Oberseminar Algebraic Geometry Summer Semester 2024 : Algebraic Groups

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Overview

The topic of this semester's Oberseminar is the theory of algebraic groups, i.e. group schemes over a field which are of finite type. Just as groups are used to rigorously define abstract concepts of symmetry and are ubiquitous in mathematics, algebraic groups often occur when dealing with symmetry within algebraic geometry.

Our first goal in the seminar will be to familiarize us with the geometric and algebraic properties of algebraic groups and see many examples. We will then study how arbitrary algebraic groups can be decomposed into pieces belonging to more specialized classes of algebraic groups such as *linear algebraic groups*, *Abelian varieties*, *anti-affine algebraic groups*, *unipotent (linear algebraic) groups* or *reductive (linear algebraic) groups*. In the end we will catch a glimpse on the classification of *split reductive linear algebraic groups* in terms of their root data.

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

Schedule

Date	**Speaker**	**Title**
08.04.2024		
15.04.2024		
22.04.2024		
29.04.2024		
06.05.2024		
13.05.2024		
20.05.2024	–	*Whitsun*
27.05.2024		
03.06.2024		
10.06.2024		
17.06.2024		
24.06.2024		
01.07.2024		
08.07.2024		
15.07.2024		Programme discussion

All dates are tentative.

Talks

Talk 1: Introduction

Summary: Discuss the basic definitions, give plenty of examples and discuss some basic facts on algebraic groups.
Main source: [Miln17, §§ 1-2]

Talk 2: Isomorphism Theorems

Summary: Discuss quotients of algebraic groups and prove the Noether isomorphism theorems for algebraic groups. In particular emphasize for which parts of the theory it is important to allow non-smooth algebraic groups.
Main source: [Miln17, §§ 5.b-5.g]

Talk 3: Affine Algebraic Groups are Linear

Summary: Discuss representations of (affine) algebraic groups, show that every affine algebraic group is linear and discuss tori and their representations.
Main source: [Miln17, §§ 4.a, 4.c, 4.d & 4.g]

Talk 4: Anti-Affine Algebraic Groups

Summary: Discuss the definition and some properties of anti-affine algebraic groups and show that every algebraic group is an extension of an affine by an anti-affine one. Show that every algebraic group in characteristic zero is smooth. If there is time, explain the construction of the three-step filtration in [L.S.24, §7].
Main source: [Miln17, §§ 8.d, 8.e & 8.i] (or [Brio17, §3])

Talk 5: Abelian varieties and the Barsotti-Chevalley Theorem

Summary: Define Abelian and pseudo-Abelian varieties and discuss the Barsotti-Chevalley theorem [Miln17, Thm. 8.27 & 8.28] (or [Brio17, Thm. 2, Thm. 4.3.4 & Cor. 4.3.7] respectively).
Main source: [Miln17, §§ 8.b, 8.c & 8.h] (or [Brio17, §4])

Talk 6: Semisimple and Reductive Linear Algebraic Groups

Summary: Define solvable, unipotent, semisimple and reductive linear algebraic groups and discuss their main properties and examples.
Main source: [Miln17, §§ 6 & 19.a-19.b]

Talk 7: Root Data of Reductive Linear Algebraic Groups

Summary: Define split reductive linear algebraic groups as well as their Lie algebras, Weyl groups and root data. Moreover discuss several examples.
Main source: [Miln17, §§ 10.a-10.d, 21.a & 21.c]

Talk 8: Classification of Split Reductive Linear Algebraic Groups

Summary: Sketch the classification of split reductive linear algebraic groups in terms of their root data. In particular discuss [Miln17, Thm. 23.25 & 23.55 & Cor. 23.56] and discuss all relevant definitions.
Main source: [Miln17, §§23.a, 23.c & 23.g]

Bonus talk: Pseudo-Abelian Varieties

Main source: [Tota13]

References

[Brio17] M. Brion, *Some structure theorems for algebraic groups*, In: M. Can (ed.), Algebraic groups: structure and actions, pp. 53–126. Amer. Math. Soc., Providence, RI, 2017.

[B.S.U.13] M. Brion, P. Samuel & V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, Hindustan Book Agency, New Delhi, 2013.

[L.S.24] B. Laurent & S. Schröer, *Para-abelian varieties and Albanese maps*, Bull. Braz. Math. Soc. (N.S.) 55 (2024), no. 1, 39 pp.

[Miln17] J. Milne, *Algebraic Groups – The Theory of Group Schemes of Finite Type over a Field*, Cambridge Studies in Advanced Mathematics 170, Cambridge University Press, 2017.

[Stacks] The Stacks project authors, *The Stacks project*, `https://stacks.math.columbia.edu` .

[Tota13] B. Totaro, *Pseudo-abelian varieties*, Annales scientifiques de l'École Normale Supérieure, Serie 4, Volume 46 (2013) no. 5, pp. 693-721.

Suggested plans

Talk 1: Introduction

Summary: Discuss the basic definitions, give plenty of examples and discuss some basic facts on algebraic groups.
Main source: [Miln17, §§ 1-2]
Suggested plan:

- Define algebraic groups both in terms of commutative diagrams and in terms of their functor of points. [Miln17, 1.1 & 1.4]
  alternative reference: [Brio17, 2.1.1, 2.1.5 & 2.1.2(ii)]
- Discuss the examples $\mathbb{G}_{\mathrm{a}}, \mathbb{G}_{\mathrm{m}}, \mathbf{GL}_n, \mathbf{SL}_n, \mathbf{GL}_V$ and μ_n. [Miln17, 1.1, 1.6, 2.1, 2.2, 2.4 & 2.8]
  alternative reference: [Brio17, 2.1.7, 2.1.8 & 2.1.9]
  needed in: Talk 2, 3, 6
- Discuss constant algebraic groups. [Miln17, 2.3]
  needed in: Talk 7
- Give more examples of algebraic groups, in particular a non-affine one, a non-smooth one and a non-connected one. [Miln17, §§ 1 & 2]
- Define homomorphisms of algebraic groups, algebraic subgroups and normal algebraic subgroups. [Miln17, Def. 1.2, 1.3 & 1.51]
  alternative reference: [Brio17, 2.1.3 & 2.1.4]
- Show that algebraic groups are separated. [Miln17, Prop. 1.22]
  needed in: Talk 4
- Discuss the characterization of smooth algebraic groups. [Miln17, Prop. 1.28]
  alternative reference: [Brio17, 2.1.12]
- Define the identity component G°. [Miln17, §1.b] and discuss the characterization of connected algebraic groups. [Miln17, Prop. 1.36]
  alternative reference: [Brio17, 2.4.1]
  needed in: Talk 6

- Discuss [Miln17, Prop. 1.38].
  needed in: Talk 4
- Show that algebraic subgroups are closed. [Miln17, Prop. 1.41].
  alternative reference: [Brio17, 2.7.1]
- Discuss that G° is a normal algebraic subgroup. [Miln17, Prop. 1.52]
  needed in: Talk 4
- Discuss [Miln17, Prop. 1.54].
  needed in: Talk 5
- Define kernels and short exact sequences of algebraic groups. [Miln17, §1.e & Def. 1.61]
  needed in: Talk 2, 4, 6
  alternative reference: [Brio17, 2.1.4 & 2.8.1-2.8.2]
- Present [Miln17, Prop. 1.62].
  alternative reference: [Brio17, 3.1.3]
  needed in: Talk 4, 5
- Define actions of algebraic groups. [Miln17, §1.f]
  alternative reference: [Brio17, 2.2.1]
  needed in: Talk 2, 4
- Show that the image of a homomorphism of algebraic groups is closed. [Miln17, Prop. 1.68 & Def. 1.73]
  alternative reference: [Brio17, 2.7.1]
- Sketch the definition of normalizers of algebraic subgroups. [Miln17, Prop. 1.83]
  alternative reference: [Brio17, 2.2.6]
  needed in: Talk 7
- Define the component group $\pi_0(G)$ and discuss (some of) its main features. [Miln17, 2.37-2.39]
  needed in: Talk 2, 4, 7
- If there is time, sketch [Miln17, Cor. 2.70].
  alternative reference: [Brio17, 3.1.2]

Talk 2: Isomorphism Theorems
Summary: Discuss quotients of algebraic groups and prove the Noether isomorphism theorems for algebraic groups. In particular emphasize for which parts of the theory it is important to allow non-smooth algebraic groups.
Main source: [Miln17, §§ 5.b-5.g]
Suggested plan:

- Define quotient maps of algebraic groups. [Miln17, Def. 5.5]
- Discuss [Miln17, Prop. 1.70].
- Discuss the notion of fat subfunctors.[1] [Miln17, Def. 5.6]
- Prove [Miln17, Prop. 5.7] and use it to show that quotient maps of algebraic groups have the universal property of quotients. [Miln17, Thm. 5.13]
  alternative reference: [Brio17, 2.7.4]

[1]If you have time, you can elaborate on the following: Note that the functor of points of a scheme is a sheaf with respect to the fpqc topology [Stacks, Tag 023Q] and that if a subfunctor D of an fpqc sheaf F is fat, then F is an fpqc sheafification of D. [Miln17, Exam. 5.71] See also [Miln17, §§5.j-5.m] and [Stacks, Tags 022H & 00W1, in part. 00WB] for more on this.

- Discuss the existence of quotients by normal algebraic subgroups without a proof. [Miln17, Thm. 5.14, Def. 5.20, Thm. 5.28, Prop. 5.43 & Prop. 5.45]
  alternative reference: [Brio17, 2.7.2]
  needed in: Talk 4 and 6
- Discuss examples of quotients of algebraic groups. (e.g. [Miln17, Exam. 5.49])
- Discuss monomorphisms of algebraic groups. [Miln17, Prop. 5.31, Def. 5.32 & Thm. 5.34]
  alternative reference: [Brio17, 2.7.1]
  needed in: Talk 3 and 4
- Discuss the homomorphism theorem [Miln17, Thm. 5.39 & Rem. 5.40] and [Miln17, Rem. 5.42].
  alternative reference: [Brio17, 2.7.3(iv) & 2.7.4]
  needed in: Talk 7
- Discuss [Miln17, Cor. 5.11] and use it to prove the isomorphism theorems. [Miln17, Thm. 5.52 & 5.55]
  alternative reference: [Brio17, 2.8.4 & 2.8.5]
  needed in: Talk 6
- Discuss the notes at the end [Miln17, §5.g]. In particular use [Miln17, Thm. 5.52] to show that $\mathbf{SL}_n/\mu_n = \mathbf{PGL}_n = \mathbf{GL}_n/\mathbb{G}_m$ for all n.
- Discuss [Miln17, Prop. 5.59] and [Miln17, Prop. 6.42].
  alternative reference: [Brio17, 3.1.3-3.1.4]
  needed in: Talk 5 and 6

Talk 3: Affine Algebraic Groups are Linear

Summary: Discuss representations of (affine) algebraic groups, show that every affine algebraic group is linear and discuss tori and their representations.
Main source: [Miln17, §§ 4.a, 4.c, 4.d & 4.g]
Suggested plan:

- Define representations and faithful representations of affine algebraic groups.[Miln17, §4.a]
  alternative reference: [Brio17, 2.3.1]
  needed in: Talk 4, 6 and 7
- Discuss why it is unnecessary to consider representations of non-affine algebraic groups. [Miln17, §4.0]
- Give the examples of the standard representation of (closed subgroups of) $\mathbf{GL}_n$ and the regular representation. [Miln17, Exam. 4.2 & §4.d]
  alternative reference: [Brio17, 2.3.2]
- Define Hopf algebras [Miln17, Def. 3.3] and briefly discuss [Miln17, Prop. 3.6].
- Define comodules and subcomodules of Hopf algebras and discuss their relation to representations. [Miln17, 4.0 & Rem. 4.1]
- Show that every comodule of a Hopf algebra is the union of its finite dimensional subcomodules. [Miln17, Prop. 4.7]
  alternative reference: [Brio17, 2.3.4]
- Prove [Miln17, Thm. 4.9 & Cor. 4.10] and discuss [Miln17, Rem. 4.11].
  alternative reference: [Brio17, 2.3.5 & 3.1.1]

- Discuss that every Hopf algebra is the union of its finitely generated Hopf subalgebras. [Miln17, Prop. 8.34]
  needed in: Talk 4
- Discuss that affine algebraic groups in characteristic zero are smooth. [Miln17, Thm. 3.23]
  needed in: Talk 4
- Define characters of affine algebraic groups and eigenspaces of representations of affine algebraic groups. [Miln17, §4.g]
  needed in: Talk 7
- Discuss [Miln17, Prop. 4.22].
- Define tori and split tori. [Miln17, 2.11 & Def. 12.14]
  needed in: Talk 7
- Discuss that every representation of a split torus is the direct sum of its eigenspaces. [Miln17, Thm. 12.12]
  needed in: Talk 7

Talk 4: Anti-Affine Algebraic Groups

Summary: Discuss the definition and some properties of anti-affine algebraic groups and show that every algebraic group is an extension of an affine by an anti-affine one. Show that every algebraic group in characteristic zero is smooth. If there is time, explain the construction of the three-step filtration in [L.S.24, §7].
Main source: [Miln17, §§ 8.d, 8.e & 8.i] (or [Brio17, §3])
Suggested plan:

- Define anti-affine algebraic groups. [Miln17, §2.0]
  alternative reference: [Brio17, 3.3.1]
- Define the centralizer of an algebraic subgroup and sketch how to prove its existence. [Miln17, Prop. 1.92] Define the center of an algebraic group.
  alternative reference: [Brio17, 2.2.6]
  needed in: Talk 7
- Discuss [Miln17, Cor. 8.10].
  alternative reference: [Brio17, 2.3.6]
  needed in: Talk 5
- Discuss that homomorphisms from anti-affine algebraic groups to affine algebraic groups are trivial. [Miln17, §8.e] Use this to prove that anti-affine algebraic groups are commutative. [Miln17, Cor. 8.14]
- Show that anti-affine algebraic groups are smooth and connected. [Miln17, Prop. 8.37]
  alternative reference: [Brio17, 3.3.2]
- Discuss that the ring $\mathcal{O}(X \times_k Y)$ of global section of a product of quasi-compact separated k-schemes X, Y is just given by the tensor product. [Brio17, Lemma 2.3.3] Deduce that $\mathcal{O}(G)$ is a Hopf algebra for every algebraic group G.
- Define the affinization of an algebraic group and present [Miln17, Prop. 8.36].
  alternative reference: [Brio17, 3.2.1]
- Explain the short exact sequence

$$e \to G_{\text{ant}} \to G \to G^{\text{aff}} \to e$$

- Discuss that algebraic groups in characteristic zero are smooth. [Miln17, Cor. 8.39]
  alternative reference: [Brio17, 2.1.6]
- If there is time, explain the construction of the three-step filtration in [L.S.24, §7].

Talk 5: Abelian varieties and the Barsotti-Chevalley Theorem
Summary: Define Abelian and pseudo-Abelian varieties and discuss the Barsotti-Chevalley theorem [Miln17, Thm. 8.27 & 8.28] (or [Brio17, Thm. 2, Thm. 4.3.4 & Cor. 4.3.7] respectively).
Main source: [Miln17, §§ 8.b, 8.c & 8.h] (or [Brio17, §4])
Suggested plan:

- Define Abelian varieties as a smooth connected proper algebraic group. [Brio17, §1]
- Define pseudo-Abelian varieties. [Miln17, Def. 8.3]
  alternative reference: [Brio17, 5.6.6]
- Discuss that Abelian varieties are pseudo-Abelian. [Miln17, Exam. 8.4]
- Give examples of Abelian and pseudo-Abelian varieties.
- Discuss [Miln17, Prop. 8.2].
- Show that being pseudo-Abelian is preserved by separable extension of the base field. [Miln17, Prop. 8.5]
- Discuss that over a perfect field all pseudo-Abelian varieties are Abelian. [Miln17, Thm. 8.26] Skip the proof entirely if there is no time for it.
- State and prove the Barsotti-Chevalley theorem. [Miln17, Thm. 8.27]
  alternative reference: [Brio17, 4.3.4]
- State [Miln17, Thm. 8.28] without a proof.
  alternative reference: [Brio17, 4.3.7]

Talk 6: Semisimple and Reductive Linear Algebraic Groups
Summary: Define solvable, unipotent, semisimple and reductive linear algebraic groups and discuss their main properties and examples.
Main source: [Miln17, §§ 6 & 19.a-19.b]
Suggested plan:

- Define solvable algebraic groups and unipotent linear algebraic groups. [Miln17, 6.1, 6.26 & 6.45].
- Give examples of solvable and unipotent linear algebraic groups (e.g. $\mathbb{G}_a$, commutative linear algebraic groups, ...)
- Discuss that $\mathbb{T}_n$ is solvable and $\mathbb{U}_n$ is unipotent. [Miln17, 2.9 & 6.49]
- Discuss that *solvable* and *unipotent* are preserved by extensions and quotients. [Miln17, 6.27 & 6.45]
- Define semisimple and reductive linear algebraic groups. [Miln17, 6.44 & 6.46]
  needed in: Talk 7
- Give several examples of semisimple and reductive linear algebraic groups. [Miln17, 19.19]
- Discuss that unipotent linear algebraic groups are solvable. [Miln17, Prop. 14.21]

- Present the example in [Miln17, 6.48].
- Discuss [Miln17, Prop. 19.1 & 19.2] and [Miln17, Prop. 19.9 & 19.11]
  alternative reference: [Brio17, 3.1.5]
- Sketch how to prove the claims in [Miln17, 19.19]. [Miln17, 19.16-19.18]

Talk 7: Root Data of Reductive Linear Algebraic Groups

Summary: Define split reductive linear algebraic groups as well as their Lie algebras, Weyl groups and root data. Moreover discuss several examples.

Main source: [Miln17, §§ 10.a-10.d, 21.a & 21.c]

Suggested plan:

- Define Lie algebras [Miln17, Def. 10.1].
  alternative reference: [Brio17, §2.3]
- Define the Lie algebra of a linear algebraic group [Miln17, 10.6] and discuss examples. (e.g. [Miln17, 10.2, 10.7 & 10.8])
  alternative reference: [Brio17, §2.3]
- Introduce the adjoint representation $\mathrm{Ad} : G \to \mathbf{GL}_{\mathfrak{g}}$ and use it to define the Lie algebra structure on $\mathfrak{g}$. [Miln17, 10.20 & 10.22]
  alternative reference: [Brio17, §2.3]
- Briefly discuss [Miln17, Thm. 10.23].
- Define split reductive linear algebraic groups and their homomorphisms. [Miln17, Def. 19.22]
  needed in: Talk 8
- Define the character lattice $X(T) = X^*(T)$ and the cocharacter lattice $X_*(T)$ of a split reductive linear algebraic group. [Miln17, §§12.a & 12.g]
- Define the Weyl group of a split reductive linear algebraic group [Miln17, Def. 17.41] and discuss [Miln17, Prop. 21.1].
- Define the set $\Phi(G, T)$ of roots of a split reductive linear algebraic group and discuss the action of the Weyl group on the character lattice $X^*(T)$ and the induced action on $\Phi(G, T)$. [Miln17, Prop. 21.2]
- Define the torus T_α and the subgroup G_α induced by a root $\alpha \in \Phi(G, T)$. [Miln17, §21.c]
- Sketch the definition of the map $\Phi(G, T) \to X_*(T), \alpha \mapsto \alpha^\vee$. [Miln17, Thm. 21.11]
- Define root data and reduced root data. [Miln17, Def. C.28].
  needed in: Talk 8
- Discuss that the root datum of a split reductive linear algebraic group satisfies the definition of a reduced root datum. [Miln17, Cor. 21.12]
- Give several examples of root data and Weyl groups of some split reductive groups. (e.g. [Miln17, Exam. 21.3-21.6, 21.15-21.16])
- If there is time, sketch the definition of the root groups U_α. [Miln17, Def. 21.10] (see also [Miln17, §12.a, §13.c & Prop. 13.29])
  needed in: Talk 8
- If there is time, discuss (parts of) [Miln17, §10.c].

Talk 8: Classification of Split Reductive Linear Algebraic Groups

Summary: Sketch the classification of split reductive linear algebraic groups in terms of their root data. In particular discuss [Miln17, Thm. 23.25 & 23.55 & Cor.

23.56] and discuss all relevant definitions.
Main source: [Miln17, §§23.a, 23.c & 23.g]
Suggested plan:

- Define isogenies of split reductive linear algebraic groups. [Miln17, Def. 6.6 & Prop. 23.5]
- Recall the definition of a (reduced) root datum from Talk 7.
- Define isogenies and isomorphisms of root data. [Miln17, Def. 23.1-23.2]
- Discuss how isogenies (resp. isomorphisms) of split reductive groups induce isogenies (resp. isomorphisms) of their root data. [Miln17, Prop. 23.5]
- Give examples of isogenies of split reductive groups and the induced isogenies of their root data.
- Sketch [Miln17, Prop. 23.7].
- Present the isogeny theorem and the isomorphism theorem [Miln17, Thm. 23.9 & 23.25, Rem. 23.26].
- Present the existence theorem [Miln17, Thm. 23.55] and the corollary [Miln17, Cor. 23.56] without giving proofs.
- If there is time, you can additionally discuss central isogenies. [Miln17, Def. 18.1 & 23.2]
- If there is time, you can talk about the relation of root data and root systems and say something about their classification. [Miln17, Appendix C]