Mathematisches Institut
Heinrich-Heine-Universität Düsseldorf
Prof. Dr. Stefan Schröer

WS 21/22

Algebraic Geometry I

Sheet 1

Excercise 1. Verify explicitly that for each ring R the topological space $X = \mathrm{Spec}(R)$ is quasicompact. Give an example where X is not Hausdorff.

Exercise 2. Set $X = \mathrm{Spec}(\mathbb{Q})$ and $Y = \mathrm{Spec}(\mathbb{Z})$. Determine all morphisms $f : X \to Y$ of ringed spaces. Which of them are morphisms of schemes?

Exercise 3. Recall that the ring elements $e \in R$ with $e^2 = e$ are called *idempotent*. Let X be a scheme. Show that

$$e \longmapsto X_e = \{x \in X \mid e(x) \neq 0 \text{ in the residue field } \kappa(a) = \mathscr{O}_{X,a}/\mathfrak{m}_a\}$$

gives a bijection between the set of idempotents in the ring $R = \Gamma(X, \mathscr{O}_X)$ of global sections and the open-and-closed sets $U \subset X$.

Exercise 4. The group $G = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, which is cyclic of order two, acts on the ring $\mathbb{C}[T]$ by complex conjugation. In turn, we get a G-action on the affine scheme $X = \mathrm{Spec}(\mathbb{C}[T])$. Set $Y = \mathrm{Spec}(\mathbb{R}[T])$ and consider the canonical morphism

$$f : X \longrightarrow Y$$

corresponding to the inclusion $\mathbb{R}[T] \subset \mathbb{C}[T]$. Prove that for the underlying sets this is the quotient map for the G-action.

Abgabe: Bis Donnerstag, den 28. Oktober um 23:59 Uhr über ILIAS.

Die Lösungen müssen handschriftlich und individuell sein und in Form einer einzigen pdf-Datei vorliegen, mit der Bezeichnung `NameVorname--Abgabe01.pdf` beim ersten Blatt. Die Aufgaben werden in den Übungsgruppen vor- und nachbesprochen. Es gibt keine Korrekturen, daher werden auch keine Punkte vergeben.

Quorum: Um zur mündlichen Prüfung zugelassen zu werden, müssen sie insgesamt 8 mathematisch sinnvolle Abgaben gemacht haben.

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Algebraic Geometry I

Sheet 2

Excercise 1. Let X be a scheme, R be a ring, and $\varphi : R \to \Gamma(X, \mathscr{O}_X)$ be a homomorphism. For $x \in X$ we define $f(x) \in \operatorname{Spec}(R)$ as the point corresponding to the kernel for the composition

$$R \xrightarrow{\varphi} \Gamma(X, \mathscr{O}_X) \xrightarrow{\text{res}} \mathscr{O}_{X,x} \xrightarrow{\text{pr}} \mathscr{O}_{X,x}/\mathfrak{m}_x = \kappa(x).$$

Check that the resulting map $f : X \to \operatorname{Spec}(R)$ is continuous.

Exercise 2. Let X be a scheme. Show that X is quasiseparated if and only if there is an affine open covering $X = \bigcup_{\lambda \in L} W_\lambda$ such that the intersections $W_{\lambda\mu} = W_\lambda \cap W_\mu$ are quasicompact.

Exercise 3. Let $(U, \mathscr{O}_U)$ and $(V, \mathscr{O}_V)$ be two schemes. Suppose we have open sets $U' \subset U$ and $V' \subset V$, together with an isomorphism of schemes

$$(f, \varphi) : (U', \mathscr{O}_{U'}) \longrightarrow (V', \mathscr{O}_{V'}).$$

Endow the set-theoretic union $X = U \cup V$, where the points $x \in U'$ are identified with $f(x) \in V'$, with a canonical scheme structure, by declaring a topology and defining the structure sheaf.

Exercise 4. Let T_0 and T_1 be indeterminates. The *projective line* $X = \mathbb{P}^1_R$ over a ring R is defined by the above gluing construction for $U = \operatorname{Spec} R[T_0]$ and $V = \operatorname{Spec} R[T_1]$, and

$$U' = \operatorname{Spec} R[T_0^{\pm 1}] \quad \text{and} \quad V' = \operatorname{Spec} R[T_1^{\pm 1}]) \quad \text{and} \quad \varphi(T_1) = T_0^{-1}.$$

Prove that the affinization of this scheme is given by

$$(\mathbb{P}^1_R)^{\text{aff}} = \operatorname{Spec}(R),$$

by computing the ring of global sections $\Gamma(X, \mathscr{O}_X)$ with the sheaf axiom.

Abgabe: Bis Donnerstag, den 4. November um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 3

Excercise 1. Let $f : X \to Y$ and $g : Y \to Z$ be morphisms of schemes. Verify the following implications:

(i) If f and g are locally of finite type, the same holds for the composition $g \circ f$.

(ii) If $g \circ f$ is locally of finite type, then f is locally of finite type.

Also give an example where $g \circ f$ is locally of finite type, but g is not.

Exercise 2. Let $f : X \to Y$ be a continuous map of topological spaces, and $\mathscr{F}$ be an abelian sheaf on X. We define an abelian sheaf $f_*(\mathscr{F})$ on Y by declaring

$$\Gamma(V, f_*(\mathscr{F})) = \Gamma(f^{-1}(V), \mathscr{F}).$$

Make the restriction maps explicit, verify that this indeed gives a presheaf, and check the sheaf axiom.

Exercise 3. Let R be a principal ideal domain and $X = \operatorname{Spec}(R)$ the resulting affine scheme. Show that every open set U is affine.

Exercise 4. Let X be a scheme. Show that the following two conditions are equivalent, by using the sheaf axiom:

(i) There is an affine open covering $X = \bigcup_{\lambda \in L} U_\lambda$ such that the rings of local sections $\Gamma(U_\lambda, \mathscr{O}_X)$ are reduced.

(ii) For every open set $V \subset X$ the ring $\Gamma(V, \mathscr{O}_X)$ is reduced.

Recall that a ring R is *reduced* if $g = 0$ is the only nilpotent element $g \in R$. In other words, the zero ideal is a radical ideal.

Abgabe: Bis Donnerstag, den 11. November um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 4

Excercise 1. Let L be an ordered set, viewed as category, and $L \to (\mathrm{Ab})$ be a contravariant functor, comprising abelian groups G_λ, $\lambda \in L$ and *transition maps* $f_{\lambda\mu} : G_\mu \to G_\lambda$, $\lambda \leq \mu$. On the disjoint union $\bigcup_{\lambda \in L} G_\lambda$, we consider the relation

$$a_\lambda \sim a_\mu \quad \Longleftrightarrow \quad f_{\lambda\eta}(a_\lambda) = f_{\mu\eta}(a_\mu) \text{ for some } \eta \geq \lambda, \mu.$$

Assume that the ordered set G is *directed*, that it, for each $\lambda, \mu \in L$ there is some $\eta \in L$ with $\lambda, \mu \leq \eta$. Check that the above is an equivalence relation, and that the set of equivalence classes

$$\varinjlim_{\lambda \in L} G_\lambda = (\bigcup_{\lambda \in L} G_\lambda)/\sim$$

inherits the structure of an abelian group. Furthermore, interpret localizations $S^{-1}R$ and stalks $\mathscr{F}_a$ as such *direct limits.*

Exercise 2. Let $(X, \mathscr{O}_X)$ be a ringed space, and $\mathscr{F}$ be a presheaf of modules. Show that the sheafification $\mathscr{F}^+$, whose groups of local sections $\Gamma(U, \mathscr{F}^+)$ comprises the compatible tuples

$$(s_a)_{a \in U} \in \prod_{a \in U} \mathscr{H}_a,$$

indeed satisfies the sheaf axiom.

Exercise 3. Let $X = \mathbb{A}^1$ be the affine line over some ground field k. Give an example of a non-zero presheaf of modules $\mathscr{F}$ whose sheafification $\mathscr{F}^+$ becomes the zero sheaf.

Exercise 4. Let $\mathscr{F}$ and $\mathscr{G}$ be $\mathscr{O}_X$-modules on some ringed space X.

(i) Define the tensor product sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$.

(ii) Suppose X is a scheme, with $\mathscr{F}$ and $\mathscr{G}$ quasicoherent. Show that the tensor product sheaf $\mathscr{F} \otimes_{\mathscr{O}_X} \mathscr{G}$ is quasicoherent as well.

Abgabe: Bis Donnerstag, den 18. November um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 5

Excercise 1. Let X be a scheme and $Z \subset X$ be a closed subscheme. Verify that the inclusion morphism $i : Z \to X$ is a *monomorphism* in the category $\mathcal{C} = (\text{Sch})$ of schemes. In other words, for each scheme T and each pair of morphisms $f, g : T \to Z$ with $i \circ f = i \circ g$ we already have $f = g$.

Exercise 2. Let X be a scheme, and $X_{\text{red}} \subset X$ be its reduction. Show that for each reduced scheme T, the canonical map

$$\text{Hom}(T, X_{\text{red}}) \longrightarrow \text{Hom}(T, X)$$

is bijective.

Exercise 3. Let X be a noetherian scheme. Verify that every ascending chain $\mathscr{I}_0 \subset \mathscr{I}_1 \subset \ldots$ of quasicoherent sheaves of ideals is stationary. Conclude that every descending chain $Z_0 \supset Z_1 \supset \ldots$ of closed subschemes is stationary.

Exercise 4. Let X be a scheme and $A, B \subset X$ be two closed subschemes. Prove that among the closed subschemes $Z \subset X$ contained in both A and B there is a largest one, which is then written as $Z = A \cap B$.

Abgabe: Bis Donnerstag, den 25. November um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 6

Excercise 1. Let R be a ring and $n \geq 1$ be some integer. We consider the polynomial ring $S = R[u, v]$ as graded with respect to the monoid $\Lambda = \mathbb{Z}/n\mathbb{Z}$ of congruence classes modulo n, by declaring

$$\deg(u) = 1 \quad \text{and} \quad \deg(v) = -1.$$

Show that the subring S_0 is isomorphic to $R[x, y, z]/(z^n - xy)$.

Exercise 2. Let $S = \bigoplus_{\lambda \in \Lambda} S_\lambda$ be a Λ-graded ring, and $\mathfrak{a} \subset S$ be an ideal. Verify that the subgroup

$$\mathfrak{a}^{\text{hgs}} = \bigoplus_{\lambda \in \Lambda} (\mathfrak{a} \cap S_\lambda)$$

inside S is a homogeneous ideal contained in $\mathfrak{a}$, and indeed the largest such ideal. Furthermore, check that $\mathfrak{p}^{\text{hgs}}$ is prime if $\mathfrak{p}$ is prime, provided $\Lambda = \mathbb{N}$.

Exercise 3. Let $S = \bigoplus_{i \geq 0} S_i$ be a graded ring. Show that the homogeneous spectrum $\text{Proj}(S)$ is empty if and only if the irrelevant ideal S_+ consists only of nilpotent elements.

Exercise 4. Let $S = \bigoplus_{i \geq 0} S_i$ be a graded ring. Prove that S is noetherian if and only if S_0 is noetherian and the S_0-algebra S is of finite type.

Abgabe: Bis Donnerstag, den 2. Dezember um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 7

Exercise 1. Let $S = \bigoplus_{i\geq 0} S_i$ be a graded ring, and $M = \bigoplus_{j\in\mathbb{Z}} M_j$ be a graded module. Suppose that M is finitely generated as an ungraded module. Prove that there is a surjection of graded modules

$$\varphi : \sum_{i=0}^{r} S(-d_i) \longrightarrow M$$

for some $r \geq 0$ and some $d_i \geq 0$. What does this statement mean for quasicoherent sheaves $\mathscr{F}$ on $\mathbb{P}^n_R$?

Exercise 2. Let $S = \bigoplus_{i\geq 0} S_i$ be a graded ring, and $M = \bigoplus_j M_j$ be a graded module. Suppose that $S = S_0[S_1]$ and that M is generated by elements of degree $\leq d$. Show that

$$M_{i+j} = S_i M_j$$

for every $j \geq d$ and $i \geq 0$.

Exercise 3. Let X be a scheme. Show the the subset $\operatorname{Pic}^+(X) \subset \operatorname{Pic}(X)$ comprising the invertible sheaves that are globally generated is a submonoid. Verify that for $X = \operatorname{Spec}(R)$ affine, every invertible sheaf is globally generated.

Exercise 4. Let $f : X \to Y$ be a morphism of ringed spaces. For an abelian sheaf $\mathscr{G}$ on Y, we define the abelian sheaf $f^{-1}(\mathscr{G})$ as the sheafification of the presheaf

$$U \longmapsto \varinjlim_{V} \Gamma(V, \mathscr{G}),$$

where the direct limit runs over all open neighborhoods V of the subset $f(U) \subset Y$. Describe the restriction maps, check

$$f^{-1}(\mathscr{G})_a = \mathscr{G}_{f(a)}$$

for $a \in X$, and deduce that the functor $\mathscr{G} \mapsto f^{-1}(\mathscr{G})$ is exact.

Abgabe: Bis Donnerstag, den 9. Dezember um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 8

Exercise 1. Let X be a quasicompact scheme, and $\mathscr{L}$ be an invertible sheaf. Suppose for each point $a \in X$ there is some $n \geq 0$ and a global section $s \in H^0(X, \mathscr{L}^{\otimes n})$ with $s(a) \neq 0$. Deduce that $\mathscr{L}^{\otimes d}$ is globally generated for some exponent $d \geq 0$.

Excercise 2. Let R be a ground ring and $n, d \geq 0$ be two integers. Construct a canonical map

$$R[T_0, \ldots, T_n]_d \longrightarrow H^0(\mathbb{P}^n, \mathscr{O}_{\mathbb{P}^n}(d))$$

by using the basic open sets $D_+(T_j)$, and infer with the sheaf axiom that the map is bijective.

Exercise 3. Fix a ground field k. Let X be a scheme, $\mathscr{O}_X^{\oplus n+1} \to \mathscr{L}$ be an invertible quotient, and

$$f : X \longrightarrow \mathbb{P}^n$$

the resulting morphism with $\mathscr{L} = f^*\mathscr{O}_{\mathbb{P}^n}(1)$. Let $Z \subset X$ be a closed subscheme with $H^0(Z, \mathscr{O}_Z) = k$ and $\mathscr{L}_Z \simeq \mathscr{O}_Z$. Show that Z is non-empty and connected, and that the image $f(Z)$ contains but one point, which must be rational.

Exercise 4. Let $\mathcal{C}$ be a category, $P \in \mathcal{C}$ be an object, and $F : \mathcal{C} \to (\text{Set})$ be a contravariant functor. Show that the map

$$\operatorname{Hom}_{\widehat{\mathcal{C}}}(h_P, F) \longrightarrow F(P), \quad (\Phi_X)_{X \in \mathcal{C}} \longmapsto \Phi_P(\operatorname{id}_P)$$

is bijective.

Abgabe: Bis Donnerstag, den 16. Dezember um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 9

Exercise 1. Let $f : X \to Y$ be a continuous map of topological space, and $\mathscr{F}$ be a flasque sheaf on X. Verify that the direct image $f_*(\mathscr{F})$ is flasque as well.

Exercise 2. Prove Cartan–Serre Vanishing in detail for noetherian affine schemes $X = \operatorname{Spec}(R)$, by using the fact that each R-module M can be embedded into some R-module F whose sheafification $\mathscr{F} = \tilde{F}$ is flasque.

Exercise 3. Let k be a ground field and $n \geq 0$. Show that the map

$$\mathbb{Z} \longrightarrow \mathbb{Z}, \quad t \longmapsto \chi(\mathscr{O}_{\mathbb{P}^n}(t)) = \sum_{i=0}^{n} (-1)^i h^i(\mathscr{O}_{\mathbb{P}^n}(t))$$

can be expressed as a polynomial belonging to $\mathbb{Q}[t]$. Verify that this does not hold in general for the maps $t \mapsto h^i(\mathscr{O}_{\mathbb{P}^n}(t))$.

Exercise 4. Let k be a ground field and $n \geq 1$. Suppose that $Z \subset \mathbb{P}^n$ is a finite subscheme containing at least two points, and let $\mathscr{I} \subset \mathscr{O}_{\mathbb{P}^n}$ be the corresponding quasicoherent sheaf of ideals. Use the long exact sequence in cohomology to prove $H^1(\mathbb{P}^n, \mathscr{I}) \neq 0$.

Abgabe: Bis Donnerstag, den 6. Januar um 23:55 Uhr über ILIAS.

Frohe Weihnachten und guten Rutsch!

Mathematisches Institut
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Algebraic Geometry I

Sheet 10

Exercise 1. Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of scheme. Use the universal properties of fiber products to construct a continuous map

$$|X \times_Y Z| \longrightarrow |X| \times_{|Z|} |Y|$$

on the underlying space of the fiber product to the fiber product of the underlying spaces. Furthermore, give an example with affine schemes where the above map is not bijective.

Exercise 2. Consider the morphism

$$f : \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[T] \longrightarrow \operatorname{Spec} \mathbb{C}[T] = \mathbb{A}^1$$

defined by the complex polynomial $P(T) = T^5 - 1$. Describe for each point $a \in \mathbb{A}^1$ the scheme-theoretic fibers $f^{-1}(a)$ and determine which of them are reduced or irreducible. Do not forget the case $a = \eta$.

Exercise 3. Let $X = \operatorname{Spec}(R)$ be an affine scheme. Show that the diagonal morphism $\Delta : X \to X \times X$ corresponds to the homomorphism

$$R \otimes R \longrightarrow R, \quad f \otimes g \longmapsto fg,$$

and that its kernel $\mathfrak{a} \subset R \otimes R$ is generated by the tensors $1 \otimes g - g \otimes 1$ with $g \in R$.

Exercise 4. Let $f : X \to Y$ be a morphism of schemes, and $g : Y' \to Y$ be a closed embedding. Show that the base-change

$$X' = X \times_Y Y \xrightarrow{\text{pr}_1} X$$

is also a closed embedding.

Abgabe: Bis Donnerstag, den 20. Januar um 23:55 Uhr über ILIAS.

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Algebraic Geometry I

Sheet 11

Exercise 1. Let $C, C' \subset \mathbb{P}^2$ be two curves of degrees $d, d' \geq 0$, respectively. What is the genus $g = h^1(\mathscr{O}_X)$ of the union $X = C \cup C'$?

Exercise 2. Suppose $n \geq 2$, and let $f \in k[T_0, \ldots, T_n]$ be a homogeneous polynomial of degree $d \geq 1$, and $X \subset \mathbb{P}^n$ be the resulting hypersurface. Compute the invariants

$$h^i(\mathscr{O}_X) = \dim_k H^i(X, \mathscr{O}_X)$$

and deduce that the scheme X is geometrically connected.

Exercise 3. (i) Suppose $k = \mathbb{R}$. Give a curve $C \subset \mathbb{P}^2$ of degree $d = 2$ that is irreducible but not geometrically irreducible.

(ii) Suppose now $k = \mathbb{F}_2(T)$. Find some $C \subset \mathbb{P}^2$ that is reduced but not geometrically reduced.

Exercise 4. Consider the curve $C = V_+(xy, x^2)$ in the projective plane $\mathbb{P}^2 = \operatorname{Proj}(k[x, y, z])$. Show that C is irreducible but not reduced, and verify that there is no effective Cartier divisor $D \subset C$ containing the point $a = (0 : 0 : 1)$.

Abgabe: Bis Donnerstag, den 27. Januar um 23:55 Uhr über ILIAS.