OBERSEMINAR ALGEBRAIC GEOMETRY WS 2020/21: ABELIAN VARIETIES

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1. Overview

An *abelian variety* A over a ground field k is a smooth proper connected group scheme. These are truly foundational objects in algebraic geometry. In dimension $g = 1$, this are the *elliptic curves*, which can be described in terms of *Weierstraß equations*

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with discriminant $\Delta \neq 0$. In higher dimensions $g \geq 2$, it is much more difficult to describe equations for abelian varieties explicitly. However, each smooth proper curve C of genus $g \geq 0$ yields an abelian variety $A = \text{Pic}^0_{C/k}$. The same actually holds for any proper scheme X, under suitable assumptions on the ground field and the Picard scheme. Over the complex numbers, any abelian variety A can be seen as a *complex torus* $A(\mathbb{C}) = \mathbb{C}^g/\Gamma$, and the theory goes back to the 19th century, via abelian functions. Note, however, that for $g \geq 2$ most complex tori have algebraic dimension $a < g$, hence do not carry an algebraic structure. Families of abelian varieties over a base schemes are also called *abelian schemes*, although this is perhaps an unfortunate terminology [20].

Much of the modern theory of abelian varieties, in the language of schemes, was developed by David Mumford. To date, his monograph [16] remains the unsurpassed standard reference. The main goal of this seminar is to work through the central *Chapter III, Algebraic Theory Via Schemes*, in order to learn how to handle abelian varieties, purely in terms of schemes. In addition, we could also have more specialized talks, for example on the *Fourier–Mukai transform* on abelian varieties [14], recent work on the *Kummer construction* [11] or the *Moret–Bailly pencil* [22], or any other subject related to abelian varieties.

Time and Place: Monday, 12:30-14:00 in 25.22.O2.81 (maximal 8 participants)

Program:

October 26, Talk 1: Stefan Schröer
Generalities on the notion of abelian varieties.
([16], [19], [5] and [1])

November 2, Talk 3: Bruno Laurent
The theorem of the cube, Basic theory of group schemes.
([16], Sections 10 and 11, pp. 89–107)

November 9, Talk 2: Thuong Tuan Dang
The classifying stack of an abelian variety.

November 16, Talk 4: Jakob Bergvist
Quotients by finite group schemes.
([16], Section 12, pp. 108–122)

November 23, Talk 5: Thuong Tuan Dang
The dual abelian variety in any characteristic.
([16], Section 13, pp. 122–131)

November 30, Talk 6: Johannes Fischer
Duality theory of finite commutative group schemes.
([16], Section 14, pp. 132–142)

December 14, Talk 7: Siddharth Mathur
Cohomology of line bundles.
([16], Section 16, page 150–163)

January 18, Talk 8: Daniel Harrer
Applications to abelian varieties.
([16], Section 15, pp. 143–149)

January 25, Talk 9: Stefan Schröer
Automorphism group schemes for surfaces of general type.

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