Oberseminar Algebraic Geometry Summer Semester 2023 : Bridgeland Stability Conditions

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Overview

Stability conditions are a tool in moduli theory to split up a given moduli problem into simpler pieces. Forming moduli spaces of *stable objects* often provides better behaved moduli spaces. Notions of stability conditions are particularly well established when the objects of the moduli problem are contained in an Abelian category $\mathcal{A}$. Bridgeland's approach passes from $\mathcal{A}$ to its derived category $D^b(\mathcal{A})$. His framework often allows to study $D^b(\mathcal{A})$ by geometric methods and provides applications to the cohomology theory of $\mathcal{A}$. Moreover there are intimate connections to string theory.

While it is theoretically possible to define Bridgeland stability conditions for more or less arbitrary Abelian or triangulated categories, we will focus on categories of coherent sheaves $\mathrm{Coh}(X)$ and their bounded derived categories $D^b(X) := D^b(\mathrm{Coh}(X))$. For technical purposes it will be crucial to restrict ourselves to smooth projective k-schemes X. Moreover we should restrict to the case X irreducible and k algebraically closed to avoid less important technicalities. For practical reasons it is often suitable to restrict to the case $k = \mathbb{C}$, however most things should also work in positive characteristic if we replace Betti cohomology by étale cohomology, $\mathbb{Z}$-lattices by $\mathbb{Z}_\ell$-lattices etc. These restrictions might look severe, but it turns out that even in this setting constructing Bridgeland stability conditions is a very tough business and becomes more and more difficult for higher dimensional X.

Time and Place: Monday, 12:30-13:30, seminar room 25.22.03.73.

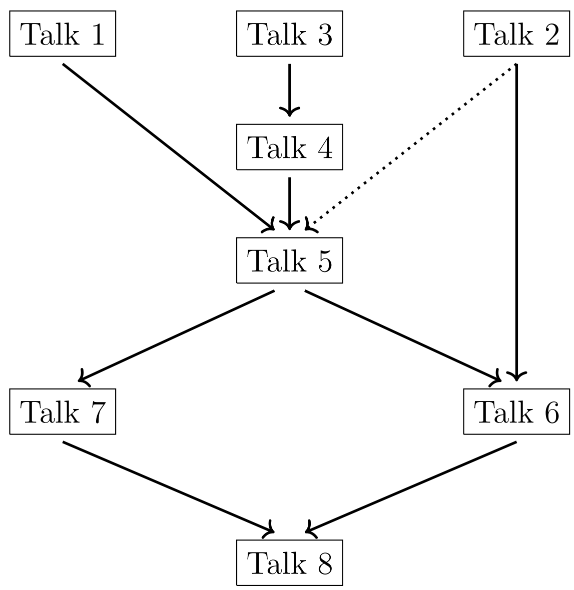
Schedule

Date	Speaker	Title
03.04.2023	Fabian Korthauer	Stability for coherent sheaves on curves
10.04.2023	–	*Easter*
17.04.2023	Thor Wittich	Chern classes and $K_{\mathrm{num}}(X)$
24.04.2023	Cesar Hilario	Recollection on derived categories
01.05.2023	–	*Labour Day*
08.05.2023	Jan Hennig	Tilting & bounded t-structures
15.05.2023	Otto Overkamp	Bridgeland stability conditions
22.05.2023	Jakob Bergqvist	Kummer Constructions in Families
29.05.2023	–	*Whitsun*
05.06.2023	Jakob Bergqvist	Stability conditions on surfaces
12.06.2023	Stefan Schröer	Bridgeland's deformation result
19.06.2023	Sabrina Pauli	Tropical methods in refined enumerative geometry
26.06.2023	Otto Overkamp	Jacobians of singular curves and their Néron models
03.07.2023	Ivo Kroon	Stability conditions on K3 surfaces
10.07.2023		Programme discussion

All dates are tentative. (last updated: 22 May 2023)

Talks

The following directed graph sketches the interdependencies between the talks. If Talk Y requires preknowledge from Talk X, then there is an oriented path from Talk X to Talk Y. A dotted arrow indicates that only little preknowledge is required.



Talk 1: (03. April), Fabian Korthauer: Stability for coherent sheaves on curves

Summary: Discuss the classical notion of stability for coherent sheaves on curves with several examples and generalize the definition to Abelian categories.

Main source: [Baye11, §2]

- Define the degree $\deg(\mathcal{F}) = \chi(\mathcal{F}) - \mathrm{rk}(\mathcal{F})\chi(\mathcal{O}_X)$ and the slope $\mu(\mathcal{F}) \in \mathbb{Q} \cup \{\infty\}$ of a coherent sheaf $\mathcal{F}$ on an irreducible smooth projective curve as well as stability and semistability of coherent sheaves. [Baye11, §2.1]
- Present the See-Saw Lemma. [Baye11, Lemma 2.1.1]
- Give examples for semistable and stable coherent sheaves, e.g. torsion sheaves, simple torsion sheaves and line bundles. [Baye11, Exam. 2.1.4]
- Present [Baye11, Lemma 2.1.5].
- Present Schur's Lemma for $\mathrm{Coh}(C)$. [Huyb14, 1.5]
- Recall the definition of the Grothendieck group of an Abelian category. [Huyb14, §2.2]
- Define stability functions and weak stability functions as well as stable and semistable objects for arbitrary Abelian categories and discuss that this generalizes the notions for coherent sheaves on curves. [Baye11, Def. 2.2.1], [M.S.17, Rem. 4.14]
- Define Harder-Narasimhan filtrations and sketch how to prove uniqueness and existence for (weak) stability functions on Noetherian Abelian categories with discrete image. [Baye11, Thm. 2.1.6], [M.S.17, Prop. 4.10 & Rem. 4.14] In particular draw some pictures like [Baye11, p.4] or [Huyb14, p.5].

- Discuss that the See-Saw Lemma and Schur's Lemma also hold for stability functions on Abelian categories (the latter for k-linear Abelian categories with finite dimensional Hom-sets).

Talk 2: (17. April), Thor Wittich: Chern classes and $K_{\text{num}}(X)$
Summary: Recall some basic facts on Ext groups of coherent sheaves and define the numerical Grothendieck group and the Chern character for surfaces.
Main source: [Hart77, III.§6 & Appendix A.§§3-4]
Let X be an irreducible smooth projective k-scheme. Feel free to assume $k = \mathbb{C}$.

- Recall the construction of Ext groups of coherent sheaves, present [Hart77, Prop. III.6.3(c)] and define the homological Euler form

$$\chi(\mathcal{F}, \mathcal{G}) = \sum_{i=0}^{\dim(X)} \dim_k \operatorname{Ext}^i(\mathcal{F}, \mathcal{G})$$

- Present [Hart77, Exer. III.6.9] for irreducible smooth projective k-schemes.
- Use [Hart77, Prop. III.6.4] and Serre duality to show that $\chi(-,-)$ defines a $\mathbb{Z}$-bilinear form on $K_0(X)$ for X smooth.
- Define the numerical Grothendieck group. [Brid14, §2.1]
- Use the Riemann-Roch Theorem to show that for an irreducible smooth curve C, $K_{\text{num}}(C)$ is a rank 2 lattice generated by $[\mathcal{O}_C]$ and the skyscraper sheaf $[\mathcal{O}_x]$ for a rational point $x \in C$.
- Use [Hart77, Exer. III.6.9] to define the first Chern class $c_1 : K_0(X) \to \operatorname{Num}(X)$ for an irreducible smooth surface X. [Hart77, Appendix A.§3]
- Define the second Chern class $c_2 : K_0(X) \to \mathbb{Z}$ via the Riemann-Roch theorem [Hart77, Appendix A.§4] and define the Chern character ch. [Brid14, §2.1][1]
- Use [Hart77, Prop. III.6.7] and the (Hirzebruch-)Riemann-Roch theorem to write down a numerical formula for the homological Euler form $\chi(-,-)$. Use this formula to identify $K_{\text{num}}(X)$ with the image of ch and to show that $-\chi(-,-)$ restricted to $\operatorname{Num}(X)$ recovers the intersection pairing.

Talk 3: (24. April), Cesar Hilario: Recollection on derived categories
Summary: Recall the construction of the derived category[2] $D^b(X)$ and some of its basic properties.
Main source: [Huyb06, §§1.2 & 2.1] or the respective chapters of your favourite homological algebra book
Let X be an irreducible smooth projective k-scheme and $\mathcal{A} = \operatorname{Coh}(X)$ be the Abelian category of its coherent sheaves.

- Define the category of bounded (cochain) complexes $\operatorname{Kom}^b(\operatorname{Coh}(X))$ and briefly mention the homotopy category of bounded complexes $K^b(\operatorname{Coh}(X))$. [Huyb06, Def. 2.2, 2.12 & 2.29]

[1] see also [M.S.17, Def. 3.1 & Exam. 5.7]
[2] Historical Remark: After Grothendieck came up with the idea of derived categories in the early 1960s, he instructed his student Verdier to rigorously define them and lay out the theory of derived categories as his PhD project. Verdier obtained his PhD in 1967 without ever completing his thesis. His incomplete dissertation was finally published in 1996 seven years after Verdier died in a car accident.

- Define quasi-isomorphisms of complexes. [Huyb06, Def. 2.9]
- Define the objects and morphisms of the bounded derived category $D^b(X) := D^b(\mathrm{Coh}(X))$ and sketch how to compose morphisms. [Huyb06, after Rem. 2.14]
- Discuss that $D^b(X)$ is an additive category, but almost never Abelian. [Huyb06, Exer. 2.21]
- Briefly mention [Huyb06, Cor. 2.11(iii)].
- Define the shift functor $[1] : D^b(X) \to D^b(X)$, mapping cones and the distinguished triangles of $D^b(X)$. [Huyb06, Def. 2.4, 2.15 & 2.23]
- Briefly mention the axioms of a triangulated category and state that $D^b(X)$ satisfies them. [Huyb06, Def. 1.32 & Prop. 2.24]
- Discuss [Huyb06, Exer. 2.27].
- Define the notion of cohomological functors [Weib94, Def. 10.2.7], discuss the example H^0 briefly and discuss [Weib94, Exam. 10.2.8] in detail.
- Define the Grothendieck group $K_0(D^b(X))$ and argue that $[\mathcal{F}^\bullet[1]]$ is the inverse of $[\mathcal{F}^\bullet]$ in $K_0(D^b(X))$. [Brid14, §1.4]
- State [Huyb06, Prop. 2.56] without a proof. Briefly discuss that one can define Ext groups also for complexes $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(\mathcal{A})$ and state [Huyb06, (2.2) in Rem. 2.57].[3]
- Present [Baye11, Prop. 3.3.1] and use it to show that $K_0(D^b(X))$ is isomorphic to $K_0(X)$ and to prove $\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \chi(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)$ for $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$.[4] Also briefly discuss the last paragraph in [Baye11, §3.3] and [Baye11, Exer. 3.7.8].

Talk 4: (08. May), Jan Hennig: Tilting & bounded t-structures

Summary: Introduce the notion of bounded t-structures and the concept of tilting from homological algebra.
Main source: [Huyb14, §§1.2-1.4][5]
Let X be an irreducible smooth projective k-scheme.

- Define bounded t-structures on $D^b(X)$, their hearts and their cohomology functors. [Huyb14, §1.3] Present [Huyb14, Exam. 1.14]. Emphasize the difference between the ordinary cohomology functors $H^i : D^b(X) \to \mathrm{Coh}(X)$ and those given by a different bounded t-structure.
- Recall [Baye11, Prop. 3.3.1] from Talk 3, state [Huyb14, Rem. 1.16] and discuss how a bounded t-structure can be recovered from its heart.
- Discuss that hearts of bounded t-structures are Abelian categories and describe their short exact sequences. [Huyb14, §1.3], [Baye11, Exer. 3.7.10]
- Prove [M.S.17, Exer. 5.4] analogous to $K_0(D^b(X)) \cong K_0(X)$ in Talk 3.

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[3] See also [Huyb06, Rem. 3.7(i)]. Note that $\mathrm{Coh}(X)$ usually does not have enough injectives. With some effort it is still possible to define all kinds of derived functors, e.g. by using [Huyb06, Prop. 3.5]. However, we won't have to bother with these technicalities in this seminar.
[4] Use that $- \otimes \omega_X$ is an automorphism of $D^b(X)$ as a triangulated category, i.e. preserves distinguished triangles, use [Baye11, Prop. 3.3.1] to reduce the claim to (classical) Serre duality. See also [Huyb06, Thm. 3.12].
[5] The main source lacks motivation of the content. So maybe first skim [Baye11, §§3.3-3.6] which does not cover all the material, but motivates the concepts quite well.

- Define torsion pairs on Abelian categories [Huyb14, Def. 1.10][6] and discuss examples. [Baye11, Exam. 3.6.(1)&(5)][7]
- Define tilts of hearts of bounded t-structures and discuss [M.S.17, Lemma 6.3].[8]
- If there is time left, discuss the picture on [Huyb14, p.12] and possibly [M.S.17, Exer. 6.4 & 6.5].

Talk 5: (15. May), Otto Overkamp: Bridgeland stability conditions
Summary: Define Bridgeland stability conditions and relate them to stability conditions on Abelian categories.
Main source: [Baye11, §4]
Let X be an irreducible smooth projective k-scheme.

- Define slicings of $D^b(X)$ [Baye11, Def. 4.1.1].
- Discuss [Baye11, Rem. 4.1.2] and draw analogies to Talk 1.
- Define Bridgeland stability conditions on $D^b(X)$. [Baye11, Def. 4.1.3]
- Discuss [Baye11, Prop. 4.1.4] in detail.
- Discuss the example [Baye11, §4.2].
- Recall the definition of the homological Euler form and the numerical Grothendieck group from Talk 2.
- Define numerical Bridgeland stability conditions [Baye11, (1) in §5.1] and the support property. [Baye11, §5.1] Verify that the example [Baye11, §4.2] is numerical and has the support property.[9]
- State that one may endow the set $\mathrm{Stab}(D^b(X))$ of numerical Bridgeland stability conditions with support property with the structure of a complex manifold.[10] [Baye11, Thm. 5.1.1]
- If there is time left, present parts of [Huyb14, Cor. 2.8][11], [M.S.17, Exer. 5.9] and/or [Huyb14, Exam. 2.11].

Talk 6: (19. June), Jakob Bergqvist: Stability conditions on surfaces
Summary: Construct examples of stability conditions on surfaces.
Main source:[M.S.17, §§6.2 & 6.3]
Let X be an irreducible smooth projective surface. Feel free to assume $k = \mathbb{C}$.

- Briefly recall the definition of a numerical Bridgeland stability condition and [Baye11, Prop. 4.1.4] from Talk 5.
- Present [Toda09, Lemma 2.7].
- Define the slope function $\mu_{\omega,B}$ as well as slope stable and slope semistable coherent sheaves for $B \in N^1(X) := \mathrm{Num}(X) \otimes \mathbb{R}$ an $\mathbb{R}$-divisor and $\omega \in \mathrm{Amp}(X)$ an ample $\mathbb{R}$-divisor. [M.S.17, Def. 3.3][12]

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[6] see also [Baye11, Def. 3.5.1]
[7] see also [Huyb14, Rem. 1.8]
[8] see also [Baye11, Prop. 3.6.1] & [Huyb14, Prop. 1.17]
[9] Nowadays being numerical and satisfying the support property is usually included in the definition of a Bridgeland stability condition.
[10] This will be discussed in detail in Talk 7.
[11] The proof relies on the See-Saw Lemma.
[12] see also [Huyb14, §3.0] & [Brid08, §5]

- Discuss that slope stability is only a weak stability condition on $\mathrm{Coh}(X)$ and relate this to [Toda09, Lemma 2.7]. [M.S.17, §4.3(2)][13]
- Recall from Talk 1 that weak stability conditions admit Harder-Narasimhan filtrations. [M.S.17, Prop. 4.10 & Rem. 4.14]
- Define the torsion pair $(\mathcal{T}_{\omega,B}, \mathcal{F}_{\omega,B})$ and the tilt $\mathrm{Coh}^{\omega,B}(X)$. [M.S.17, Def. 6.6]
- Define the stability function $Z_{\omega,B}$ and state [M.S.17, Thm. 6.10].
- Prove [M.S.17, Thm. 7.4].
- Sketch as much of the proof of [M.S.17, Thm. 6.10] as possible. In particular discuss how the Bogomolov inequality defined by ω, B shows up. [M.S.17, §§6.2 & 6.3]

Talk 7: (26. June), Stefan Schröer: Bridgeland's deformation result

Summary: The main feature of Bridgeland stability conditions is that the space of all stability conditions on a triangulated category has the structure of a complex manifold. This allows to deform stability conditions along paths in this manifold. Present the manifold structure on $\mathrm{Stab}(D^b(X))$.

Main source: [Baye11, §5]

Let X be an irreducible smooth projective k-scheme.

- Recall once again the definition of numerical Bridgeland stability conditions with support property from Talk 5.
- Define the generalized metric on $\mathrm{Stab}(D^b(X))$.
- State [Baye11, Thm. 5.1.1] and sketch the proof. [Baye11, §5.5]
- If there is time left, discuss [M.S.17, Prop. 5.27] (feel free to specialize e.g. to $X = \mathbb{P}^1$ and/or S a single sheaf like $S = \{\mathcal{O}(n)\}$).[14] Maybe have a look at [Okad06, Prop. 5.1] and try to draw a picture of the wall and chamber structure of $\mathrm{Stab}(\mathbb{P}^1) = \mathbb{C}^2$.

Talk 8: (03. July), Ivo Kroon: Stability conditions on K3 surfaces

Summary: Recall the definition of the Mukai pairing, the Mukai lattice and the Mukai vector we saw in the seminar on Fourier-Mukai transformation. Discuss the stability conditions on K3 surfaces originally constructed by Bridgeland and sketch his study of the stability manifold of a K3 surface.

Main source: [Brid08]

Let X be a (projective) K3 surface.[15] Feel free to assume $k = \mathbb{C}$.

- Recall the numerical formula for the homological Euler form from Talk 2 and argue that $\chi(-,-)$ is symmetric, i.e. $(K_{\mathrm{num}}(X), -\chi(-,-))$ is a lattice extension of $\mathrm{Num}(X)$.
- Recall the definition of the Mukai pairing on $H^*(X,\mathbb{Z})$ and discuss that $(K_{\mathrm{num}}(X), -\chi(-,-))$ is nothing but the Mukai sublattice $N(X) = H^*_{\mathrm{alg}}(X,\mathbb{Z}) \subseteq H^*(X,\mathbb{Z})$. [Brid08, §1.1 & §5][16]

[13]see also [Huyb14, Rem. 1.9]
[14]see also [Baye11, §§5.2 & 5.4]
[15]A lot of the material below should also work for Abelian or bielliptic surfaces.
[16]see also [Huyb14, §§3.1 & 5.1], [M.S.17, §6.6] or [Muka87, §1]

- Recall the definition of the Mukai vector $v : K_0(X) \to K_{\mathrm{num}}(X)$. [Brid08, §1.1 & §5][17]
- Define the stability function Z given by [Brid08, ($\star$) in §6][18] and discuss that it differs from the stability function $Z_{\omega,B}$ from Talk 6 by $Z_{\omega,B} - Z = \mathrm{rk}$.
- Present [Huyb14, Prop. 3.7] and sketch the proof.[19]
- Present [Huyb14, Prop. 3.8 & Cor. 3.9]. Sketch the proof only if time permits.[20]
- Recall the definition of the group homomorphism $\mathrm{Aut}\left(D^b(X)\right) \to O(H^*(X,\mathbb{Z}))$. [Huyb06, Cor. 10.7][21]
- Define $\mathcal{P}_0^+(X)$, state [Huyb14, Thm. 5.3][22] and compare it to [Baye11, Thm. 5.1.1] from Talk 7.
- If there is time left, you could do one of the following things:
  - Discuss how [Huyb14, Thm. 5.3] can be used to study $\mathrm{Aut}\left(D^b(X)\right)$. [Huyb14, §§5.1 & 5.2]
  - Present [Brid08, Thm. 15.2].
  - Discuss [Huyb14, Rem. 3.6(ii)].
  - Sketch the actions of $\mathrm{Aut}\left(D^b(X)\right)$ and $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on $\mathrm{Stab}(D^b(X))$ as well as the definition of $U(X) = \widetilde{\mathrm{GL}}^+(2,\mathbb{R}).V(X) \subseteq \mathrm{Stab}^\circ(D^b(X))$ and state that $\mathrm{Aut}_0^\circ(D^b(X)).\overline{U(X)} = \mathrm{Stab}^\circ(D^b(X))$. [Huyb14, §§2.3 & 5.3], [Brid08, §§10, 11 & 13]

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[17] see also [Huyb14, §§3.1 & 3.2] or [Muka87, §1 & Def. 2.1]
[18] see also [Huyb14, §3.2]
[19] see also [Brid08, Lemmata 5.1 & 6.2]
[20] see also [Brid08, Prop. 7.1]
[21] see also [Huyb14, §5.1] and [Brid08, §1.1]
[22] see also [Brid08, Thm. 1.1]

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