_\lambda, \mathscr{O}_{X_\lambda}))_{\lambda \in L}$ of fields is essentially constant. $\square$

Let us call $k' = \text{“}\varinjlim\text{”} H^0(X_\lambda, \mathscr{O}_{X_\lambda})$ *the essential field of constants* for the algebraic space U. The morphism $U \subset X_\lambda \to (X_\lambda)^{\text{aff}} = \operatorname{Spec}(k')$, with sufficiently large index λ, endows the algebraic space U over k with a canonical k'-structure. After replacing k by k', the ground field and the essential field of constants coincide. In what follows, we usually make this assumption, because the theory of Albanese maps for proper algebraic spaces was developed in [32] under this condition.

Proposition 5.2. *Suppose the ground field k equals the essential field of constants for U. Then the ind-object $(\operatorname{Pic}^\alpha_{X_\lambda/k})_{\lambda \in L}$ of abelian varieties is essentially constant.*

Proof. By passing to a cofinal subset, we may assume that L contains a smallest member (Y, i). Combining Proposition 3.3 and Lemma 4.1, we see that our ind-object is essentially constant. $\square$

Corollary 5.3. *Suppose the ground field k equals the essential field of constants for U. Then there is an index $\lambda \in L$ such that for all $\mu \geq \lambda$ the transition map $f : X_\mu \to X_\lambda$ induces an isomorphism $f_* : \operatorname{Alb}_{X_\mu/k} \to \operatorname{Alb}_{X_\lambda/k}$ of para-abelian varieties.*

Proof. Passing to a cofinal index set, we may assume that the homomorphisms $f^* : \operatorname{Pic}^\alpha_{X_\lambda/k} \to \operatorname{Pic}^\alpha_{X_\mu/k}$ are isomorphisms, for all $\mu \geq \lambda$ in L. Let $P_\lambda = \operatorname{Alb}_{X_\lambda/k}$ be the Albanese varieties, and $g_\lambda : X_\lambda \to P_\lambda$ be the Albanese maps. By our definition of Albanese maps ([32], Section 8), the induced map $g_\lambda^* : \operatorname{Pic}^\alpha_{P_\lambda/k} \to \operatorname{Pic}^\alpha_{X_\lambda/k}$ of abelian varieties is an isomorphism. It follows that $f_* : P_\mu \to P_\lambda$ induces an isomorphism $\operatorname{Pic}^\tau_{P_\lambda/k} \to \operatorname{Pic}^\tau_{P_\mu/k}$. According to Proposition 1.3, the former is an isomorphism as well. $\square$

We now come to the main result of the paper:

Theorem 5.4. *Suppose the ground field k equals the essential field of constants for U. Then there is a para-abelian variety P and a morphism $f : U \to P$ such that for each other para-abelian variety Q with a morphism $g : U \to Q$, there is a unique morphism $h : P \to Q$ such that $g = h \circ f$.*

Proof. Choose a cofinal index set $L \subset \mathrm{Cpt}(V)^{\mathrm{op}}$ such that $f_* : \mathrm{Alb}_{X_\mu/k} \to \mathrm{Alb}_{X_\lambda/k}$ are isomorphisms for all transition maps $f : X_\mu \to X_\lambda$, and that there is a smallest member (Y, j). Let $P = \mathrm{Alb}_{Y/k}$ be the Albanese variety of the proper algebraic space Y. We obtain a morphism $f : U \to P$ as the composition of the Albanese map $a : Y \to P$ with the inclusion $j : U \to Y$.

We have to verify the universal property. Let $g : U \to Q$ be a morphism to some other para-abelian variety. It can be seen as a rational map $g : Y \dashrightarrow Q$. Taking the closure of the graph $\Gamma_g \subset Y \times Q$ we obtain another compactification (X, i). The projection $\mathrm{pr} : X \to Y$ is a morphism of compactifications, and the rational map $g : Y \dashrightarrow Q$ extends to a morphism $\tilde{g} : X \to Q$. We have the following commutative diagram:

$$\begin{tikzcd}
U \arrow[r, "j"] \arrow[rr, bend left, "i"] \arrow[ddr, "g"'] & Y \arrow[d, "a"] & X \arrow[l, "\mathrm{pr}"'] \arrow[ddl, "\tilde{g}"] \\
& P \arrow[d, dashed, "h"] & \\
& Q &
\end{tikzcd}$$

By construction, $a \circ \mathrm{pr} : X \to P$ is the Albanese map for the proper algebraic space X, hence there is a unique morphism $h : P \to Q$ with $h \circ a \circ \mathrm{pr} = \tilde{g}$. We then also have $h \circ a \circ j = h \circ a \circ \mathrm{pr} \circ i = \tilde{g} \circ i = g$, so the whole diagram is commutative.

It remains to verify uniqueness: Suppose there is another morphism $h' : P \to Q$ with $h' \circ a \circ j = g$. Again the whole diagram is commutative, and $h \circ a \circ j = h' \circ a \circ j$. The compactification $j : U \to Y$ is an epimorphism by schematic density. To see this, choose an étale surjection $\tilde{U} \to U$ from some scheme U', and apply [41], Lemma 2.1.1 to the resulting morphisms of schemes $\tilde{U} \to Y$. It follows $h \circ a = h' \circ a$, and in particular $h \circ a \circ \mathrm{pr} = h' \circ a \circ \mathrm{pr}$. Now recall that $a \circ \mathrm{pr} : X \to P$ is the Albanese map for X. Its universal property ensures $h = h'$. ☐

By the Yoneda Lemma, the pair (P, f) is unique up to unique isomorphism. We then write $P = \mathrm{Alb}_{U/k}$ and call it the *Albanese variety* of the algebraic space U. Moreover, the morphism $f : X \to \mathrm{Alb}_{U/k}$ is called the *Albanese map*. Note that if U is already proper, the category $\mathrm{Cpt}(U)$ is equivalent to a singleton, hence our new construction for algebraic spaces that are separated and of finite type coincides with the old construction for proper algebraic spaces.

Proposition 5.5. *Let $g : U' \to U$ be a morphism between algebraic spaces that are separated and of finite type. Suppose the ground field k coincides with the essential field of constants for both U and U', and that their affine hulls are connected and reduced. Then there is a unique morphism $g_* : \mathrm{Alb}_{U/k} \to \mathrm{Alb}_{U/k}$ making the diagram*

$$\begin{tikzcd}
U' \arrow[r, "g"] \arrow[d, "f'"'] & U \arrow[d, "f"] \\
\mathrm{Alb}_{U/k} \arrow[r, "g_*"'] & \mathrm{Alb}_{U/k}
\end{tikzcd}$$

commutative, where the vertical arrows are the Albanese maps.

Proof. This is an immediate consequence of the universal property for Albanese maps. Note that the assumptions are made to ensure the existence of the Albanese varieties for U and U'. $\square$

In particular, the action of the automorphism group $\mathrm{Aut}(U)$ induces an action on $\mathrm{Alb}_{U/k}$ such that the Albanese map is equivariant. It would be interesting to understand to what extend this holds true for group scheme actions. Let us close the section with the following observation:

Proposition 5.6. *Suppose that U is connected and reduced, and that all its irreducible components are geometrically integral. Then the ground field k coincides with the essential field of constants for U.*

Proof. Choose a sequence of irreducible components $U_i \subset U$, $1 \leq i \leq r$ so that the successive intersections $U_i \cap U_{i+1}$ are non-empty, and $U = U_1 \cup \ldots \cup U_r$, with repetitions allowed. Then the base-changes $U_i \otimes k^{\mathrm{alg}}$ remain integral, and if follows that U is geometrically connected and geometrically reduced. So the ring $R = H^0(U, \mathscr{O}_U)$ is geometrically indecomposable and geometrically reduced.

We proceed by showing that $k \subset R$ is integrally closed. Suppose there is some intermediate field $k \subset k' \subset R$. Then $R \otimes k^{\mathrm{alg}}$ contains $k' \otimes k^{\mathrm{alg}}$. If $[k' : k] > 1$, the ring $k' \otimes k^{\mathrm{alg}}$ contains idempotent elements $e \neq 0, 1$ or nilpotent elements $f \neq 0$, so the same holds for the over-ring $R \otimes k^{\mathrm{alg}}$, contradiction. Thus k is integrally closed in R, and thus must coincide with the essential ground field for U. $\square$

6. Behavior under base change

Let k_0 be a ground field of characteristic $p \geq 0$, and U_0 be an algebraic space that is separated and of finite type. Given a field extension $k_0 \subset k$, we consider the base-change $U = U_0 \otimes k$. Suppose that U^{aff} is reduced and connected, and that k is the essential field of constants for U. Then the analogous statement holds for U_0, and we have Albanese maps

$$f_0 : U_0 \longrightarrow \mathrm{Alb}_{U_0/k_0} \quad \text{and} \quad f : U \longrightarrow \mathrm{Alb}_{U/k},$$

and also the base-change of the Albanese map $f_{0,k} : U \to \mathrm{Alb}_{U_0/k_0} \otimes k$. The universal property of f gives a *comparison map*

$$c : \mathrm{Alb}_{U/k} \longrightarrow \mathrm{Alb}_{U_0/k_0} \otimes k, \tag{6}$$

such that $c \circ f = f_{0,k}$. This is an isomorphism, provided $U_0 = X_0$ is proper, according to [32], Corollary 10.5. In general, the situation is more complicated, because the base-change functor $\mathrm{Cpt}(U_0) \to \mathrm{Cpt}(U)$ usually is not an equivalence of categories. In fact, we shall see in the next section examples of algebraic curves over imperfect fields where the comparison map fails to be an isomorphism.

In this section we want to establish a positive result. Recall that the field extension $k_0 \subset k$ is called *separable* if for each reduced k_0-algebra A_0, the base-change $A = A_0 \otimes k$ remains reduced.

Theorem 6.1. *In the above setting, the comparison map* (6) *is a finite universal homeomorphism. It is actually an isomorphism provided the field extension $k_0 \subset k$ is separable.*

Before entering the proof, let us simplify notation and examine the assertions from various angles. Set

$$P_0 = \mathrm{Alb}_{U_0/k} \quad \text{and} \quad P = \mathrm{Alb}_{U/k} \quad \text{and} \quad P_{0,k} = \mathrm{Alb}_{U_0/k_0} \otimes k.$$

So our Albanese maps are $f_0 : U_0 \to P_0$ and $f : U \to P$, and the comparison map becomes $c : P \to P_{0,k}$. Let $G \subset \mathrm{Aut}_{P/k}$ be the subgroup scheme that fixes $\mathrm{Pic}^\tau_{P/k}$. Then G is an abelian variety, its action on P is free and transitive, and we have an identification $\mathrm{Pic}^\tau_{G/k} = \mathrm{Pic}^\tau_{P/k}$, according to [32], Section 5. Similarly, we form $G_0 \subset \mathrm{Pic}_{P_0/k_0}$ and its base-change $G_{0,k} = G_0 \otimes k$. Then there is a unique homomorphism $c_* : G \to G_{0,k}$ making the comparison map $c : P \to P_{0,k}$ equivariant, by loc. cit. Proposition 5.4. We see that the assertion holds for c if and only if the corresponding statement holds for c_*. The latter respects the group laws, so the assertion of the theorem means that c_* is surjective and has local kernel. Furthermore, c and c_* induce the same homomorphism

$$\mathrm{Pic}^\tau_{G_0/k_0} \otimes k = \mathrm{Pic}^\tau_{P_0/k_0} \otimes k \xrightarrow{c^*} \mathrm{Pic}^\tau_{P/k} = \mathrm{Pic}^\tau_{G/k}\,.$$

In light of Proposition 1.3, the assertion of the theorem means that c^* is surjective and has unipotent kernel.

Choosing appropriate compactifications $i_0 : U_0 \to X_0$ and $i : U \to X$, we get a commutative diagram

$$\begin{tikzcd} U \arrow[r, "i"] \arrow[dr, "i_{0,k}"'] & X \arrow[r, "\bar f"] \arrow[d, "g"] & P \arrow[d, "c"] \\ & X_{0,k} \arrow[r, "\bar f_{0,k}"'] & P_{0,k} \end{tikzcd} \tag{7}$$

of k-schemes, where $\bar f_0 : X_0 \to P_0$ is the Albanese map for the proper scheme X_0, such that $f_0 = \bar f_0 \circ i_0$, and likewise for $\bar f : X \to P$. We choose the latter so that it dominates the base-change $X_{0,k}$, which yields the vertical map $g : X \to X_{0,k}$ in the middle. The vertical map to the right is the comparison map. By definition of Albanese maps in the proper case, the pull-back maps $\bar f_0^*$ and $\bar f^*$ give identifications $\mathrm{Pic}^\alpha_{X_0/k_0} = \mathrm{Pic}^\alpha_{P_0/k_0}$ and $\mathrm{Pic}^\alpha_{X/k} = \mathrm{Pic}^\alpha_{P/K}$. So the assertion of the theorem means that

$$g^* : \mathrm{Pic}^\alpha_{X_0/k_0} \otimes k \longrightarrow \mathrm{Pic}^\alpha_{X/k} \tag{8}$$

is surjective with unipotent kernel.

Proof of Theorem 6.1. We proceed in six steps. Note that we may replace X_0 by any other compactification $\tilde X_0$ of U_0 that dominates X_0, and simultaneously replace X by some $\tilde X$ that dominates the schematic closure of U inside the fiber product $X \times_{X_0} \tilde X_0$. We call this process *passing to dominating compactifications*, and do this several times throughout to improve the situation.

Step 1: *The comparison map $c : P \to P_{0,k}$ is surjective.* In light of Lemma 1.3, this equivalently means that the kernel of (8) is finite. By construction, $g : X \to X_{0,k}$ is surjective. According to [6], Exposé XII, Corollary 1.5 the induced map on Picard schemes has affine kernel. Its intersection with the maximal abelian subvariety is proper. In turn, $\mathrm{Ker}(g^*)$ is finite.

Step 2: *Reduction to the case that the field extension $k_0 \subset k$ is finitely generated.* Choose an intermediate field $k_0 \subset k_1 \subset k$ that is finitely generated over k_0 and that there is a proper algebraic space X_1 and a para-abelian variety P_1 over k_1, together with a morphisms $i_1 : U_1 \to X_1$ and $\bar{f}_1 : X_1 \to P_1$ inducing the upper row in the diagram (9), where we set $U_1 = U_0 \otimes k_1$. By enlarging k_1 if necessary, we may assume that there are morphisms that make the diagram

$$\begin{tikzcd}
U \arrow[r, "i"] \arrow[d] & X \arrow[r, "\bar{f}"] \arrow[d] & P \arrow[d] \\
U_1 \arrow[r, "i_1"] \arrow[d] & X_1 \arrow[r, "\bar{f}_1"] \arrow[d] & P_1 \arrow[d] \\
U_0 \arrow[r, "i_0"'] & X_0 \arrow[r, "\bar{f}_0"'] & P_0
\end{tikzcd} \tag{9}$$

commutative. According to [32], Proposition 8.2 the morphism $\bar{f}_1 : X_1 \to P_1$ is an Albanese map for the proper k_1-scheme X_1. We claim that the composite map $f_1 : U_1 \to P_1$ is an Albanese map. It suffices to check that each morphism $h_1 : U_1 \to Q_1$ to some para-abelian variety, viewed as a rational map $X_1 \dashrightarrow Q_1$, is defined everywhere. In other words, the schematic closure of the graph Γ_{h_1} inside $X_1 \times Q_1$ remains a graph. By construction, this holds after base-changing along $k_1 \subset k$. Since the formation of schematic closure commutes with flat base-change, we we infer that $X_1 \dashrightarrow Q_1$ is defined everywhere.

Step 3: *The case that the extension $k_0 \subset k$ is finite and separable.* We use the universal property of Albanese maps to deduce that the comparison map (6) is an isomorphism. There is a finite extension $k \subset k'$ such that k' is Galois over both k_0 and k. Thus it suffices to treat the case that $k_0 \subset k$ is finite and Galois. Write $G = \mathrm{Gal}(k/k_0)$ for the Galois group, and let $s : P \to \mathrm{Spec}(k)$ be the structure morphism. Fix some $\sigma \in G$, and consider the commutative diagram

$$\begin{tikzcd}
U_0 \otimes k \arrow[r, "\mathrm{id}_{U_0} \otimes \sigma"] \arrow[d, "f"'] \arrow[dr] & U_0 \otimes k \arrow[d, "f"] \\
P \arrow[r, dashed, "\psi_\sigma"] \arrow[d, "s"'] & P \arrow[d, "s"] \arrow[dl] \\
\mathrm{Spec}(k) \arrow[r, "\mathrm{Spec}(\sigma)"'] & \mathrm{Spec}(k).
\end{tikzcd}$$

We now observe that the upper diagonal arrow $f \circ (\mathrm{id}_{U_0} \otimes \sigma)$ is a k-morphism to the para-abelian variety P, provided that the latter is endowed with the lower diagonal arrow $\mathrm{Spec}(\sigma)^{-1}\circ s$ as new structure morphism. By the universal property of the Albanese map f, there is a unique dashed arrow $\psi_\sigma : P \to P$ making the triangles on the left commutative. It follows that the whole diagram is commutative. The uniqueness of ψ_σ ensures that the map $G \to \mathrm{Aut}_{k_0}(P)$ given by $\sigma \mapsto \psi_\sigma$ respects the group laws, and that the structure morphism $s : U \to \mathrm{Spec}(k)$ is equivariant. Since P is a projective k_0-scheme, the quotient P/G exists as a projective k_0-scheme. We then have a canonical identification $P = (P/G) \otimes_{k_0} k$. In particular, P/G is a para-abelian variety over k_0. By the above commutative diagram, the Albanese map $f : U \to P$ over k descends to a k_0-morphism $U_0 \to P/G$. Arguing as above, one sees that the latter has the universal property of the Albanese map, and infer that the comparison map must be an isomorphism.

Step 4: *Reduction to the cases $k = k_0(t)$ and $k = k_0(\lambda^{1/p})$.* In light of step 2, it suffices to treat the case that $k_0 \subset k$ is finitely generated. Choose a transcendence basis $t_1, \ldots, t_n$. Set $k_1 = k_0(t_1, \ldots, t_n)$ and let k_2 be its relative separable closure in k. The extension $k_2 \subset k$ is an equality in characteristic zero. For $p > 0$, it can be written as the successive adjunction of certain elements $\lambda_1^{1/p}, \ldots, \lambda_m^{1/p}$, according to [8], Chapter V, §7, No. 7, Proposition 13. Using inductions on $n \geq 0$ and $m \geq 0$, together with step 3, we get the desired reduction.

Step 5: *The case $k = k_0(t)$.* This involves the passage to a relative setting. For the sake of exposition, we write F for the field $k = k_0(t)$, and regard it as the function field of the affine line $\mathbb{A}^1_{k_0}$. Choose a localization $k_0[t] \subset R$ by a non-zero polynomial so that X extends to a proper morphism $s : \mathfrak{X} \to \mathrm{Spec}(R)$. Localizing further, we may assume that s is flat, and also cohomologically flat in degree zero. In turn, the the numerically trivial part $\mathrm{Pic}^\tau_{\mathfrak{X}/R}$ exists ([32], Theorem 2.1). Doing another localization, we may assume that P extends to a family of para-abelian varieties $\mathfrak{P} \to \mathrm{Spec}(R)$, and that the morphism $\bar{f} : X \to P$ extends to a morphism $\bar{f}_R : \mathfrak{X} \to \mathfrak{P}$. Localizing further, we may assume that the diagram (7) spreads out to commutative diagram

$$\begin{tikzcd} U_0 \otimes R \arrow[r, "i_R"] \arrow[dr, "i_{0,R}"'] & \mathfrak{X} \arrow[r, "\bar{f}_R"] \arrow[d, "g_R"] & \mathfrak{P} \arrow[d, "c_R"] \\ & X_0 \otimes R \arrow[r, "\bar{f}_{0,R}"'] & P_0 \otimes R, \end{tikzcd} \tag{10}$$

where all tensor products are over k_0. Moreover, we may assume that for each prime ideal $\mathfrak{p} \subset R$, the fiberwise morphisms $U_0 \otimes_{k_0} \kappa(\mathfrak{p}) \to \mathfrak{X} \otimes_R \kappa(\mathfrak{p})$ are compactifications that dominate $U_0 \otimes_{k_0} \kappa(\mathfrak{p}) \to X_0 \otimes_{k_0} \kappa(\mathfrak{p})$. Making a final localization, we can achieve that the cokernel of c_R is a family of abelian varieties, and that the kernel is an extension of a family of finite group schemes by a family of abelian varieties.

Now recall that R is a localization of the polynomial ring. Write $S = \mathrm{Spec}(R)$. Then the closed points $\sigma \in S$ whose residue field $\kappa = \kappa(\sigma)$ is separable over k form a Zariski dense set. For such points, the morphism $c_\kappa : \mathfrak{P} \otimes_R \kappa \to P_0 \otimes \kappa$ are isomorphisms by step 3. Using flatness, we infer that c_R is an isomorphism, and in particular $c = c_F$ is an isomorphism.

Step 6: *The case $k = k_0(\lambda^{1/p})$ in characteristic $p > 0$.* This is the most interesting and most challenging part. Recall that by step 1 we already know that the the comparison map $c : P \to P_{0,k}$ is surjective. It remains to check that it is injective.

First, consider the finitely many codimension-one points $\zeta_1, \ldots, \zeta_r \in X_0$ that do not belong to U_0. Passing to dominating compactifications, we may assume that the local rings $\mathscr{O}_{X_0,\zeta_i}$ and $\mathscr{O}_{X,\zeta_i}$ are *unibranch*. In other words, their henselizations have integral reductions. Here we regard ζ_i as points in both X_0 and X, which indeed have the same underlying topological space. It follows that the modification $g : X \to X_{0,k}$ is a universal homeomorphism over some open set $V_0 \subset X_0$ that contains $U_0 \cup \{\zeta_1, \ldots, \zeta_r\}$. Making a further passage to dominating compactifications, we may assume that $P_0 = \mathrm{Alb}_{U_0/k_0} = \mathrm{Alb}_{V_0/k_0} = \mathrm{Alb}_{X_0/k_0}$.

Next, consider the sheaf of $\mathscr{O}_X$-algebras $\mathscr{A}$ given by $\Gamma(W_0, \mathscr{A}) = \Gamma(W_0 \cap V_0, \mathscr{O}_{X_0})$. This is a coherent sheaf, by [25], Proposition 5.11.1 and coincides with $\mathscr{O}_{X_0}$ on V_0. Now pass to dominating compactifications, where X_0 is replaced by the relative spectrum of $\mathscr{A}$. It then follows that the local rings $R = \mathscr{O}_{X_0,a}$ satisfy Serre's condition (S_2), at each boundary point $a \in X_0 \smallsetminus U_0$. In other words, the local cohomology group $H^i_{\mathfrak{m}}(R)$ vanishes for $i \leq 1$. Thus the restriction map $k = H^0(X_0, \mathscr{O}_{X_0}) \to H^0(V_0, \mathscr{O}_{V_0})$ is bijective.

Now consider the kernels $G_n = G[F^n]$ for the iterated relative Frobenius maps $G \to G^{(p^n)}$ and the resulting universal homeomorphisms $P \to P/G_n$. For sufficiently large $n \geq 0$, the fibers of $g^{-1}(V_{0,k}) \to V_{0,k}$ over R-valued points in $V_{0,k}$ are contained in the corresponding orbits of the translations action of G_n on P. Using that the action is free, we obtain a factorization

$$\begin{tikzcd}
g^{-1}(V_{0,k}) \arrow[r, "f"] \arrow[d] & P \arrow[d] \\
V_{0,k} \arrow[r, dashed, "f'"'] & P',
\end{tikzcd}$$

where we set $P' = P/G_n$. Note that the latter is a torsor with respect to the abelian variety $G' = G/G_n$, and thus a para-abelian variety.

To proceed we use the existence of *Weil restrictions* along $\mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$. Indeed, for each k-scheme Y of finite type, the functor

$$(\mathrm{Aff}/k) \longrightarrow (\mathrm{Set}), \quad R_0 \longmapsto Y(R_0 \otimes_{k_0} k)$$

is representable by a k_0-scheme $\mathrm{Res}_{k/k_0}(Y)$ of finite type. We refer the monograph of Conrad, Gabber and Prasad [15], Appendix A.5 for a comprehensive treatment of Weil restrictions. The functor respects products and closed embeddings, and therefore also group structures, torsor structures, and separability. In particular $\mathrm{Res}_{k/k_0}(P')$ is a torsor with respect to the group scheme $\mathrm{Res}_{k/k_0}(G')$. Moreover, the morphism $f' : V_0 \otimes k \to P'$ corresponds to a morphism $f'_0 : V_0 \to \mathrm{Res}_{k/k_0}(P')$.

By construction, $h^0(\mathscr{O}_{V_0}) = 1$, whence the image of V_0 in the affine hull of the Weil restriction is a rational point. According to Lemma 6.2 below, the kernel of the affinization map of the group scheme $\mathrm{Res}_{k/k_0}(G')$ is an abelian variety. It follows that f'_0 factors over some para-abelian variety inside $\mathrm{Res}_{k/k_0}(P')$. It thus also uniquely factors over a morphism $P_0 = \mathrm{Alb}_{V_0/k_0} \to \mathrm{Res}_{k/k_0}(P')$, by the universal property of Albanese maps. In turn, $g^{-1}(V_{0,k}) \to P'$ factors over some morphism $P_0 \otimes k \to P'$. Passing to some dominating compactifications, we may assume that the composition $X \to P \to P'$ factors over $\mathrm{Alb}_{V_0/k_0} \otimes k$. Summing up, we have a commutative diagram of para-abelian varieties

$$\begin{tikzcd} & P \arrow[dl, "c"'] \arrow[d, "\mathrm{can}"] \\ P_{0,k} \arrow[r] & P'. \end{tikzcd}$$

where the vertical map is bijective. It follows that the comparison map $c : P \to P_{0,k}$ is injective. $\square$

In the above proof we have used the following fact:

Proposition 6.2. *Suppose $p > 0$. Let $k_0 \subsetneqq k$ be a purely inseparable field extension, A be an abelian variety over k, and $G_0 = \mathrm{Res}_{k/k_0}(A)$ the Weil restriction. Then the affinization G_0^{aff} is a smooth connected unipotent group scheme of dimension $n \geq 1$, and the kernel of $G_0 \to G_0^{\mathrm{aff}}$ is an abelian variety of dimension $g = \dim(A)$.*

Proof. By descend, it suffices to verify the assertion with $E = G_0 \otimes_{k_0} k$. According to [15], Proposition A.5.11 the canonical homomorphism $f : E = \mathrm{Res}_{k/k_0}(A) \otimes_{k_0} k \to A$ is smooth and surjective, with geometrically connected fibers, and $U = \mathrm{Ker}(f)$ is unipotent and non-zero. It follows that E and whence also its affinization are smooth and connected. Write $h : E \to E^{\mathrm{aff}}$ for the affinization map. Its kernel N is smooth with $h^0(\mathscr{O}_N) = 1$, according to [19], Chapter III, §3, 8.2 It must be an extension of some abelian variety by a torus, by [9], Proposition 2.2. In our situation, the torus is trivial, so N is an abelian variety. The cokernel for $U \to E^{\mathrm{aff}}$ is affine (loc. cit., Chapter III, §3, Theorem 5.6). This cokernel is also the quotient of the abelian variety $A = E/U$. It follows that $U \to E^{\mathrm{aff}}$ is surjective, whence E^{aff} is unipotent, of some dimension $n \geq 1$. The kernel for $U \to E^{\mathrm{aff}}$ is affine and belongs to N, hence is proper, and therefore finite. In turn, $\dim(N) = \dim(A)$. $\square$

7. The case of algebraic curves

Let k be a ground field of characteristic $p \geq 0$. In this section, C denotes an *algebraic curve*, that is, a scheme that is separated, of finite type, equi-dimensional, and of dimension $d = 1$. Write $C_1, \ldots, C_r$ for the irreducible component. We regard each C_i as the schematic images of the local Artin schemes $\mathrm{Spec}(\mathscr{O}_{C,\eta_i})$, where $\eta_i \in C$ denote the generic points. Note that we do not assume that C is proper, and that each C_i is either proper or affine. We start by describing the affinization map for C:

Proposition 7.1. *The k-algebra $\Gamma(C, \mathscr{O}_C)$ is of finite type, the affinization map $f : C \to C^{\mathrm{aff}}$ is projective with $f_*(\mathscr{O}_C) = \mathscr{O}_{C^{\mathrm{aff}}}$, and the exceptional locus $\mathrm{Exc}(C/C^{\mathrm{aff}})$ is the union of the irreducible components C_i that are proper.*

Proof. Choose some compactification $X = \bar{C}$, together with some effective Cartier divisor $D \subset X$ whose support is the locus at infinity $X \smallsetminus C$. The ensuing short exact sequence $0 \to \mathscr{L}^{\otimes n-1} \to \mathscr{L}^{\otimes n} \to \mathscr{L}_D^{\otimes n} \to 0$ induces a long exact sequence

$$H^0(X, \mathscr{L}^{\otimes n}) \longrightarrow H^0(X, \mathscr{L}_D^{\otimes n}) \longrightarrow H^1(X, \mathscr{L}^{\otimes n-1}) \longrightarrow H^1(X, \mathscr{L}^{\otimes n}) \longrightarrow 0.$$

We conclude that $h^1(\mathscr{L}^{\otimes n})$ is decreasing in n, hence becomes constant for $n \gg 0$. Then the map on the left must be surjective, and it follows that $\mathscr{L}$ is semi-ample. Passing to a multiple we may assume that $\mathscr{L}$ is globally generated. The homogeneous spectrum $Y = P(X, \mathscr{L})$ of the graded ring $R(X, \mathscr{L}) = \bigoplus_{n\geq 0} \Gamma(X, \mathscr{L}^{\otimes n})$ is a projective scheme and defines a morphism $f : X \to Y$ with $\mathscr{O}_Y = f_*(\mathscr{O}_X)$. Moreover, $\mathscr{L}$ is the preimage of the ample invertible sheaf $\mathscr{O}_Y(1)$. The map contracts the irreducible components $X_i = \bar{C}_i$ that are disjoint from D; these are exactly the C_i that are proper. Let $s \in \Gamma(Y, \mathscr{O}_Y(1))$ be the global section whose preimage $f^*(s) \in \Gamma(X, \mathscr{L})$ vanishes precisely at D. Then the non-zero locus Y_s is an affine open set, with preimage $C = X \smallsetminus D = f^{-1}(Y_s)$. Using $\mathscr{O}_Y = f_*(\mathscr{O}_X)$ we infer that Y_s must be the affinization C^{aff}. □

Suppose that the affine irreducible components C_i are generically reduced. Applying [24], Corollary 7.4.11 to the normalization of C_{red}, we see that there is a compactification $\bar{C}$ such that for each point at infinity $a \in \bar{C}$ the local ring $\mathscr{O}_{\bar{C},a}$ is a discrete valuation ring. We call it the *canonical compactification.*

Proposition 7.2. *Assumptions as above. Then the canonical compactification $\bar{C}$ yields an initial object in the category* $\mathrm{Cpt}(C)$ *of all compactifications. Moreover, the formation of $\bar{C}$ commutes with separable ground field extensions $k \subset k'$.*

Proof. Let $C \subset X$ be any compactification. By assumption, the finite set $Z = X \smallsetminus C$ admits an open neighborhood U such that $U \smallsetminus Z$ is regular. It then follows that the canonical compactification $\bar{C}$ arises from X by normalization on the open set U, and making no change on the open set $X \smallsetminus Z$. In particular, there is a morphism $\bar{C} \to X$ between compactifications, so $\bar{C}$ yields an initial object in $\mathrm{Cpt}(C)$.

For the second assertion, write the locus at infinity as $\bar{C} \smallsetminus C = \{a_1, \ldots, a_r\}$, and consider the residue fields $k_i = \kappa(a_i)$. Suppose the field extension $k \subset k'$ has the property that the finite k'-algebra $k_i \otimes k'$ is regular, that is, a product of fields. Then $\bar{C} \otimes k'$ remains regular over each $a_i \in \bar{C}$, hence must coincide with the canonical compactification of $C \otimes k'$. This happens in particular if $k \subset k'$ is separable. □

We see that the canonical compactification $X_\lambda = \bar{C}$ yields a final object in the opposite category $\mathrm{Cpt}(C)^{\mathrm{op}}$. Choosing the singleton $L = \{\lambda\}$ as a cofinal index set, the proof for Theorem 5.4 immediately gives:

Proposition 7.3. *Assumptions as in the previous proposition. Suppose C^{aff} is connected and reduced, and $k = H^0(\bar{C}, \mathscr{O}_{\bar{C}})$. Then the composition $C \subset \bar{C} \to \mathrm{Alb}_{\bar{C}/k}$ is the Albanese map for the algebraic curve C.*

Note that there is no final object in $\mathrm{Cpt}(C)^{\mathrm{op}}$ for more general curves C. For example, the infinitesimal extensions $\mathbb{P}^1 \oplus \mathscr{O}_{\mathbb{P}^1}(n)$ of the projective line, with n arbitrary, can be seen as compactifications of the non-reduced curve $C = \mathbb{A}^1_{k[\epsilon]}$, where $\epsilon^2 = 0$.

Let us unravel the condition that C^{aff} is connected and reduced. Write $\Gamma(C)$ for the *dual graph* of the scheme C. Recall that its vertices correspond to the irreducible components C_i, and two vertices are joined by an edge if $C_i \cap C_j$ is non-empty. Let $C' \subset C$ be the union of the proper irreducible components, and $C'' \subset C$ be the union of the affine irreducible components. These correspond to coherent sheaf of ideals $\mathscr{I}' \subset \mathscr{O}_C$ and $\mathscr{I}'' \subset \mathscr{O}_C$, respectively. Under the assumption that C has no embedded components, we have $\mathrm{Supp}(\mathscr{I}'') = C'$, and may regard the sheaf of ideals $\mathscr{I}''$ also as abelian sheaf on C', sitting in a short exact sequence $0 \to \mathscr{I}' \cap \mathscr{I}'' \to \mathscr{I}'' \to \mathscr{I}''\mathscr{O}_{C'} \to 0$, where the outer terms are $\mathscr{O}_{C'}$-modules.

Proposition 7.4. *Notation as above. Suppose that the algebraic curve C is not proper. Then the affine hull C^{aff} is connected and reduced if and only if the following conditions hold:*

(i) *The dual graph $\Gamma(C)$ is connected.*
(ii) *The scheme C has no embedded components.*
(iii) *The affine curve C'' is generically reduced.*
(iv) *The group of global sections $\Gamma(C', \mathscr{I}'')$ vanishes.*

Proof. First, suppose that that C^{aff} is connected and reduced. Then the noetherian scheme C must be connected, so the same holds for the dual graph $\Gamma(C)$. Moreover, the structure sheaf $\mathscr{O}_C$ has no global sections whose support is zero-dimensional. Since $\dim(C) = 1$ we infer that C has no embedding components. According to Proposition 7.1, the induced map $C'' \to C^{\mathrm{aff}}$ is a modification, and it follows that C'' is generically reduced. The short exact sequence $0 \to \mathscr{I}'' \to \mathscr{O}_C \to \mathscr{O}_{C''} \to 0$ gives an exact sequence

$$0 \longrightarrow H^0(C, \mathscr{I}'') \longrightarrow H^0(C, \mathscr{O}_C) \longrightarrow H^0(C'', \mathscr{O}_{C''}),$$

hence the term on the left vanishes, and this group can be written as $\Gamma(C', \mathscr{I}'')$.

Conversely, suppose that conditions (i)–(iv) hold. The scheme C is connected by (i). The map $C \to C^{\mathrm{aff}}$ is surjective, according to Proposition 7.1, so C^{aff} is connected as well. In the above exact sequence, the term on the left vanishes by (iv), and the ring on the right is reduced by (iii). Consequently $H^0(C, \mathscr{O}_C)$ is reduced. □

Let $k \subset k'$ be a field extension, and set $C' = C \otimes k'$. Suppose that C' is connected and reduced, and $k' = H^0(\bar{C}', \mathscr{O}_{\bar{C}'})$, and consider the resulting comparison map

$$c' : \mathrm{Alb}_{C'/k'} \longrightarrow \mathrm{Alb}_{C/k} \otimes k' \tag{11}$$

We shall see now that this may fail to be an isomorphism. As in [23], Section 2 we write $\mathrm{Sing}(\bar{C}/k)$ for the *locus of non-smoothness*. It carries a scheme structure, defined by the first Fitting ideal $\mathrm{Fitt}_1(\Omega^1_{\bar{C}/k})$. Note that it contains the *singular locus* $\mathrm{Sing}(\bar{C})$, but over imperfect fields may be much larger.

Theorem 7.5. *In the above setting, suppose that for the curve C and the extension $k \subset k'$ the following holds:*

(i) *The locus of non-smoothness $\mathrm{Sing}(\bar{C}/k)$ is non-empty and contained in the locus at infinity $\bar{C} \smallsetminus C$.*
(ii) *The normalization X for the base-change $Y = \bar{C} \otimes k'$ is a smooth curve of genus $g \geq 1$.*

Then the comparison map (11) is not an isomorphism.

Proof. Our assumption ensure that X is the canonical compactification of C', and we have a commutative diagram

$$\begin{tikzcd} Y \arrow[d, "g"'] & X \arrow[l, "\nu"'] \arrow[d, "f"] \\ \mathrm{Alb}_{Y/k'} & \mathrm{Alb}_{X/k'} \arrow[l, "c"] \end{tikzcd}.$$

By definition, $\mathrm{Alb}_{Y/k'}$ is the base-change of $\mathrm{Alb}_{C/k}$, whereas $\mathrm{Alb}_{X/k} = \mathrm{Alb}_{C'/k'}$. The Albanese map $f : X \to \mathrm{Alb}_{X/k'}$ is a closed embedding, in light of (ii).

Seeking a contradiction, we assume that the comparison map is an isomorphism. Then the composition $g \circ \nu$ is affine, and the same holds for the normalization map $\nu : X \to Y$. Using the Leray–Serre spectral sequence and Serre's Criterion ([24], Corollary 5.2.2), we infer that $g : Y \to \mathrm{Alb}_{Y/k}$ is affine. Since the composition

$$\mathscr{O}_{\mathrm{Alb}_{Y/k'}} \to g_*(\mathscr{O}_Y) \to (g \circ \nu)_*(\mathscr{O}_X)$$

is surjective, it follows that $\nu : X \to Y$ is a closed embedding. Hence this is an isomorphism, consequently $\bar{C}$ is smooth, in contradiction to (i). $\square$

Let us discuss explicit examples. Suppose the ground field k is imperfect, and consider the affine plane curve

$$Z : \quad y^l = \prod_{i=1}^{n} (x^{q_i} - \lambda_i),$$

where $l \geq 2$ is prime to the characteristic, and $q_i = p^{\nu_i} > 1$ are powers of the characteristic, and $\lambda_i \in k$ are pairwise different scalars that are no p-powers, and $n \geq 4$ is some even integer. The right-hand side is an inseparable square-free polynomial of degree $nq \geq 2$, and we see with Eisenstein's Criterion that Z is integral. Note that such curves where used by Totaro to construct *pseudo-abelian varieties* that are not abelian varieties ([48], Example 3.1).

To simplify the exposition, we also assume that $l \mid q_1 + \ldots + q_n$. Then the projection $Z \to \mathbb{A}^1 = \operatorname{Spec} k[x]$ easily gives the canonical compactification $h : \bar{Z} \to \mathbb{P}^1$, via the equation $y'^l = \prod(1 - \lambda_i x'^{q_i})$ in the new indeterminates $y' = y/x^{(q_1+\ldots+q_n)/l}$ and $x' = 1/x$. One sees that $h^{-1}(\infty)$ is étale, so $\bar{Z}$ is smooth around this fiber.

The ideal for $\mathfrak{a} \subset k[x, y]$ for the locus of non-smoothness $\mathrm{Sing}(Z/k) \subset \mathbb{A}^2$ is generated by the defining equation and its partial derivatives. One easily computes that the underlying closed set is given by $y = 0$ and $\prod(x^{q_i} - \lambda_i) = 0$. The equation $y = 0$ defines an effective Cartier divisor $D \subset Z$ whose coordinate rings is the product of the residue fields $k(\lambda_i^{1/q_i})$, and it follows that the affine plane curve Z is regular.

Let $k \subset k'$ be an extension field that contains the roots λ_i^{1/q_i}, and set $C' = C \otimes k'$. At each singularity $a_i \in C'$, the complete local ring becomes $R = k'[[u, v]]/(u^l - v^m)$, where $u = x$ and $v = y - \lambda_i^{1/q_i}$ and $m = q_i$. Its normalization is $k[[t]]$, with normalization map determined by $u = t^m$ and $v = t^l$. We conclude that the normalization X of the base-change $Y = \bar{Z} \otimes k'$ is smooth, of certain genus $g = h^1(\mathscr{O}_X)$. It comes with a branched covering $X \to \mathbb{P}^1_{k'}$ of degree l. In the

Riemann–Hurwitz Formula $2g-2 = l\cdot(-2)+\sum_{i=1}^n(l-1)$, the term on the right is

$$l\cdot(-2)+\sum_{i=1}^n(l-1) = n(l-1)-2l \geq 4(l-1)-2l = 2l-4\geq 0,$$

and it follows that X is a curve of genus $g\geq 1$. Now consider the affine curve $C = Z\smallsetminus \mathrm{Sing}(Z/k)$. Then $\bar{C}=\bar{Z}$, and the two conditions in Theorem 7.5 are satisfied. We see that the comparison map (11) is not an isomorphism.

8. The case of algebraic groups

Let k be a ground field of characteristic $p\geq 0$. Throughout this section, G denotes a group scheme of finite type, which are also called *algebraic groups* in the literature. First note that G is connected and reduced if and only if the respective properties hold for the *affinization* $G^{\mathrm{aff}} = \mathrm{Spec}\,\Gamma(G,\mathscr{O}_G)$. Throughout, we will assume that these equivalent conditions hold.

Since the neutral element $e\in G$ is a rational point, the ground field k coincides with the essential field of constants for G. By Theorem 5.4, there is an Albanese map $f: G\to \mathrm{Alb}_{G/k}$. The Albanese variety is a para-abelian variety endowed with a rational point $f(e)$. The latter becomes the zero element $0\in\mathrm{Alb}_{G/k}$ for a unique group law, according to [32], Proposition 4.3. In what follows we regard $\mathrm{Alb}_{G/k}$ as an *abelian variety.*

The goal of this section is to analyze how the various group laws interact with the Albanese map. We shall see that f does not necessarily respect the group laws. However, the following key fact ([10], Proposition 4.1.4) ensures that it does so on many subgroup schemes:

Lemma 8.1. *The restriction of the Albanese map $f:G\to\mathrm{Alb}_{G/k}$ to any smooth connected subgroup scheme $H\subset G$ respects the group laws.*

If k is perfect, then the reduced group scheme G itself is smooth, so the Albanese map is the universal homomorphism to an abelian variety.

The following terminology, which applies to any closed subscheme $E\subset G$ containing the origin $e\in G$, will be useful: We say that $f|E$ is *trivial* if it factors over the zero element $0\in\mathrm{Alb}_{G/k}$, viewed as a reduced closed subscheme. More generally, we say that $f|E$ is *set-theoretically trivial* if $f(g)=0$ for every $g\in E$. In this case, the *schematic image* $Z\subset\mathrm{Alb}_{G/k}$ is some closed subscheme supported by the zero element. In turn, $Z=\mathrm{Spec}(R)$ for some finite local k-algebra R with residue field $R/\mathfrak{m}_R = k$.

Proposition 8.2. *The restriction $f|H$ to any reduced connected affine subgroup scheme $H\subset G$ is trivial.*

Proof. We start with the case that $f|H$ respects the group law. According to [19], Chapter II, Proposition 5.1 the set-theoretical image of $f|H$ is a closed set. It must be connected, because H is connected. Write E for this closed set, endowed with the reduced scheme structure. Since H is reduced, $f:H\to\mathrm{Alb}_{G/k}$ factors over E. By loc. cit. E is a subgroup scheme, and the homomorphism $f:H\to E$ is faithfully flat. In turn, we have $E=H/H'$ where $H'=\mathrm{Ker}(f|H)$, and the quotient must be affine, according to [19], Chapter III, Theorem in 7.2. Since the Albanese variety is proper, the same holds for E. Being proper and affine, the group scheme E must be finite. Since it is connected it must be a singleton, thus $f|H$ is trivial.

In light of Lemma 8.1, our assertion holds if H is smooth. Suppose now that H is not smooth. Then we are in characteristic $p > 0$. Consider the *Frobenius kernels* $H[F^n]$ for the iterated relative Frobenius maps $F^n : H \to H^{(p^n)}$. According to [19], Chapter III, Lemma 6.10 the quotient $H/H[F^n]$ is smooth for sufficiently large $n \geq 0$. Set $A = \mathrm{Alb}_{G/k}$ and consider the induced morphism $f_n : G^{(p^n)} \to A^{(p^n)}$. Its restriction to the subgroup scheme $H/H[F^n]$ inside $H^{(p^n)} \subset G^{(p^n)}$ is trivial, by the preceding paragraph, and we infer that $f|H$ is at least set-theoretically trivial. Let $Z = \mathrm{Spec}(R)$ be its schematic image. Then R is a finite local k-algebra with residue field $R/\mathfrak{m}_R = k$, and the canonical map $R \to \Gamma(H, \mathscr{O}_H)$ is injective. Since H is reduced, the same holds for R, and thus $R = k$. $\square$

Note that the above arguments also give the fact that each subscheme $H \subset G$ that inherits a group law must be a closed subscheme. Each such $H \subset G$ acts via translations on G, from both sides. If $f|H$ respects the group law, H likewise acts on $\mathrm{Alb}_{G/k}$ via translations, also from both sides.

Proposition 8.3. *Let $H \subset G$ be a smooth connected subgroup scheme. Then the Albanese map $f : G \to \mathrm{Alb}_{G/k}$ is equivariant with respect to the H-actions.*

Proof. We have to verify that the graph $\Gamma_f \subset G \times \mathrm{Alb}_{G/k}$ is H-stable, where H acts diagonally. In light of Theorem 6.1, it suffices to treat the case that k is separably closed. Let $H' \subset H$ be the stabilizer of the graph. From the universal property of Albanese maps, we see that the inclusion $H'(k) \subset H(k)$ is an equality. Since H is smooth, the set $H(k)$ is dense, and thus the inclusion of the closed set H' is an equality. $\square$

Note that for reduced but non-smooth H, the regular locus $\mathrm{Reg}(H)$ is open and dense, but contains no rational point ([22], Corollary 2.6), and it can easily happen that the group $H(k)$ is trivial ([45], Section 8).

Corollary 8.4. *Suppose in addition that the subgroup scheme H is affine. Then the Albanese map $f : G \to \mathrm{Alb}_{G/k}$ uniquely factors over G/H.*

Proof. By Proposition 8.2, the homomorphism $f|H$ is trivial, hence the H-action on the Albanese variety is trivial. Passing to quotients we get $G/H \to \mathrm{Alb}_{G/k}$. Such a factorization is unique because the projection $G \to G/H$ is an epimorphism. $\square$

Since affinization of schemes that are separated and of finite type commutes with products, G^{aff} inherits a group law, and the affinization map $p : G \to G^{\mathrm{aff}}$ is a homomorphism. Let N be its kernel and write $i : N \to G$ for the inclusion map. By the Affinization Theorem ([19], Chapter III, 8.2), we get a central extension

$$0 \longrightarrow N \xrightarrow{i} G \xrightarrow{p} G^{\mathrm{aff}} \longrightarrow 1. \tag{12}$$

Furthermore, N is smooth and connected, with $h^0(\mathscr{O}_N) = 1$. Such group schemes are called *anti-affine*. Their structure was analyzed by Brion [9]. Note that on G and G^{aff}, the group laws are written in a multiplicative way, whereas for N we choose additive notation. Let us say that the group scheme G *weakly splits* if there is a morphism of schemes $s : G^{\mathrm{aff}} \to G$ with $p \circ s = \mathrm{id}_{G^{\mathrm{aff}}}$. Then

$$N \times G^{\mathrm{aff}} \longrightarrow G, \quad (x, y) \longmapsto i(x) \cdot s(y) \tag{13}$$

is an isomorphism of schemes. If the section s additionally respects the group laws, we say that G *strongly splits*; then the above is actually an isomorphism of group schemes.

We now can formulate our main result on the Albanese variety of group schemes G of finite type that are reduced and connected. It unravels how the Albanese map $f : G \to \mathrm{Alb}_{G/k}$ is related to the central extension $0 \to N \to G \to G^{\mathrm{aff}} \to 0$. Note that by Proposition 8.2, the restriction $f|N$ respects the group law, and we set $N' = \mathrm{Ker}(f|N)$.

Theorem 8.5. *In the above situation, the kernel $N' \subset N$ inside the anti-affine group scheme N is the smallest subgroup scheme such that N/N' is proper and G/N' weakly splits. Moreover, for any section $s : G^{\mathrm{aff}} \to G/N'$, the composition*

$$G \xrightarrow{\mathrm{can}} G/N' \xrightarrow{(i,s)^{-1}} N/N' \times G^{\mathrm{aff}} \xrightarrow{\mathrm{pr}_1} N/N'$$

is an Albanese map for G.

Proof. Let B be the cokernel of $f|N$, which is an abelian variety. According to Proposition 8.3, the Albanese map $f : G \to \mathrm{Alb}_{G/k}$ induces a morphism $G^{\mathrm{aff}} = G/N \to B$. The latter is trivial, by Proposition 8.2, and the universal property of f reveals that $B = 0$. Thus $f|N$ is surjective, and we get $\mathrm{Alb}_{G/k} = N/N'$. In particular, N/N' is proper.

Set $A = \mathrm{Alb}_{G/k}$. The Albanese map $f : G \to A$ factors over G/N'. The induced morphism $g : G/N' \to A$ is equivariant with respect to the actions of N/N' and the restriction of g to N/N' is an isomorphism of abelian varieties. In turn, $r = (g|N/N')^{-1} \circ g$ is an *equivariant retraction* for the inclusion $j : N/N' \to G/N'$. It follows that the composition

$$G^{\mathrm{aff}} \longrightarrow \{e\} \times G^{\mathrm{aff}} \longrightarrow N/N' \times G^{\mathrm{aff}} \xrightarrow{(r,p)^{-1}} G$$

is a section for the projection $G/N' \to G^{\mathrm{aff}}$. Thus G/N' weakly splits.

Next, we describe the Albanese map in the special case that G is weakly split. Since Albanese maps depend only on the underlying scheme, it suffices to treat the case that G is strongly split. Let $N_1 \subset N$ be the maximal smooth connected affine subgroup scheme. Using that the Albanese map factors over N/N_1, we reduce to the case $N_1 = 0$. Then N is an abelian variety, (this depends on Brion's result [9], see also [32], Proposition 7.1). We now have to verify that $\mathrm{pr}_1 : N \times G^{\mathrm{aff}} \to N$ is an Albanese map. Consider the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[dr] & N \times G^{\mathrm{aff}} \arrow[d, "f"] \arrow[dr, "\mathrm{pr}_1"] & \\
& \mathrm{Alb}_{G/k} \arrow[r, "g"'] & N,
\end{tikzcd}$$

where g comes from the universal property of the Albanese map. The left diagonal arrow $f\circ i$ is surjective, by the preceding paragraph, and has trivial kernel, because $g\circ f\circ i=\mathrm{id}_N$. It follows that g is an isomorphism, whence pr_1 is an Albanese map.

We come back to the general case. Let $H\subset N$ be any subgroup scheme with G/H weakly split and N/H proper. It only remains to check that $N'\subset H$. Choose a section s for the projection $G/H\to G^{\mathrm{aff}}$ and consider the commutative diagram

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[d, "\mathrm{can}"'] & G \arrow[r, "\mathrm{can}"] \arrow[d, "f"] & G/H \arrow[r, "(i,s)^{-1}"] & N/H\times G^{\mathrm{aff}} \arrow[d, "\mathrm{pr}_1"] \\
N/N' \arrow[r] & \mathrm{Alb}_{G/k} \arrow[rr, "g"'] & & N/H,
\end{tikzcd}$$

where g comes from the universal property of the Albanese map f. We see that the composite map $N\to N/H$ is equivariant with respect to the action of N, and factors over N/N', which ensures $N'\subset H$. $\square$

Suppose G is weakly split, and choose a section s for the projection p. Then the identification $i\cdot s:N\times G^{\mathrm{aff}}\to G$ from (13) is an isomorphism of schemes, and the group law on G arises from the product group law by a modification with a Hochschild two-cocycle $\varphi:(G^{\mathrm{aff}})^2\to N$. In fact, the isomorphism classes of central extensions $0\to N\to E\to G^{\mathrm{aff}}\to 0$ where the projection $E\to G^{\mathrm{aff}}$ admits a section correspond to classes in the Hochschild cohomology group $H^2_0(G^{\mathrm{aff}},N)$, where N is viewed as a module over G^{aff} with trivial action, see the discussion in [19], Chapter II, §3. This yields the following:

Corollary 8.6. *Suppose our group scheme G is weakly split and that N is proper. Then the Albanese map $f:G\to\mathrm{Alb}_{G/k}$ respects the group law if and only if G strongly splits.*

Proof. We may assume that $G=N\times G^{\mathrm{aff}}$ as schemes, and choose the two-cocycle $\varphi:(G^{\mathrm{aff}})^2\to N$ so that the inclusion $i:N\to G$ is given by $x\mapsto(x,e)$. The projection pr_1 is an Albanese map, by the theorem. If it respect the group law, then the identity morphism $G\to N\times G^{\mathrm{aff}}$ respects the group laws, hence G is strongly split. Conversely, if G is strongly split, then pr_1 respects the group law. $\square$

We now construct examples where the Albanese map does not respect the group laws. Recall that a group scheme U is *unipotent* if it is of finite type, and the base-change $U\otimes k^{\mathrm{alg}}$ admits a composition series whose quotients can be embedded into the additive group $\mathbb{G}_{a,k^{\mathrm{alg}}}$. Note that at least one composition series is already defined over k. We refer to [20], Exposé XVII for an comprehensive treatment.

Proposition 8.7. *Let U be a reduced connected unipotent group scheme, and N be an abelian variety. Suppose we are in characteristic $p>0$, and that there is an epimorphism $U\otimes k^{\mathrm{alg}}\to\alpha_{p,k^{\mathrm{alg}}}$ and a monomorphism $\alpha_{p,k^{\mathrm{alg}}}\to N\otimes k^{\mathrm{alg}}$. Then the following holds:*

(i) *The Hochschild cohomology group $H^2_0(U,N)$ is non-zero.*
(ii) *For all $\alpha\in H^2_0(U,N)$ the resulting extension G is reduced and connected.*
(iii) *If $\alpha\neq 0$, the Albanese map $G\to\mathrm{Alb}_{G/k}$ does not respect the group law.*

Proof. To see (ii), observe that in the extension $0 \to N \to G \to U \to 1$ the projection $G \to U$ has the structure of a principal N-bundle. By our assumptions, all fibers are smooth and connected, and the base is reduced and connected. So the latter also holds for the total space G. This gives (ii). Assertion (iii) is a direct consequence of Corollary 8.6.

It remains to verify (i). For this we first recall the computation of the cocycle group $Z^2(\mathbb{G}_a, \mathbb{G}_a)$ for Hochschild cohomology. The cochains are just morphism $\mathbb{G}_a^2 \to \mathbb{G}_a$, which corresponds to a homomorphisms of k-algebras $k[Z] \to k[X] \times k[Y]$, which is nothing but a polynomial in X and Y. By the computation in [19], Chapter II, §3, Subsection 4.6 the polynomials

$$Q(X,Y) = \sum_{i=1}^{p-1} \binom{p}{i} X^i Y^{p-i} \quad \text{and} \quad XY^{p^r} \quad (r \geq 1)$$

satisfy the cocycle condition. We remark in passing that these freely generate a complement for $B^2 \subset Z^2$, viewed as modules over the associative polynomial ring $k[F]$. The polynomial $Q(X,Y)$ induces a non-zero homomorphism

$$k[Z]/(Z^p) \longrightarrow k[X]/(X^p) \times k[Y]/(Y^p),$$

giving a non-trivial element in $Z^2(\alpha_p, \alpha_p)$. Using the epimorphism $U \otimes k^{\text{alg}} \to \alpha_{p,k^{\text{alg}}}$ over k^{alg} and applying descend, we conclude that $Z^2(U, \alpha_p)$ is non-zero.

Now let us examine the group of one-cochains $C^1(U, \alpha_p)$, whose elements are morphism of schemes $h : U \to \alpha_p$. Since U is reduced, any such morphism factors over $e \in \alpha_p$, viewed as a reduced closed subscheme. Thus the coboundary operator $C^1(U, \alpha_p) \to Z^2(U, \alpha_p)$ is the zero map, and we conclude $H^2_0(U, \alpha_p) \neq 0$.

Now consider the abelian variety N. Its Lie algebra $\text{Lie}(N)$ has trivial brackets, and comes with an additional structure given by the p-map $x \mapsto x^{[p]}$. By the Demazure–Gabriel Correspondence ([19], Chapter II, §7, Theorem 3.5) the homomorphisms $\alpha_p \to N$ correspond to the vectors $x \in \text{Lie}(N)$ with $x^{[p]} = 0$. Since the brackets are trivial, the p-map is additive. Our assumption on $N \otimes k^{\text{alg}}$ ensures that there is a non-zero homomorphism $\alpha_p \to N$. In turn, we get an inclusion $Z^2(U, \alpha_p) \to Z^2(U, N)$, so these groups are non-zero. Again we consider $C^1(U, N)$. Fix an element $h : U \to N$. By Proposition 8.2, this morphisms factors over a rational point $a \in N$. In turn, the coboundary map $C^1(U, N) \to Z^2(U, N)$ is trivial, and we conclude again that $H^2_0(U, N) \neq 0$. □

It is easy to make this concrete, over any imperfect field k of characteristic $p > 0$: Choose an element $t \in k$ that is not a p-power, and let U be the kernel of the homomorphism $\mathbb{G}_a^2 \to \mathbb{G}_a$ given by $(x, y) \longmapsto x^p - ty^p$. Then U is integral, $\text{pr}_1 |U$ is surjective, and the kernel is isomorphic to α_p. Over the field extension $k' = k(t^{1/p})$, the section given by $z \mapsto (z, t^{1/p}z)$ defines the desired splitting $U \otimes k' \simeq (\mathbb{G}_a \times \alpha_p) \otimes k'$. Furthermore, there is a supersingular elliptic curve N over k (use for example [42], Lemma 3.1), which indeed contains a copy of α_p.

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