# COMPLEX ANALYTIC TORSION FORMS FOR TORUS FIBRATIONS AND MODULI SPACES

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Abstract. We construct analytic torsion forms for line bundles on holomorphic fibrations by tori, which are not necessarily Kähler fibrations. This is done by double transgressing the top Chern class. The forms are given in terms of Epstein zeta functions. Also, we establish a corresponding double transgression formula and an anomaly formula. The forms are investigated more closely for the universal bundle over the moduli space of polarized abelian varieties and for the bundle of Jacobians over the Teichmüller space.

## 0. Introduction.

Let $Z$ be the polarized elliptic curve given by the quotient of $\mathbf{C}$ by the lattice $\Lambda := \mathbf{Z}+\tau\mathbf{Z}$ with $\tau$ in the upper half plane. $Z$ has a canonical projective embedding given by the equation $y^2 = 4x^3 - g_2 x - g_3$. Let $\zeta$ denote the zeta function defined as the holomorphic continuation of

$$\zeta(s) := \sum_{\substack{\mu\in\Lambda^* \\ \mu\neq 0}} \left(\|\mu\|^2\right)^{-s} \qquad\qquad (\mathrm{Re}\ s > 1)$$

with $\|\mu\|^2 := |\mu|^2/\mathrm{Im}\ \tau$ (hence $\zeta$ is $\mathrm{SL}(2,\mathbf{Z})$-invariant). The Kronecker limit formula (1853) states that

$$\zeta'(0) + \log \mathrm{Im}\ \tau = -\frac{1}{12} \log |g_2^3 - 27 g_3^2|^2\ . \tag{0.0}$$

Here $\zeta'(0)$ is just the analytic torsion of $Z$, as $\{\|\mu\|^2 \mid \mu \in \Lambda^*\}$ is the spectrum of the Laplace operator on $Z$. The expression $g_2^3 - 27g_3^2$ on the other side is the discriminant of the elliptic curve. Assume that $g_2$ and $g_3$ are rational. Then $Z$ has an arithmetic model over Spec $\mathbf{Z}$ and the discriminant describes the places in Spec $\mathbf{Z}$ where the fibres of the elliptic curve are singular.

In this case, formula (0.0) may be regarded as a special case of the arithmetic Riemann-Roch theorem [Bo]. One aim of this paper is to construct the analog

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of the left hand side of this formula for abelian varieties of higher dimension and for curves of higher genus. More general, the main purpose of this paper is to construct analytic torsion forms for torus fibrations which do not need to be Kähler fibrations. Torsion forms are the main ingredient of a direct image construction for an Hermitian $K$-theory, which has been developed by Gillet and Soulé [GS1] in the context of Arakelov geometry. Elements of this $K$-theory are represented by holomorphic Hermitian vector bundles and real differential forms on $B$ which are sums of forms of type $(p,p)$, defined modulo $\partial$- and $\overline{\partial}$-coboundaries [S, 4.8].

Let $\pi : M \to B$ be a holomorphic submersion with complex manifolds $M$ and $B$, compact fibres $Z$ and a Kähler metric $g^{TZ}$ on the fibres. Let $\xi$ be a holomorphic vector bundle on $M$, equipped with a Hermitian metric $h^\xi$. Then one can try to define a direct image $\pi_!(\xi, h)$ which will be an element in the Hermitian $K$-theory of $B$. If the cohomology groups $H^q(Z, \xi_{|Z})$ form vector bundles then this direct image should consist of the virtual vector bundle

$$\sum_q (-1)^q (R^q \pi_* \xi, h^q_{L^2}) \tag{0.1}$$

(where $h^q_{L^2}$ is a $L^2$-metric constructed by representing $H^q(Z, \xi_{|Z})$ by harmonic forms) and a certain class $T_{\pi, g^{TZ}}(\xi, h^\xi)$ of forms, which is called the analytic torsion form. These torsion forms have to satisfy a particular double transgression formula and when the metrics $g^{TZ}$ and $h^\xi$ change, they have to change in a special way to make the forms "natural" in Arakelov geometry. They must not depend on metrics on $B$, and their component in degree zero should be the logarithm of the ordinary Ray-Singer torsion [RS].

Such forms were first constructed by Bismut, Gillet and Soulé [BGS2, Th.2.20] for locally Kähler fibrations under the condition that $H^\bullet(Z_x, \xi|_{Z_x}) = 0$ for all $x \in B$. Gillet and Soulé [GS2] and, implicitly, Faltings [F] suggested definitions for more general cases. Then Bismut and the author gave in [BK] an explicit construction of torsion forms $T$ for Kähler fibrations with $\dim H^\bullet(Z_b, \xi|_{Z_b})$ constant on $B$. Let $\int_Z$ denote the integral along the fibres. For a Chern-Weil polynomial $\phi$ and a Hermitian holomorphic vector bundle $(\xi, h^\xi, \overline{\partial}^\xi)$, we shall denote by $\phi(\xi, h^\xi, \overline{\partial}^\xi)$ or $\phi(\xi, h^\xi)$ the Chern-Weil form associated to the canonical Hermitian holomorphic connection on $F$. By $\phi(\xi, \overline{\partial}^\xi)$ or $\phi(\xi)$ we shall denote the corresponding cohomology class. The form $T$ satisfies the double transgression formula

$$\frac{\overline{\partial}\partial}{2\pi i} T_{\pi, g^{TZ}}(\xi, h^\xi) = \operatorname{ch}\left(H^\bullet(Z, \xi|_Z), h^{H^\bullet(Z, \xi|_Z)}\right) - \int_Z \operatorname{Td}(TZ, g^{TZ}) \operatorname{ch}(\xi, h^\xi) \tag{0.2}$$

and for two pairs of metrics $(g_0^{TZ}, h_0^\xi)$ and $(g_1^{TZ}, h_1^\xi)$, $T$ satisfies the anomaly formula

$$\begin{aligned} T_{\pi, g_1^{TZ}}(\xi, h_1^\xi) - T_{\pi, g_0^{TZ}}(\xi, h_0^\xi) &= \widetilde{\operatorname{ch}}\left(H^\bullet(Z, \xi|_Z), h_0^{H^\bullet(Z,\xi|_Z)}, h_1^{H^\bullet(Z,\xi|_Z)}\right) \\ &\quad - \int_Z \Big( \widetilde{\operatorname{Td}}(TZ, g_0^{TZ}, g_1^{TZ}) \operatorname{ch}(\xi, h_0^\xi) + \operatorname{Td}(TZ, g_1^{TZ}) \widetilde{\operatorname{ch}}(\xi, h_0^\xi, h_1^\xi) \Big) \end{aligned} \tag{0.3}$$

modulo $\partial$- and $\overline{\partial}$-coboundaries. Here Td and ch are the Chern-Weil forms corresponding to the Todd class and the Chern class and $\widetilde{\operatorname{Td}}$ and $\widetilde{\operatorname{ch}}$ denote Bott-Chern forms as constructed in [BGS1, §1f].

In this paper, we shall show that the construction of the analytic torsion forms $T$ extends to the following situation: consider a $n$-dimensional holomorphic Hermitian vector bundle $\pi : (E^{1,0}, g^E) \to B$ on a compact complex manifold. Let $\Lambda$ be a lattice, spanning the underlying real bundle $E$ of $E^{1,0}$, so that local sections of $\Lambda$ are holomorphic sections of $E^{1,0}$. Then the fibration $\pi : E^{1,0}/\Lambda^{1,0} \to B$ is a holomorphic torus fibration which is not necessarily flat as a complex fibration.

In this situation, $R^\bullet\pi_*\mathcal{O}_M = (\bigwedge^\bullet E^{*0,1}, \overline{\partial}^{\overline{E}})$, where $\mathcal{O}_M$ is the trivial line bundle and $\overline{\partial}^{\overline{E}}$ is a holomorphic structure canonically induced by the flat and the holomorphic structure on $E^{1,0}$. This vector bundle may be equipped with a Hermitian metric induced by Hodge theory, which is the original metric if the volume of the fibres $Z$ is equal to 1. Classically, the formula

$$\mathrm{ch}\,(\textstyle\bigwedge^\bullet E^{*0,1}) = \frac{c_{\max}}{\mathrm{Td}}(E^{0,1}) \tag{0.4}$$

holds on the cohomological level (see e.g. [H, Th.10.11]) with $c_n$ the top Chern class. Thus, (0.1) suggests that $T$ should satisfy

$$\frac{\overline{\partial}\partial}{2\pi i} T_{\pi,g^E}(\mathcal{O}) = \frac{c_{\max}}{\mathrm{Td}}(E^{0,1}, g^E, \overline{\partial}^{\overline{E}}) \; . \tag{0.5}$$

For two Hermitian structures $g_0^E$ and $g_1^E$ on $E$, one should find the following anomaly formula

$$\begin{aligned} T_{\pi,g_1^E}(\mathcal{O}) - T_{\pi,g_0^E}(\mathcal{O}) = {} & \widetilde{\mathrm{Td}^{-1}}(E^{0,1}, g_0^E, g_1^E, \overline{\partial}^{\overline{E}}) c_{\max}(E^{0,1}, g_0^E, \overline{\partial}^{\overline{E}}) \\ & + \mathrm{Td}^{-1}(E^{0,1}, g_1^E, \overline{\partial}^{\overline{E}}) \widetilde{c_{\max}}(E^{0,1}, g_0^E, g_1^E, \overline{\partial}^{\overline{E}}) \end{aligned} \tag{0.6}$$

modulo $\partial$- and $\overline{\partial}$-coboundaries. In this paper, $T$ shall be constructed by explicitly double transgressing the top Chern class of $E^{0,1}$, which was proven to be 0 in cohomology by Sullivan [Su]. Also we shall derive the corresponding formulas for an equivariant case and the direct images of certain line bundles over $M$. It is well known that the analytic torsion equals 1 for complex tori of dimension greater than 1; we find that in fact the part of degree less than $n-1$ of $T$ vanishes.

Our method closely follows an article of Bismut and Cheeger [BC], in which they investigate eta invariants on real $SL(2n, \mathbf{Z})$ vector bundles. In this article, they consider a quotient of a Riemannian vector bundle by a lattice bundle. Then they find a Fourier decomposition of the infinite-dimensional bundle of sections on the fibres $Z$, which allows them to transgress the Euler class explicitly via an Eisenstein series $\gamma$, i.e.

$$d\gamma = \mathrm{Pf}\left(\frac{\Omega^E}{2\pi}\right) \; ,$$

where $\mathrm{Pf}$ denotes the Pfaffian and $\Omega^E$ the curvature.

The case considered here is a bit more sophisticated because neither the metric nor the complex structure necessarily have any direct relation with the flat structure. Also, it turns out that the direct image holomorphic structure $\overline{\partial}^{\overline{E}}$ on $E^{0,1}$ is not the structure induced by the metric and the original holomorphic structure as in the Kähler case considered in [BK]. In contrast to [BC, Th. 1.13] we avoid the use of some special formulas on Berezinians. We want to emphasize that as in [BC] the use of certain formulas in the Mathai-Quillen calculus [MQ] is crucial in this paper. The formulas which we are using were established by Bismut, Gillet and Soulé in [BGS5].

In the last sections we investigate more closely the case when $\pi$ is a Kähler fibration. In this situation, the formulas by which we construct the torsion forms are far simpler than in the general case. We investigate them in particular for the universal bundle of polarized abelian varieties over their moduli space and for the bundle of Jacobians over the Teichmüller space (where they take a remarkebly simple form). Also we explain their relation with arithmetic characteristic classes given by the arithmetic Riemann-Roch theorem, which gives for $n = 1$ the formula (0.0). In particular, we deduce a formula for the action of Hecke operators on some arithmetic classes.

The first four sections of this article are contained in the author's thesis [K]. Recently, Bismut and Lott investigated their real torsion forms in a similar situation [BL]. In [Be], Berthomieu investigates Torsion forms for the Poincare bundle over the product of a torus and its dual.

## I. Holomorphic Hermitian torus bundles.

Let $\pi : E^{1,0} \to B$ be a $n$-dimensional complex vector bundle on a compact complex manifold $B$, with underlying real bundle $E$. We call $J$ both the complex structure acting on $E$ and on $TB$, with $J \circ J = -1$. Assume we have a lattice bundle $\Lambda \subset E$ spanning $E$. Let the real manifold $M$ be the total space of the fibration $E/\Lambda$, where the fibre $Z_x$ over a point $x \in B$ is given by the torus $E_x/\Lambda_x$.

Let $E^*$ be the dual bundle to $E$, equipped with the complex structure

$$(J\mu)(\lambda) := \mu(J\lambda) \ \ \forall \mu \in E^* , \ \lambda \in E \ .$$

In the same way, one defines $T^*B$ and $T^*M$. We get

$$E^{1,0} = \{\lambda \in E \otimes \mathbf{C} | J\lambda = i\lambda\} \ ,$$

$$E^{0,1} = \{\lambda \in E \otimes \mathbf{C} | J\lambda = -i\lambda\} \ ,$$

and similar equations for $E^{*\,1,0}$, $E^{*\,0,1}$, $T^{1,0}M$, $T^{0,1}M$, etc.

For $\lambda \in E$, we define

$$\lambda^{1,0} := \tfrac{1}{2}(\lambda - iJ\lambda) \ \text{ and } \ \lambda^{0,1} := \tfrac{1}{2}(\lambda + iJ\lambda) \ ,$$

and in the same manner maps $E^* \to E^{*\,1,0}$, $TB \to T^{1,0}B$, etc. Let $\Lambda^* \in E^*$ be the dual lattice bundle

$$\Lambda^* := \{\mu \in E^* | \mu(\lambda) \in 2\pi \mathbf{Z} \, \forall \lambda \in \Lambda\} \ .$$

We set $\Lambda^{1,0} := \{\lambda^{1,0} | \lambda \in \Lambda\}$, similar for $\Lambda^{0,1}$, $\Lambda^{*\,1,0}$ and $\Lambda^{*\,0,1}$. The lattices $\Lambda$ and $\Lambda^*$ induce flat connections $\nabla$ on $E$ and $E^*$ by $\nabla\lambda := 0$ for all local sections $\lambda$ of $\Lambda$ (resp. $\nabla\mu := 0$ for $\mu \in \Gamma^{\,\mathrm{loc}}(\Lambda)$). These connections are dual to each other. We shall always use the same symbol for a connection on $E^{1,0}$, its conjugate on $E^{0,1}$, its realisation on $E$ and the dual induced connections on $E^{*\,1,0}$, $E^{*\,0,1}$ and $E^*$.

Generally, the connection $\nabla$ is not compatible with the complex structure $J$ (i.e. $\nabla J \neq 0$), so it does not extend to $E^{1,0}$. Instead we associate in a canonical way a complex connection $\nabla^{\mathrm{hol}}$ to $\nabla$ and $J$, namely

$$\nabla^{\mathrm{hol}} := \nabla - \frac{1}{2} J \nabla J.$$

The connection $\nabla$ induces a splitting

$$TM = \pi^* E \oplus T^H M \tag{1.0}$$

of the tangent space of $M$. The horizontal lift of $Y \in TB$ to $T^H M$ will be denoted by $Y^H$. By $\nabla'\lambda$, $\nabla''\lambda$ we shall denote the restrictions of $\nabla.\lambda : TB \otimes \mathbf{C} \longrightarrow E \otimes \mathbf{C}$ to $T^{1,0}B$ and $T^{0,1}B$ (we will use the same convention for all connections and for $\mathrm{End}\,(E \otimes \mathbf{C})$-valued one forms on $B$).

**Lemma 1.0.** *The following statements are equivalent:*

*1a) There is a holomorphic structure* $\overline{\partial}^E$ *on* $E^{1,0}$ *such that* $\overline{\partial}^E \lambda^{1,0} = 0$ *for all* $\lambda \in \Gamma^{\mathrm{loc}}(\Lambda)$.

*1b) There is a holomorphic structure* $\overline{\partial}^{\overline{E}}$ *on* $E^{*\,0,1}$ *such that* $\overline{\partial}^{\overline{E}} \mu^{0,1} = 0$ *for all* $\mu \in \Gamma^{\mathrm{loc}}(\Lambda^*)$.

*2a) The complex structure extends to* $M$ *and* $\pi : M \to B$ *is a holomorphic map.*

*2b) The complex structure extends to* $E^*/\Lambda^*$ *and* $\pi : E^{*\,0,1}/\Lambda^{*\,0,1} \to B$ *is a holomorphic map.*

*3a)* $E^{1,0}$ *is a holomorphic vector bundle and* $T^H E^{1,0}$ *is a complex subbundle of* $TE^{1,0}$.

*3b)* $E^{*\,0,1}$ *is a holomorphic vector bundle and* $T^H E^{*\,0,1}$ *is a complex subbundle of* $TE^{*\,0,1}$.

*4a)* $\nabla_{JY} J = J \nabla_Y J$ *on* $E$ *for* $Y \in TB$.

*4b)* $\nabla_{JY} J = -J \nabla_Y J$ *on* $E^*$ *for* $Y \in TB$.

*Proof.* 2) is just a reformulation of 1).

1a)$\Rightarrow$3a): At a point $(x, \Sigma \alpha_i \lambda_i) \in M$, $x \in B$, $\alpha_i \in \mathbf{R}$, $\lambda_i \in \Lambda_x$, $T^H M$ is equal to the image of the homomorphism

$$\Sigma \alpha_i \, T_x \lambda_i \, : \, TB \longrightarrow TM \; .$$

The latter commutes with $J$ by the holomorphy condition on $\Lambda$. Thus, $T^H M$ is invariant by $J$.

3a)$\Rightarrow$4a): For $Y \in TB, \lambda \in \Gamma^{\mathrm{loc}}(\Lambda)$

$$\pi^*(\nabla_{Y^{1,0}} \lambda^{1,0}) = (\pi^* \nabla)_{Y^{H^{1,0}}} (\pi^* \lambda^{1,0}) = [Y^{H^{1,0}}, \pi^* \lambda^{1,0}] \in T^{1,0} Z,$$

thus $\nabla_{Y^{H^{1,0}}} \lambda^{1,0} \in E^{1,0}$. This implies

$$\begin{aligned} 0 &= (1 + iJ) \nabla_{(1-iJ)Y} (1 - iJ) \lambda \\ &= -i \nabla_{(1-iJ)Y} J \lambda + J \nabla_{(1-iJ)Y} J \lambda \qquad\qquad . \\ &= (-i \nabla_Y J - iJ \nabla_{JY} J - \nabla_{JY} J + J \nabla_Y J) \lambda \end{aligned}$$

4a)⇒1b): Set

$$\overline{\partial}^{\overline{E}} := \nabla^{\mathrm{hol}''} \text{ on } E^{*\,0,1};$$

then one verifies that for $Y \in TB, \mu \in \Gamma^{\,\mathrm{loc}}\,(\Lambda^*), \lambda \in \Gamma^{\,\mathrm{loc}}\,(\Lambda^*)$

$$\begin{aligned}(\nabla^{\mathrm{hol}''}\mu^{0,1})(\lambda) &= \overline{\partial}(\mu^{0,1}(\lambda^{0,1})) - \mu^{0,1}(\nabla^{\mathrm{hol}''}\lambda^{0,1}) \\ &= \mu(\nabla''\lambda^{0,1}) - \mu^{0,1}(\nabla^{\mathrm{hol}''}\lambda) \\ &= \mu(\frac{i}{2}\nabla''J\lambda) + \mu^{0,1}(\frac{1}{2}J\nabla''J\lambda) \\ &= \frac{i}{2}\mu^{1,0}(\nabla''J\lambda) \\ &= \frac{i}{2}\mu^{1,0}((\nabla J\lambda)^{0,1}) = 0\end{aligned}$$

The proofs 1b)⇒3b)⇒4b)⇒1a) proceed analogously. □

Note that the connection $\nabla^{\mathrm{hol}}$ induces both the holomorphic structures on $E^{1,0}$ and $E^{*\,0,1}$. Hence its curvature is a $(1,1)$-form. We shall assume for the rest of the article that the conditions in Lemma 1.0 are satisfied.

**Lemma 1.1.** *$\overline{\partial}^{\overline{E}}$ is the holomorphic structure on $E^{*0,1}$ induced by the first direct image sheaf $R^1\pi_*\mathcal{O}$ of the trivial sheaf on the total space of $(E^{1,0}, \overline{\partial}^{E})$.*

*Proof.* Consider the 1-form $\pi^*\mu \in T^*E$ on $E$, $\mu \in \Gamma^{\,\mathrm{loc}}\,(\Lambda^*)$. Then $d^{TE}\pi^*\mu = 0$, as $\mu$ is flat. Hence $\overline{\partial}^{TE^{1,0}}\pi^*\mu^{0,1} = 0$. □

We fix a Hermitian metric $g^E = \langle\, , \,\rangle$ on $E$, i.e. a Riemannian metric with the property

$$\langle J\lambda, J\eta\rangle = \langle\lambda, \eta\rangle \;\; \forall \lambda, \eta \in E\ .$$

This induces a Hermitian metric canonically on $E^*$. We define $\|\lambda\|^2 := \langle\lambda, \overline{\lambda}\rangle$ for $\lambda \in E \otimes \mathbf{C}$. Thus $\|\lambda^{1,0}\|^2 = \frac{1}{2}\|\lambda\|^2$ for $\lambda \in E$. We need to assume that the volume of the fibres $Z$ of $M$ is constant; for simplicity we take it to be equal to 1, as the value of this constant shall not have much effect on our results. The metric induces an isomorphism of real vector bundles $\mathfrak{i} : E \to E^*$, so that $\mathfrak{i} \circ J = -J \circ \mathfrak{i}$.

**Definition 1.0.** Let $\nabla^{\overline{E}}$ be the Hermitian holomorphic connection on $E^{*\,0,1}$ associated to the canonical holomorphic structure in Lemma 1.0.1b). Let ${}^t\theta^* : TB \otimes \mathbf{C} \to \mathrm{End}\,(E^* \otimes \mathbf{C})$ denote the one-form given by

$$ {}^t\theta^* := \nabla - \nabla^{\overline{E}} \tag{1.1}$$

and let $\vartheta$ be the one-form on $B$ with coefficients in $\mathrm{End}\,(E^*)$

$$\vartheta_Y := \mathfrak{i}^{-1}\nabla_Y\mathfrak{i} \;\; \forall Y \in TB\ .$$

$\nabla^{\overline{E}}$ should not be confused with the Hermitian holomorphic connection on $E^{*\,0,1}$ induced by the metric and the holomorphic structure in Lemma 1.0.1a), which we shall not use in this article.

With respect to the natural pairing $E \otimes E^* \to \mathbf{R}$, the transpose of ${}^t\theta^*$ will be denoted by $\theta^*$, thus

$$({}^t\theta^*\mu)(\lambda) = \mu(\theta^*\lambda) \ \forall \mu \in E^*\ ,\ \lambda \in E\ .$$

The adjoints of ${}^t\theta^*$ and $\theta^*$ will be denoted by ${}^t\theta$ and $\theta$. This notation is chosen to be compatible with the notation in [BC]. By definition, ${}^t\theta^*$ satisfies

$$\begin{aligned} {}^t\theta^{*\prime\prime} &: E \otimes \mathbf{C} \longrightarrow E^{1,0}\ , \\ {}^t\theta^{*\prime} &: E \otimes \mathbf{C} \longrightarrow E^{0,1}\ . \end{aligned} \tag{1.2}$$

Notice that the connection $\nabla + \vartheta$ on $E^*$ is just the pullback of $\nabla$ by the isomorphism $\mathfrak{i}^{-1}$.

**Lemma 1.2.** *The Hermitian connection $\nabla^{\overline{E}}$ on $E^{*\,0,1}$ is given by*

$$\nabla^{\overline{E}} = (\nabla + \vartheta)' + \overline{\partial}^{\overline{E}} = \nabla^{\mathrm{hol}} + \vartheta'\ . \tag{1.3}$$

*Its curvature on $E^{*\,0,1}$ is given by*

$$\Omega^{\overline{E}} = \overline{\partial}^{\overline{E}}\vartheta'\ ,$$

*and it is characterized by the equation*

$$\left\langle (\Omega^{\overline{E}} + {}^t\theta{}^t\theta^*)\mu, \nu \right\rangle = i\partial\overline{\partial}\left\langle \mu, J\nu \right\rangle \ \forall \mu, \nu \in \Gamma^{\mathrm{loc}}(\Lambda^*)\ . \tag{1.4}$$

*Proof.* The first part is classical, but we shall give a short proof to illustrate our notations. For all $\mu \in \Gamma^{\mathrm{loc}}(\Lambda^*)$, $\nu \in \Gamma(E^*)$

$$\overline{\partial}\left\langle \mu^{0,1}, \nu^{1,0} \right\rangle = \overline{\partial}((\nu^{1,0})(\mathfrak{i}^{-1}\mu)) = ((\nabla + \vartheta)''\nu^{1,0})(\mathfrak{i}^{-1}\mu)\ ;$$

but also

$$\overline{\partial}\left\langle \mu^{0,1}, \nu^{1,0} \right\rangle = \left\langle \mu^{0,1}, \nabla^{\overline{E}\prime\prime}\nu^{1,0} \right\rangle = (\mathfrak{i}^{-1}\mu)(\nabla^{\overline{E}\prime\prime}\nu^{1,0})\ ,$$

hence $(\nabla + \vartheta)' = \nabla^{\overline{E}\prime}$ on $E^{*0,1}$. To see the second part, one calculates for $\mu, \nu \in \Gamma^{\mathrm{loc}}(\Lambda^*)$

$$\begin{aligned} \partial\overline{\partial}\left\langle \mu^{0,1}, \nu^{1,0} \right\rangle &= \left\langle \nabla^{\overline{E}\prime}\mu^{0,1}, \nabla^{\overline{E}\prime\prime}\nu^{1,0} \right\rangle + \left\langle \mu^{0,1}, \nabla^{\overline{E}\prime}\nabla^{\overline{E}\prime\prime}\nu^{1,0} \right\rangle \\ &= \left\langle \nabla^{\overline{E}\prime}\mu, \nabla^{\overline{E}\prime\prime}\nu \right\rangle + \left\langle \mu^{0,1}, \Omega^{\overline{E}}\nu^{1,0} \right\rangle \\ &= -\left\langle {}^t\theta''{}^t\theta^{*\prime}\mu, \nu \right\rangle - \left\langle \Omega^{\overline{E}}\mu^{0,1}, \nu^{1,0} \right\rangle\ ; \end{aligned}$$

but also

$$\partial\overline{\partial}\left\langle \mu^{1,0}, \nu^{0,1} \right\rangle = \left\langle {}^t\theta'{}^t\theta^{*\prime\prime}\mu, \nu \right\rangle + \left\langle \Omega^{\overline{E}}\mu^{0,1}, \nu^{1,0} \right\rangle\ .$$

Substracting and using (1.2), one finds

$$\begin{aligned} i\partial\overline{\partial}\left\langle \mu, J\nu \right\rangle &= \partial\overline{\partial}\left\langle \mu^{1,0}, \nu^{0,1} \right\rangle - \partial\overline{\partial}\left\langle \mu^{0,1}, \nu^{1,0} \right\rangle \\ &= \left\langle \Omega^{\overline{E}}\mu, \nu \right\rangle + \left\langle ({}^t\theta'{}^t\theta^{*\prime\prime} + {}^t\theta''{}^t\theta^{*\prime}), \mu, \nu \right\rangle \\ &= \left\langle (\Omega^{\overline{E}} + {}^t\theta{}^t\theta^*)\mu, \nu \right\rangle\ . \end{aligned}$$

□

**II. A transgression of the top Chern class.**

In this section, a form $\gamma$ on $B$ will be constructed which transgresses the top Chern class $c_n(\frac{-\Omega^{\overline{E}}}{2\pi i})$ of $E^{0,1}$. $\gamma$, divided by the Todd class, will be shown to equal the torsion form in section IV. We shall use the Mathai-Quillen calculus [MQ] and its version described and used by [BGS5]. Mathai and Quillen observed that for $A \in \mathrm{End}\,(E)$ skew and invertible and $\mathrm{Pf}\,(A)$ its Pfaffian, the forms $\mathrm{Pf}\,(A)(A^{-1})^{\wedge k}$ are polynomial functions in $A$, so they can be extended to arbitrary skew elements of $\mathrm{End}\,(E)$. An endomorphism $A \in \mathrm{End}\,(E^{0,1})$, i.e. $A \in \mathrm{End}\,(E)$ with $J \circ A = A \circ J$, may be turned into a skew endomorphism of $E \otimes \mathbf{C}$ by replacing

$$A \mapsto \tfrac{1}{2}(A - A^*) + \tfrac{1}{2} iJ(A + A^*) \ . \tag{2.0}$$

That is, $A$ is replaced by the operator which acts on $E^{1,0}$ as $-A^*$ and on $E^{0,1}$ as $A$. This is the convention of [BGS5, p. 288] adapted to the fact that we are dealing with $E^{0,1}$ and not with $E^{1,0}$. The same conventions will be applied to $\mathrm{End}\,(TM)$.

With $I_{E^{0,1}} \in \mathrm{End}\,(E^{0,1})$ the identity map, we consider at $Y \in E$ and $b \in \mathbf{R}$

$$\alpha_t := \det{}_{\pi^* E^{0,1}} \left( \frac{-\pi^* \Omega^{\overline{E}}}{2\pi i} - b I_{E^{0,1}} \right) e^{-t(\frac{|Y|^2}{2} + (\pi^*\Omega^{\overline{E}} - 2\pi b J)^{-1})} \tag{2.1}$$

by antisymmetrization as a form on the total space of $E$.

**Definition 2.0.** Let $\overline{\beta}_t \in \bigwedge T^*B$ be the form

$$\overline{\beta}_t := \sum_{\mu \in \Lambda^*} (\mathrm{i}^{-1}\mu)^* \alpha_t \tag{2.2}$$

and define $\beta_t, \widetilde{\beta}_t \in \bigwedge T^*B$ as

$$\beta_t := \overline{\beta}_t|_{b=0} \ , \qquad \widetilde{\beta}_t := \frac{\partial}{\partial b}\Big|_{b=0} \overline{\beta}_t \ .$$

The meaning of $\overline{\beta}_t$ will become clear in the proof of Lemma 4.0, where it is shown to be related to the supertrace which defines the torsion forms. In the following two Lemmas, $c_n$ and $c_{n-1}$ shall denote the Chern polynomials evaluated on $\mathrm{End}(E^{0,1})$-valued 2-forms on $B$.

**Lemma 2.0.** *$\widetilde{\beta}_t$ is given by*

$$\widetilde{\beta}_t = \frac{\partial}{\partial b}\Big|_{b=0} det_{E^{0,1}} \left( \frac{-\Omega^{\overline{E}}}{2\pi i} - b\, I_{E^{0,1}} \right) \sum_{\mu \in \Lambda^*} e^{-\frac{t}{2}\left\langle \mathrm{i}^{-1}\mu, (1 + \theta^*(\Omega^{\overline{E}} - 2\pi b J)^{-1}\theta)\mathrm{i}^{-1}\mu \right\rangle} \tag{2.3}$$

*and*

$$\widetilde{\beta}_t = (2\pi t)^{-n} \frac{\partial}{\partial b}\Big|_{b=0} det_{E^{0,1}} \left( \frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i} - b I_{E^{0,1}} \right) \sum_{\lambda \in \Lambda} e^{-\frac{1}{2t}\left\langle \lambda, (1 + \theta^*(\Omega^{\overline{E}} + \theta\theta^* - 2\pi b J)^{-1}\theta)\lambda \right\rangle} \ . \tag{2.4}$$

*For $t \nearrow \infty$ it has the asymptotics*

$$\widetilde{\beta}_t = -c_{n-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right) + \mathcal{O}(e^{-Ct}) \tag{2.5}$$

*and for $t \searrow 0$*

$$\widetilde{\beta}_t = -(2\pi t)^{-n} c_{n-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\right) + \mathcal{O}(e^{-\frac{C}{t}}). \tag{2.6}$$

*Proof.* We recall that $\theta^* = \nabla^{\overline{E}} - \nabla$ on $E$, hence for $\mu \in \Gamma^{\,\mathrm{loc}}(\Lambda^*)$

$$\nabla^{\overline{E}}(\mathfrak{i}^{-1}\mu) = -\theta\mathfrak{i}^{-1}\mu$$

and one obtains

$$(\mathfrak{i}^{-1}\mu)^*(\pi^*\Omega^{\overline{E}} - 2\pi bJ)^{-1} = \tfrac{1}{2}\left\langle \mathfrak{i}^{-1}\mu, \theta^*(\Omega^{\overline{E}} - 2\pi bJ)^{-1}\theta\mathfrak{i}^{-1}\mu \right\rangle \;.$$

This proves (2.3). To show (2.4), we adopt the notations of [MQ]. Let $\bigwedge[\psi]$ be a $2n$-dimensional exterior algebra with fixed generators $\psi_1, \ldots, \psi_{2n}$; let $(e_i)$ be a local orthonormal basis of $E$ and set

$$\psi := \sum e_i \otimes \psi_i$$

(To avoid choosing bases one might simply take $\bigwedge E^*$ instead of $\bigwedge[\psi]$ and $e_i^*$ instead of $\psi_i$. But taking an abstract exterior algebra helps to avoid confusion in the following calculation). The Berezin integral

$$\int^{\mathcal{D}\psi} : \bigwedge[\psi] \to \mathbf{C}$$

is defined as the linear map which equals one for $\psi_1 \wedge \cdots \wedge \psi_{2n}$ and vanishes on forms of lower degree. By applying [MQ, Prop. 1.8], we get

$$\begin{aligned}
\overline{\beta}_t =& \det{}_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}}{2\pi i} - b\, I_{E^{0,1}}\right) \sum_{\mu \in \Lambda^*} e^{-\frac{t}{2}\left\langle \mathfrak{i}^{-1}\mu, (1+\theta^*(\Omega^{\overline{E}} - 2\pi bJ)^{-1}\theta)\mathfrak{i}^{-1}\mu\right\rangle} \\
=& \left(\frac{-1}{2\pi}\right)^n \mathrm{Pf}\left(\Omega^{\overline{E}} - 2\pi bJ\right) \sum_{\mu \in \Lambda^*} e^{-\frac{t}{2}\left\|\mu\right\|^2 - \frac{t}{2}\left\langle \theta\mathfrak{i}^{-1}\mu, (\Omega^{\overline{E}} - 2\pi bJ)^{-1}\theta\mathfrak{i}^{-1}\mu\right\rangle} \\
=& \left(\frac{-1}{2\pi}\right)^n \int^{\mathcal{D}\psi} \sum_{\mu \in \Lambda^*} e^{\frac{1}{2}\left\langle \psi, (\Omega^{\overline{E}} - 2\pi bJ)\psi\right\rangle + \sqrt{t}\left\langle \theta\mathfrak{i}^{-1}\mu, \psi\right\rangle - \frac{t}{2}\left\|\mu\right\|^2} \\
=& \left(\frac{-1}{2\pi t}\right)^n \int^{\mathcal{D}\psi} \sum_{\mu \in \Lambda^*} e^{\frac{t}{2}\left\langle \psi, (\Omega^{\overline{E}} + \theta\theta^* - 2\pi bJ)\psi\right\rangle - \frac{t}{2}\left\langle \theta^*\psi, \theta^*\psi\right\rangle + t\left\langle \mathfrak{i}^{-1}\mu, \theta^*\psi\right\rangle - \frac{t}{2}\left\|\mathfrak{i}^{-1}\mu\right\|^2}
\end{aligned}$$

(by rescaling the $\psi_i$'s with $\sqrt{t}$)

$$(2.7) \qquad = \left(\frac{-1}{2\pi t}\right)^n \int^{\mathcal{D}\psi} \sum_{\mu\in\Lambda^*} e^{\frac{t}{2}\left\langle \psi,(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ)\psi\right\rangle - \frac{t}{2}\left\|\theta^*\psi-\mathrm{i}^{-1}\mu\right\|^2}$$

Now we apply the Poisson summation formula to obtain

(2.8)

$$\begin{aligned}
\overline{\beta}_t &= \left(\frac{1}{2\pi it}\right)^{2n} \int^{\mathcal{D}\psi} \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\left\|\lambda\right\|^2 - i\left\langle\theta^*\psi,\lambda\right\rangle + \frac{t}{2}\left\langle\psi,(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ)\psi\right\rangle} \\
&= \left(\frac{1}{2\pi it}\right)^{2n} t^n \mathrm{Pf}\left(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ\right) \\
&\qquad \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\left\|\lambda\right\|^2 - \frac{1}{2t}\left\langle -i\theta\lambda,(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ)^{-1}(-i\theta\lambda)\right\rangle} \\
&= \left(\frac{1}{2\pi t}\right)^n \det{}_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}-\theta\theta^*}{2\pi i} - bI_{E^{0,1}}\right) \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\left\langle\lambda,(1+\theta^*(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ)^{-1}\theta)\lambda\right\rangle} .
\end{aligned}$$

□

The above proof shows also

**Lemma 2.1.** *$\beta_t$ is given by*

$$(2.9) \qquad \beta_t = det_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}}{2\pi i}\right) \sum_{\mu\in\Lambda^*} e^{-\frac{t}{2}\left\langle \mathrm{i}^{-1}\mu,(1+\theta^*\Omega^{\overline{E}-1}\theta)\mathrm{i}^{-1}\mu\right\rangle}$$

*and*

$$(2.10) \qquad \beta_t = (2\pi t)^{-n} det_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}-\theta\theta^*}{2\pi i}\right) \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\left\langle\lambda,(1+\theta^*(\Omega^{\overline{E}}+\theta\theta^*)^{-1}\theta)\lambda\right\rangle} .$$

*For $t \nearrow \infty$ it has the asymptotics*

$$(2.11) \qquad \beta_t = c_n\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right) + \mathcal{O}(e^{-Ct})$$

*and for $t \searrow 0$*

$$(2.12) \qquad \beta_t = (2\pi t)^{-n} c_n\left(E^{0,1}, \frac{-\Omega^{\overline{E}}-\theta\theta^*}{2\pi i}\right) + \mathcal{O}(e^{-\frac{C}{t}}) .$$

We define the Epstein zeta function for Re $s > n$

$$(2.13) \qquad \zeta(s) := -\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1}\left(\widetilde{\beta}_t + c_{n-1}\big(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right) dt ,$$

i.e.

$$\zeta(s) = \frac{\partial}{\partial b}\Big|_{b=0} \det{}_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}}{2\pi i} - b\, I_{E^{0,1}}\right)$$
$$\sum_{\substack{\mu\in\Lambda^* \\ \mu\neq 0}} \left\langle \mathfrak{i}^{-1}\mu, \frac{1}{2}(1+\theta^*(\Omega^{\overline{E}} - 2\pi b J)^{-1}\theta)\mathfrak{i}^{-1}\mu \right\rangle^{-s} .$$

Note that $\zeta(s)$ may be written as

(2.14)
$$\begin{aligned}
\zeta(s) =& -\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\widetilde{\beta}_t + (2\pi t)^{-n} c_{n-1}\big(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\big)\right) dt \\
&+ \frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left((2\pi t)^{-n} c_{n-1}\big(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\big) - c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right) dt \\
&- \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\widetilde{\beta}_t + c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right) dt \\
=& -\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}\left(\widetilde{\beta}_t + (2\pi t)^{-n} c_{n-1}\big(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\big)\right) dt \\
&- \frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}\left(\widetilde{\beta}_t + c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right) dt \\
&+ \frac{(2\pi t)^{-n}}{\Gamma(s)(s-n)} c_{n-1}\big(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\big) - \frac{1}{\Gamma(s+1)} c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)
\end{aligned}$$

and this expression is holomorphic for $s \neq n$ by Lemma 2.1. Hence we may define

**Definition 2.1.** Let $\gamma$ be the form on $B$

(2.15) $$\gamma := \zeta'(0)\ .$$

More explicitly, $\gamma$ is given by

(2.16)
$$\begin{aligned}
\gamma = -\int_1^\infty \left(\widetilde{\beta}_t + c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right)\frac{dt}{t} - \int_0^1 \left(\widetilde{\beta}_t + (2\pi t)^{-n} c_{n-1}\left(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\right)\right)\frac{dt}{t} \\
+ \Gamma'(1) c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big) - \frac{1}{n(2\pi)^n} c_{n-1}\left(\frac{-\Omega^{\overline{E}} - \theta\theta^*}{2\pi i}\right)\ ,
\end{aligned}$$

which results in sums over exponential integrals. The value of $\zeta$ at zero is given by

(2.17) $$\zeta(0) = -c_{n-1}\big(\frac{-\Omega^{\overline{E}}}{2\pi i}\big)\ .$$

*Remark.* The added constant $c_{n-1}(\frac{-\Omega^{\overline{E}}}{2\pi i})$ in (2.13) is needed to make the integral converge in a certain domain. It does not effect the value of $\zeta(s)$, because

$$-\frac{1}{\Gamma(s)}\int_1^\infty t^{s-1}dt = \frac{1}{\Gamma(s+1)} \qquad \text{for } -1 < \text{Re } s < 0$$

and

$$-\frac{1}{\Gamma(s)}\int_0^1 t^{s-1}dt = \frac{-1}{\Gamma(s+1)} \qquad \text{for Re } s > 0,$$

thus the sum of the holomorphic continuations of these integrals vanishes.

**Theorem 2.2.** *$\gamma$ satisfies the double-transgression formula*

$$\frac{\overline{\partial}\partial}{2\pi i}\gamma = c_n\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right) . \tag{2.18}$$

*Proof.* By [BGS5, Th. 2.10], one knows that

$$-t\frac{\partial}{\partial t}\alpha_t\Big|_{b=0} = \frac{\overline{\partial}\partial}{2\pi i}\frac{\partial}{\partial b}\Big|_{b=0}\alpha_t . \tag{2.19}$$

The minus sign occuring here in contrast to [BGS5] is caused by the different sign of $J = -iI_{E^{0,1}}$ in our formulas.

We define $\beta^0$ by $\beta_t = t^{-n}\beta^0 + \mathcal{O}(e^{-C/t})$ for $t \searrow 0$ as in Lemma 6. Then one obtains for Re $s > n$

$$\begin{aligned} (2.20)\quad \frac{\overline{\partial}\partial}{2\pi i}\zeta(s) &= \frac{1}{\Gamma(s)}\int_0^\infty t^s\frac{\partial\beta_t}{\partial t}dt \\ &= \frac{1}{\Gamma(s)}\int_0^1 t^s\frac{\partial}{\partial t}(\beta_t - t^{-n}\beta^0)dt - \frac{n}{\Gamma(s)}\int_0^1 t^{s-1-n}\beta^0 dt + \frac{1}{\Gamma(s)}\int_1^\infty t^s\frac{\partial}{\partial t}\beta_t dt \\ &= \frac{1}{\Gamma(s)}\int_0^1 t^s\frac{\partial}{\partial t}(\beta_t - t^{-n}\beta^0)dt + \frac{1}{\Gamma(s)}\frac{n}{n-s}\beta^0 + \frac{1}{\Gamma(s)}\int_1^\infty t^s\frac{\partial}{\partial t}\beta_t dt \end{aligned}$$

and hence for the holomorphic continuation of $\zeta$ to 0

$$\frac{\overline{\partial}\partial}{2\pi i}\zeta'(0) = \lim_{t\nearrow\infty}\beta_t = c_n\left(\frac{-\Omega^{\overline{E}}}{2\pi i}\right) . \tag{2.21}$$

□

**III. Calculation of the holomorphic superconnection.**

The analytic torsion forms of a fibration are defined using a certain superconnection, acting on the infinite-dimensional bundle of forms on the fibres. In this section, this superconnection will be investigated for the torus fibration $\begin{array}{c} M \\ \pi\downarrow \\ B \end{array}$.

Let $F := \Gamma(Z, \bigwedge T^{*\,0,1}Z)$ be the infinite-dimensional bundle on $B$ with the smooth antiholomorphic forms on $Z$ as fibres. By using the holomorphic Hermitian connection $\nabla^{\overline{E}}$ on $E^{*\,0,1}$, one can define a connection $\widetilde{\nabla}$ on $F$ setting

$$\widetilde{\nabla}_Y h := (\pi_* \nabla^{\overline{E}})_{Y^H} h \;\forall Y \in \Gamma(TB)\,,\; h \in \Gamma(B,F)\;.$$

The metric $\langle\,,\rangle$ on $E$ induces a metric on $Z$. Then $F$ has a natural $TZ \otimes \mathbf{C}$ Clifford module structure, given by the actions of

$$c(Z^{1,0}) := \sqrt{2}\mathrm{i}(Z^{1,0}) \wedge \text{ and } c(Z^{0,1}) := -\sqrt{2}\iota_{Z^{0,1}} \;\; \forall z \in TZ\;.$$

$\iota_{Z^{0,1}}$ denotes here interior multiplication. Hence

$$c(Z)c(Z') + c(Z')c(Z) = -2\,\langle Z, Z'\rangle \;\forall Z, Z' \in TZ \otimes \mathbf{C}\;.$$

Let $\overline{\partial}^Z$ be the Dolbeault operator, let $\overline{\partial}^{Z*}$ denote its dual on $Z$ and let

$$D := \overline{\partial}^Z + \overline{\partial}^{Z*}$$

denote the Dirac operator action on $F$. In fact, for an orthonormal basis $(e_i)$ of $TZ \otimes \mathbf{C}$ and the Hermitian connection $\nabla^Z$ on $Z$

$$D = \frac{1}{\sqrt{2}} \sum c(e_i) \nabla^Z_{e_i}\;.$$

A form $\mu = \mu^{1,0} + \mu^{0,1} \in \Lambda^*$ can be identified with a $\mathbf{R}/2\pi\mathbf{Z}$-valued function on $Z$. In particular, the $\mathbf{C}$-valued function $e^{i\mu}$ is well-defined on $Z$. Then one finds the analogue of Theorem 2.7 in [BC].

**Lemma 3.0.** *For $x \in B$, $F_x$ can be orthogonally decomposed into Hilbert spaces*

$$F_x = \bigoplus_{\mu \in \Lambda^*_x} \bigwedge E^{*\,0,1}_x \otimes \{e^{i\mu}\}\;. \tag{3.0}$$

*For $\mu \in \Lambda^*_x$, $\alpha \in \bigwedge E^{*\,0,1}_x$, $D$ acts on $\bigwedge E^{*\,0,1}_x \otimes \{e^{i\mu}\}$ as*

$$D(\alpha \otimes e^{i\mu}) = \frac{ic(\mathrm{i}^{-1}\mu)}{\sqrt{2}} \alpha \otimes e^{i\mu} \tag{3.1}$$

*and*

$$D^2(\alpha \otimes e^{i\mu}) = \tfrac{1}{2}\,|\mu|^2\,\alpha \otimes e^{i\mu}\;. \tag{3.2}$$

*Proof.* The first part of the Lemma is standard Fourier analysis, using that $\mathrm{vol}(\Lambda) = 1$. The second part is obtained by calculating

$$\begin{aligned} &\overline{\partial}^Z(\alpha \otimes e^{i\mu^{1,0}}) = 0\,,\;\; \overline{\partial}^Z(\alpha \otimes e^{i\mu^{0,1}}) = i\mu^{0,1} \wedge \alpha \otimes e^{i\mu^{0,1}}\,,\\ &\overline{\partial}^{*\,Z}(\alpha \otimes e^{i\mu^{0,1}}) = 0\,,\;\; \overline{\partial}^{Z\,*}(\alpha \otimes e^{i\mu^{1,0}}) = -i\,\iota_{\mathrm{i}^{-1}\mu^{1,0}}\alpha \otimes e^{i\mu^{1,0}}\;. \end{aligned} \tag{3.3}$$

□

Now one can determine the action of $\widetilde{\nabla}$ with respect to this splitting. Define a connection on the infinite-dimensional bundle $C^\infty(Z,\mathbf{C})$ by setting

$$\nabla^\infty_Y f := Y^H.f \;\forall Y \in TB\,,\; f \in C^\infty(Z,\mathbf{C})\;.$$

**Lemma 3.1.** *The connection* $\widetilde{\nabla}$ *acts on* $F = \bigwedge E^{*\,0,1} \otimes C^\infty(Z, \mathbf{C})$ *as*

$$\widetilde{\nabla} = \nabla^{\overline{E}} \otimes 1 + 1 \otimes \nabla^\infty \; ;$$

*hence it acts on local sections of* $\bigwedge E^{*\,0,1} \otimes \{e^{i\mu}\}$ *for* $\mu \in \Gamma^{\,\mathrm{loc}}\,(\Lambda^*)$ *as* $\nabla^E \otimes 1$*. In particular,*

$$\widetilde{\nabla}^2 = \Omega^{\overline{E}} \otimes 1 \; .$$

*Proof.* This follows because $\mu$ is a flat local section. $\square$

**Definition 3.0.** The superconnection $A_t$ on $\begin{smallmatrix} F \\ \downarrow \\ B \end{smallmatrix}$, depending on $t \in \mathbf{R}$, $t \geq 0$, given by

$$A_t := \widetilde{\nabla} + \sqrt{t} D$$

is called the Levi-Civita superconnection.

In fact, this definition is the analogue to the Definition 2.1 in [BGS2]; the torsion term appearing there vanishes in the case mentioned here. By Lemma 3.0 and Lemma 3.1, it is clear that $A_t^2$ acts on $\bigwedge E^{*\,0,1} \otimes \{e^{i\mu}\}$, $\mu \in \Gamma^{\,\mathrm{loc}}\,(\Lambda^*)$, as

$$A_t^2 = (\nabla^{\overline{E}} + i\sqrt{\frac{t}{2}} c(\mathrm{i}^{-1}\mu))^2 \otimes 1 \; . \tag{3.4}$$

## IV. The analytic torsion form.

Let $N_H$ be the number operator on $B$ acting on $\bigwedge^p T^*B \otimes F$ by multiplication with $p$. $\mathrm{Tr}_{\,s}\bullet$ will denote the supertrace $\mathrm{Tr}\,(-1)^{N_H}\bullet$. Let $\varphi$ be the map acting on $\bigwedge^{2p} T^*B$ by multiplication with $(2\pi i)^{-p}$. Let $P$ denote the vector space of sums of $(p,p)$-forms and define $P'$ by

$$P' := \{\omega \in P | \exists \text{ forms } \alpha, \beta : \omega = \partial\alpha + \overline{\partial}\beta\}.$$

Let $\mathrm{Td}^{-1}$ and $(\mathrm{Td}^{-1})'$ denote the ad-invariant polynomials which are such that

$$\mathrm{Td}^{-1}(\mathrm{diag}(x_1, ..., x_n)) = \prod_1^n \frac{1 - e^{-x_i}}{x_i}$$

and

$$(\mathrm{Td}^{-1})'(\mathrm{diag}(x_1, ..., x_n)) = \frac{\partial}{\partial b}\Big|_{b=0} \prod_1^n \frac{1 - e^{-x_i - b}}{x_i + b}$$

for a diagonal matrix $\mathrm{diag}(x_1, ..., x_n)$.

**Lemma 4.0.** *In* $P$*, the following equality holds*

$$\varphi\; Tr_{\,s} N_H e^{-A_t^2} = Td^{-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right) \widetilde{\beta}_t - (Td^{-1})'\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right) \beta_t \; . \tag{4.0}$$

*In particular*

$$\varphi\, Tr_s N_H e^{-A_t^2} = Td^{-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right)\widetilde{\beta}_t \ \text{in}\ P/P'\ . \tag{4.1}$$

*Proof.* Define a form $\widehat{\alpha}_t$ on the total space of $E$ with value

$$\widehat{\alpha}_t := \varphi\,\mathrm{Tr}_s N_H \exp\left(-(\nabla^{\overline{E}} + i\sqrt{\frac{t}{2}}c(\lambda))^2\right) \tag{4.2}$$

at $\lambda \in E$. Then one observes

$$\varphi\,\mathrm{Tr}_s N_H e^{-A_t^2} = \sum_{\mu\in\Lambda^*} (\mathrm{i}^{-1}\mu)^*\widehat{\alpha}_t\ . \tag{4.3}$$

Also by [BGS5,Proof of Th. 3.17] one knows that

$$\widehat{\alpha}_t = \frac{\partial}{\partial b}\Big|_{b=0}\mathrm{Td}^{-1}\left(E^{0,1}, \frac{-\pi^*\Omega^{\overline{E}}}{2\pi i} - bI_{E^{0,1}}\right)\alpha_t\ . \tag{4.4}$$

This proves (4.0). Using (3.19), it is clear that

$$\beta_t = \frac{\partial\overline{\partial}}{2\pi i}\int_0^\infty\left(\widetilde{\beta}_t + c_{n-1}\big(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\big)\right)\frac{dt}{t} + c_n\big(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\big)\ ,$$

thus $\beta_t \in P'$. This proves (4.1). □

Lemmas 2.1, 4.0 and Theorem 2.2 show the existence of a form $\omega_\infty$ on $B$ with the property

$$\varphi\,\mathrm{Tr}_s N_H e^{-A_t^2} = \omega_\infty + O(e^{-Ct}) \text{ for } t \nearrow \infty.$$

**Definition 4.1.** As in [BK], we define the analytic torsion form $T_{\pi,g^E}$ to be the derivative at 0 of the zeta function which is given by

$$-\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\varphi\,\mathrm{Tr}_s N_H e^{-A_t^2} - \omega_\infty\right)dt \qquad (\mathrm{Re}\ s > n),$$

Note that the zeta function in [BK] needed to be defined in a more complicated way, as the above integral would generally never converge.

**Theorem 4.1.** *The analytic torsion form $T_{\pi,g^E}$ is given by*

$$T_{\pi,g^E} = Td^{-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right)\gamma \ \ \text{in}\ P/P'. \tag{4.5}$$

*Proof.* This is a consequence of Lemma 8. Using the asymptotic expansion (2.12) of $\beta_t$ and the fact that $\beta_t$ is exact, one shows the exactness of the corresponding term in $T_{\pi,g^E}$ by an explicit calculation similar to (2.20). □

In particular, we deduce from Theorem 2.2

$$\frac{\overline{\partial}\partial}{2\pi i} T_{\pi, g^E} = \left(\frac{c_n}{\mathrm{Td}}\right)\left(\frac{-\Omega^{\overline{E}}}{2\pi i}\right) \quad . \tag{4.6}$$

Now we shall investigate the dependence of $T$ on the metric $g^E$. For two Hermitian metrics $g_0^E, g_1^E$ on $E$ and a Chern-Weil polynomial $\phi$, let $\widetilde{\phi}(E^{0,1}, g_0^E, g_1^E) \in P/P'$ denote the axiomatically defined Bott-Chern class of [BGS1, Sect. 1f)]. It has the following property

$$\frac{\overline{\partial}\partial}{2\pi i}\widetilde{\phi}(E^{0,1}, g_0^E, g_1^E) = \phi(E^{0,1}, g_1^E) - \phi(E^{0,1}, g_0^E) \ .$$

**Corollary 4.2.** *Let $g_0^E, g_1^E$ be two Hermitian metrics on $E$. Then the associated analytic torsion forms change by*

(4.7)

$$T_{\pi, g_1^E} - T_{\pi, g_0^E} = \widetilde{Td^{-1}}(E^{0,1}, g_0^E, g_1^E) c_n(E^{0,1}, g_0^E) + Td^{-1}(E^{0,1}, g_1^E)\widetilde{c_n}(E^{0,1}, g_0^E, g_1^E)$$

*modulo $\partial-$ and $\overline{\partial}-$coboundaries.*

*Proof.* This follows by the uniqueness of the Bott-Chern classes. Using Theorem 2.2, Theorem 4.1 and the characterization of Bott-Chern classes in [BGS1, Th. 1.29], it is clear that

$$T_{\pi, g_1^E} - T_{\pi, g_0^E} = \widetilde{\left(\frac{c_n}{\mathrm{Td}}\right)}(E^{0,1}, g_0^E, g_1^E) \ .$$

The result follows. □

In this proof we make essential use of the fact that we do not assume a Kähler condition.

**V. The equivariant case with coefficients in a line bundle.**

Let $\mu_0$ be a flat section of $E^*$ (e.g. the zero section). $\mu_0$ induces a complex line bundle $L_{\mu_0}$ on $M$ as

$$L_{\mu_0} := E \times \mathbf{C}/\Lambda$$

via the action of $\lambda \in \Lambda$ on $E \times \mathbf{C}$

$$\lambda \cdot (\eta, z) := (\lambda + \eta, e^{i\mu_0(\lambda)} z) \ .$$

Thus, a section of $L_{\mu_0}$ may be represented as a function $s \in C^\infty(E, \mathbf{C})$ which verifies the condition

$$s(\lambda + \eta) = e^{i\mu_0(\lambda)} s(\eta) \qquad (\lambda \in \Lambda, \eta \in E). \tag{5.0}$$

The first Chern class of the restriction of $L_{\mu_0}$ to $Z$ vanishes. The holomorphic and Hermitian structures on the trivial line bundle over $E$ induce a holomorphic structure $\overline{\partial}^L$ and an Hermitian structure on $L_{\mu_0}$. As in section II, one has the Hilbert space decomposition

$$\Gamma(Z, \bigwedge T^{*\,0,1}Z \otimes L_{\mu_0}) = \bigoplus_{\mu\in\Lambda_x^*} \bigwedge E_x^{*\,0,1} \otimes \{e^{i(\mu_0+\mu)}\} \tag{5.1}$$

and the Dirac operator $D = \overline{\partial}^L + \overline{\partial}^{L^*}$ acts on $\bigwedge E_x^{*\,0,1} \otimes \{e^{i(\mu_0+\mu)}\}$ as

$$\begin{aligned} D(\alpha \otimes e^{i(\mu_0+\mu)}) =& \frac{ic(\mathrm{i}^{-1}(\mu_0+\mu))}{\sqrt{2}} \alpha \otimes e^{i(\mu_0+\mu)}, \\ D^2(\alpha \otimes e^{i(\mu_0+\mu)}) =& \tfrac{1}{2} \left|\mu_0+\mu\right|^2 \alpha \otimes e^{i(\mu_0+\mu)} \ . \end{aligned} \tag{5.2}$$

In particular, the cohomology $H^*(Z, L_{\mu_0\,|Z}) \cong \ker D^2$ vanishes for $\mu_0 \notin \Lambda^*$. The action of the curvature of the superconnection $A_{\mu_0,t}$ associated to $L_{\mu_0}$ on $\bigwedge E_x^{*\,0,1} \otimes \{e^{i(\mu_0+\mu)}\}$ is given by

$$A_t^2 = \left(\nabla^{\overline{E}} + i\sqrt{\frac{t}{2}} c(\mathrm{i}^{-1}(\mu_0+\mu))\right)^2 \otimes 1 \ . \tag{5.3}$$

Now consider a flat section $\lambda_0$ of $E$ (e.g. $\lambda_0 = 0$). $\lambda_0$ acts fibrewise as a translation on $M$. The line bundle $L_{\mu_0}$ is invariant under this action, and we let $\lambda_0$ act on $\bigwedge E_x^{*\,0,1} \otimes \{e^{i(\mu_0+\mu)}\}$ as

$$\lambda_0^*\big(\alpha \otimes e^{i(\mu_0+\mu)}\big) := e^{i(\mu_0+\mu)(\lambda_0)} \alpha \otimes e^{i(\mu_0+\mu)} \ . \tag{5.4}$$

This action is chosen in such a way that it is $\Lambda^*$-invariant. Thus, $\mu_0$ may in fact be a section of $E^*/\Lambda^*$ and $\lambda_0^*$ acts trivially on the cohomology. Alternatively one could consider the action

$$\lambda_0^*\big(\alpha \otimes e^{i(\mu_0+\mu)}\big) = e^{i\mu(\lambda_0)} \alpha \otimes e^{i(\mu_0+\mu)} \ . \tag{5.5}$$

which in $\Lambda$-invariant and allowes $\lambda_0$ to be a section of $E/\Lambda$.

The equivariant analytic torsion form with coefficients in $L_{\mu_0}$ shall be defined via the heat kernel

$$\varphi \operatorname{Tr}_s N_H \lambda_0^* e^{-A_{\mu_0,t}^2} \ . \tag{5.6}$$

Using the Hilbert space decomposition , one finds that

$$\varphi \operatorname{Tr}_s N_H \lambda_0^* e^{-A_{\mu_0,t}^2} = \sum_{\mu\in\Lambda^*} \left(\mathrm{i}^{-1}(\mu_0+\mu)\right)^* \widehat{\alpha}_t e^{i(\mu_0+\mu)(\lambda_0)} \ .$$

We set

$$\overline{\beta}_{\mu_0,\lambda_0,t} := \sum_{\mu\in\Lambda^*} \left(\mathrm{i}^{-1}(\mu_0+\mu)\right)^* \alpha_t e^{i(\mu_0+\mu)(\lambda_0)} \ . \tag{5.7}$$

As $\mu_0$ is flat, one has the equation

$$\nabla^{\overline{E}}(\mathfrak{i}^{-1}\mu_0) = -\theta\mathfrak{i}^{-1}\mu_0$$

hence

$$(5.8)\quad \overline{\beta}_{\mu_0,\lambda_0,t} = \det{}_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}}{2\pi i} - b\,I_{E^{0,1}}\right) \sum_{\mu\in\Lambda^*} e^{-\frac{t}{2}\left\langle \mathfrak{i}^{-1}(\mu_0+\mu),(1+\theta^*(\Omega^{\overline{E}}-2\pi bJ)^{-1}\theta)\mathfrak{i}^{-1}(\mu_0+\mu)\right\rangle + i(\mu_0+\mu)(\lambda_0)}$$

and

$$(5.9)\quad \overline{\beta}_{\mu_0,\lambda_0,t} = (2\pi t)^{-n}\det{}_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}-\theta\theta^*}{2\pi i} - bI_{E^{0,1}}\right) \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\left\langle \lambda-\lambda_0,(1+\theta^*(\Omega^{\overline{E}}+\theta\theta^*-2\pi bJ)^{-1}\theta)(\lambda-\lambda_0)\right\rangle + i\mu_0(\lambda)}\ .$$

In particular, one finds for $t\nearrow\infty$ the asymptotics

$$(5.10)\qquad \overline{\beta}_{\mu_0,\lambda_0,t} = \mathcal{O}(e^{-Ct}) + \begin{cases} \det_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}}{2\pi i} - b\,I_{E^{0,1}}\right) & \text{for } \mu_0\in\Lambda^* \\ 0 & \text{otherwise} \end{cases}$$

and for $t\searrow 0$
(5.11)

$$\overline{\beta}_{\mu_0,\lambda_0,t} = \mathcal{O}(e^{-\frac{C}{t}}) + \begin{cases} (2\pi t)^{-n}\det_{E^{0,1}}\left(\frac{-\Omega^{\overline{E}}-\theta\theta^*}{2\pi i} - bI_{E^{0,1}}\right)e^{-i\mu_0(\lambda_0)} & \text{for } \lambda_0\in\Lambda \\ 0 & \text{otherwise.} \end{cases}$$

Using (2.19) and the flatness of $\lambda_0$, one obtains again

$$(5.12)\qquad -t\frac{\partial}{\partial t}\overline{\beta}_{\mu_0,\lambda_0,t}\Big|_{b=0} = \frac{\overline{\partial}\partial}{2\pi i}\frac{\partial}{\partial b}\Big|_{b=0}\overline{\beta}_{\mu_0,\lambda_0,t}\ .$$

One may consider again the Epstein zeta function

$$(5.13)\qquad \zeta_{\mu_0,\lambda_0}(s) := -\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\left(\frac{\partial}{\partial b}\Big|_{b=0}\overline{\beta}_{\mu_0,\lambda_0,t} - \frac{\partial}{\partial b}\Big|_{b=0}\overline{\beta}_{\mu_0,\lambda_0,\infty}\right)dt$$

and we define

$$\gamma_{\mu_0,\lambda_0} := \zeta'_{\mu_0,\lambda_0}(0)\ .$$

We define the equivariant torsion form $T_{\pi,g^E}(L_{\mu_0},\lambda_0)$ as the derivative at 0 of the Mellin transform of the heat kernel (5.5) similar to Definition 4.1. As in section IV, one finds the following Theorem:

**Theorem 5.0.** *The analytic torsion form is given by*

$$T_{\pi,g^E}(L_{\mu_0},\lambda_0) = Td^{-1}\left(E^{0,1}, \frac{-\Omega^{\overline{E}}}{2\pi i}\right)\gamma_{\mu_0,\lambda_0} \qquad in\ P/P'. \tag{5.14}$$

*If* $\mu_0 \in \Lambda^*$ *then*

$$\frac{\overline{\partial}\partial}{2\pi i} T_{\pi,g^E}(L_{\mu_0},\lambda_0) = \left(\frac{c_n}{Td}\right)\left(\frac{-\Omega^{\overline{E}}}{2\pi i}\right) \tag{5.15}$$

*and for two metrics* $g_0^E$, $g_1^E$ *on* $E$

$$T_{\pi,g_1^E}(L_{\mu_0},\lambda_0) - T_{\pi,g_0^E}(L_{\mu_0},\lambda_0) = \widetilde{\left(\frac{c_n}{Td}\right)}(E^{0,1}, g_0^E, g_1^E)\ . \tag{5.16}$$

*If* $\mu_0 \notin \Lambda^*$ *then*

$$\frac{\overline{\partial}\partial}{2\pi i} T_{\pi,g^E}(L_{\mu_0},\lambda_0) = 0 \tag{5.17}$$

*and* $T_{\pi,g^E}(L_{\mu_0},\lambda_0)$ *is independent of* $g^E$.

For elliptic curves the function $T_{\pi,g^E}(L_{\mu_0},0)$ has been calculated by Ray and Singer [RS] in terms of theta series; see also Epstein [E,§7] for $T_{\pi,g^E}(L_{\mu_0},\lambda_0)$. For the action (5.5), the formulas for $\overline{\beta}_{\mu_0,\lambda_0,t}$, $\gamma_{\mu_0,\lambda_0}$ and $T_{\pi,g^E}(L_{\mu_0},\lambda_0)$ still hold when multiplied by a factor $e^{-i\mu_0(\lambda_0)}$.

## VI. The Kähler case.

The analytic torsion forms were only constructed in [BK] for the case where the fibration is Kähler. That means, there had to exist a closed $(1,1)$-form on the total space $M$, so that the decomposition (1.0) is an orthogonal decomposition and the form has to be positive in the vertical direction. Hence it is interesting to see when this happens for the case investigated here.

**Lemma 6.0.** *The fibration* $\begin{smallmatrix} M \\ \downarrow \\ B \end{smallmatrix}$ *is Kähler iff there exists a flat symplectic form* $\omega^E$ *on* $E$*, which is a positive* $(1,1)$*-form with respect to* $J$*, i.e.*

$$\begin{aligned} &(1) \quad \nabla\omega^E = 0\,, \\ &(2) \quad \omega^E(J\lambda, J\eta) = \omega_0^E(\lambda,\eta) \qquad && \forall\,\lambda,\eta\in E\,, \\ &(3) \quad \omega^E(\lambda, J\lambda) > 0 && \forall\,\lambda\in E\ . \end{aligned}$$

*Furthermore, a (1,1)-form on* $M$ *respecting the decomposition (1.0) can induce a Kähler metric on* $M$ *iff* $\pi$ *is a Kähler fibration and* $B$ *is a Kähler manifold.*

The local Kähler condition as posed in [BGS1],[BGS2] holds if such a positive flat symplectic structure exists locally on $B$.

*Proof.* Let $\omega$ be a $(1,1)$-form on $TM$ such that $\omega(T^H M, TZ) = 0$. By $\omega^H$ and $\omega^Z$ we denote the horizontal and the vertical part of $\omega$. Using the decomposition (1.0), the condition $d\omega = 0$ splits into four parts :

i) For $Y_1, Y_2, Y_3 \in TB$ :

$$0 = d\omega(Y_1^H, Y_2^H, Y_3^H) = d\omega^H(Y_1^H, Y_2^H, Y_3^H)\,,$$

ii) for $Y_2, Y_2 \in TB$, $Z \in TZ$ :

$$0 = d\omega(Y_1^H, Y_2^H, Z) = Z.\omega^H(Y_1^H, Y_2^H)\,,$$

iii) for $Y \in TB, Z_1, Z_2 \in TZ$ :

$$0 = d\omega(Y^H, Z_1, Z_2) = (L_{Y^H}\omega^Z)(Z_1, Z_2)\,,$$

iv) for $Z_1, Z_2; Z_3 \in TZ$ :

$$0 = d\omega(Z_1, Z_2, Z_3) = d\omega^Z(Z_1, Z_2, Z_3)\,.$$

The conditions i) and ii) just mean that $\omega^H$ is the horizonal lift of a closed $(1,1)$-form on $B$ (e.g. of 0). Of course, $\omega^H$ is positive iff it corresponds to a Kähler metric on $B$.

If there is a form $\omega^Z$ satisfying condition iii), then its restriction to the zero section of $E$ induces a Kähler form $\omega^E$ on $E$, so that the left $\pi^*\omega^E$ satisfies conditions iii) and iv). Only the following necessary condition remains

iii') There exists a Hermitian metric $g^E$ on $E$, so that for the corresponding Kähler form $\omega^E$ and all $\lambda, \eta \in \Gamma^{\mathrm{loc}}(\Lambda)$

$$\omega^E(\lambda, \eta) = \mathrm{const}\,.$$

On the other hand, $M$ is clearly Kähler if this condition is satisfied. This proves the Lemma. □

The Kähler condition simplifies the geometry considerably:

**Lemma 6.1.** *Assume that $\pi$ is Kähler. Let $g^E := \omega^E(\cdot, J\cdot)$ be induced by a positive flat symplectic structure $\omega^E$. Then*

(1) $\overline{\partial}^E = \overline{\partial}^{\overline{E}}$, *i.e.* $\overline{\partial}^{\overline{E}}$ *is the holomorphic stucture induced by* $\overline{\partial}^E$ *and by the metric.*
(2) ${}^t\theta^* = {}^t\theta = -\frac{1}{2}J\nabla J$ *on* $E^*$ *and* $\theta^* = \theta = -\frac{1}{2}J\nabla J$ *on* $E$.
(3) $\Omega^{\overline{E}} = -\theta^{\wedge 2} = -\frac{1}{4}(\nabla J)^2$.
(4) $\partial\overline{\partial}\|\lambda^{1,0}\|^2/2 = -i\omega^E(\theta\lambda, \theta\lambda) = i\omega^E(\Omega^{\overline{E}}\lambda, \lambda) = \langle \lambda^{1,0}, \Omega^{\overline{E}}\lambda^{0,1}\rangle$.

*Proof.* For $\lambda, \eta \in \Gamma^{\mathrm{loc}}(\Lambda(\Lambda)$,

$$\begin{aligned}\langle \nabla^{\mathrm{hol}}\lambda, \eta\rangle + \langle \lambda, \nabla^{\mathrm{hol}}\eta\rangle &= -\frac{1}{2}\langle J\nabla J\lambda, \eta\rangle - \frac{1}{2}\langle \lambda, J\nabla J\eta\rangle \\ &= -\omega^E(\nabla J\lambda, \eta) + \omega^E(\lambda, \nabla J\eta) \\ &= \nabla\big(\omega^E(J\lambda, \eta) - \omega^E(\lambda, J\eta)\big) = \nabla\langle \lambda, \eta\rangle\ .\end{aligned}$$

Thus the connection $\nabla^{\mathrm{hol}}$ is Hermitian and hence it equals $\nabla^{\overline{E}}$. This implies (2). Now $\nabla^{\mathrm{hol}}$ induces both $\overline{\partial}^{\overline{E}}$ and $\overline{\partial}^{E}$, hence $\overline{\partial}^{\overline{E}} = \overline{\partial}^{E}$. (3) follows by (2) and equation (1.4). Also

$$\begin{aligned}\partial\overline{\partial}\langle\lambda^{1,0},\eta^{0,1}\rangle =&\langle\lambda^{1,0},\Omega^{\overline{E}}\eta^{0,1}\rangle + \langle\nabla^{\overline{E}'}\lambda^{1,0},\nabla^{\overline{E}''}\eta^{0,1}\rangle\\ =&\langle\lambda^{1,0},\Omega^{\overline{E}}\eta\rangle + \langle\nabla^{\overline{E}}\lambda^{1,0},\nabla^{\overline{E}}\eta\rangle\\ =&\langle\lambda^{1,0},-\theta^{\wedge 2}\eta\rangle + \langle(\theta\lambda)^{1,0},\theta\eta\rangle\\ =&-2i\omega^{E}(\theta\lambda,\theta\eta)\ .\end{aligned}$$

Together with the skew-symmetry of $\theta^2$ this proves (4). □

By (1), $T$ coincides with the torsion form in [BK] in this case. Furthermore, the asymptotic terms in (2.6), (2.12) vanish by (2). For the rest of this section, we shall assume that $g^E$ is induced by a flat symplectic form. We shall not distinguish anymore between $\overline{\partial}^{E}$ and $\overline{\partial}^{\overline{E}}$ or between $\nabla^{E}$ and $\nabla^{\overline{E}}$. By Lemma 6.1, the last expression in equation (2.8) equals

$$\begin{aligned}\overline{\beta}_t &= \left(\frac{1}{2\pi t}\right)^n \det{}_{E^{0,1}}(-bI_{E^{0,1}})\sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\langle\lambda,(1+\theta^*(-2\pi bJ)^{-1}\theta)\lambda\rangle}\\ &= \left(\frac{-b}{2\pi t}\right)^n \sum_{\lambda\in\Lambda} e^{-\frac{1}{2t}\|\lambda\|^2-\frac{1}{4\pi bt}\langle\theta\lambda,J\theta\lambda\rangle}\\ &= \sum_{\lambda\in\Lambda} e^{-\frac{1}{t}\|\lambda^{1,0}\|^2}\sum_{l=0}^{n}\frac{(-b)^{n-l}}{(2\pi t)^{n+l}l!}\left(\frac{i}{2}\partial\overline{\partial}\|\lambda^{1,0}\|^2\right)^{\wedge l}\ .\end{aligned}\tag{6.0}$$

Hence

$$\beta_t = \frac{1}{(2\pi t)^{2n}n!}\sum_{\lambda\in\Lambda} e^{-\frac{1}{t}\|\lambda^{1,0}\|^2}\left(\frac{i}{2}\partial\overline{\partial}\|\lambda^{1,0}\|^2\right)^{\wedge n}\tag{6.1}$$

and

$$\widetilde{\beta}_t = \frac{-1}{(2\pi t)^{2n-1}(n-1)!}\sum_{\lambda\in\Lambda} e^{-\frac{1}{t}\|\lambda^{1,0}\|^2}\left(\frac{i}{2}\partial\overline{\partial}\|\lambda^{1,0}\|^2\right)^{\wedge(n-1)}\ .\tag{6.2}$$

One may use these expressions to give a simpler proof of the identity

$$t\frac{\partial}{\partial t}\beta_t = \frac{\partial\overline{\partial}}{2\pi i}\widetilde{\beta}_t\ .$$

The expression for $\widetilde{\beta}_t$ shows the following

**Theorem 6.2.** *The zeta function equals for Re $s<0$*

$$\begin{aligned}\zeta(s) =& \frac{-1}{\Gamma(s)}\int_0^\infty t^{s-1}\widetilde{\beta}_t dt\\ =& \frac{\Gamma(2n-s-1)}{\Gamma(s)(n-1)!}\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{\left(\frac{-\partial\bar\partial}{4\pi i}\|\lambda^{1,0}\|^2\right)^{\wedge(n-1)}}{\left(\|\lambda^{1,0}\|^2\right)^{2n-1-s}}\\ =& \frac{\Gamma(2n-s-1)}{\Gamma(s)(n-1)!}\sum_{\lambda\in\Lambda\setminus\{0\}} \frac{\left\langle\lambda^{1,0},\frac{-\Omega^E}{2\pi i}\lambda^{0,1}\right\rangle^{\wedge(n-1)}}{\left(\|\lambda^{1,0}\|^2\right)^{2n-1-s}}\ .\end{aligned}$$

**VII. Hecke operators and the relation with Arakelov geometry.**

In this section we shall investigate the action of Hecke operators on the torsion forms. For this purpose we assume that $\omega^E$ is a principal polarization, i.e. $2\pi\omega^E$ maps $\Lambda$ to $\Lambda^*$.

Let $(E,\Lambda,\omega^E)\to B$ be a bundle of principally polarized abelian varieties and let $\alpha(E,\Lambda,\omega^E)\in\bigwedge^* T^*B$ be a differential form associated to $(E,\Lambda,\omega^E)$ in a functorial way: If $f:B'\to B$ is a holomorphic map and $(f^*E,f^*\Lambda,f^*\omega^E)$ the induced bundle over $B'$, then $\alpha(f^*E,f^*\Lambda,f^*\omega^E)=f^*\alpha(E,\Lambda,\omega^E)$; in other words, $\alpha$ shall be a modular form. Choose an open cover $(U_i)$ of $B$ such that the bundle trivializes over $U_i$. To define the Hecke operator $T(p)$, $p$ prime, associated to the group $Sp(n,\mathbf{Z})$, consider on $U_i$ the set $\mathcal{L}(p)$ of all maximal sublattices $\Lambda'\subset\Lambda_{|U_i}$ such that $\omega^E$ takes values in $p\mathbf{Z}$ on $\Lambda'$. The sums

$$T(p)\alpha(E,\Lambda,\omega^E)_{|U_i} := \sum_{\Lambda'\in\mathcal{L}(p)}\alpha(E,\Lambda',\frac{\omega^E}{p}) \tag{7.0}$$

patch together to a globally defined differential form on $B$. Note that the set $\mathcal{L}(p)$ may be identified with the set of all maximal isotropic subspaces (Lagrangians) $\Lambda'/p\Lambda$ of the symplectic vector space $(\Lambda/p\Lambda,\omega^E)$ over $\mathbf{F}_p$.

**Lemma 7.0.** *The zeta function is for each $s$ an eigenfunctions for $T(p)$ with eigenvalues*

$$T(p)\zeta(s) = \prod_{i=1}^{n-1}(p^i+1)\left(p^{n-s}+p^s\right)\zeta(s)\ . \tag{7.1}$$

*In particular*

$$T(p)\gamma = \prod_{i=1}^{n}(p^i+1)\,\gamma + \prod_{i=1}^{n-1}(p^i+1)\cdot(1-p^n)\log p\cdot\zeta(0)\ . \tag{7.2}$$

*Proof.* Set $\nu_p(n):=\#\mathcal{L}(p)$ and $\nu_1^v(n):=\#\{\Lambda'\in\mathcal{L}(p)\mid v\in\Lambda'/p\Lambda\}$ for $v\in\Lambda/p\Lambda\setminus\{0\}$. Consider the space $V_v:=\{w\in\Lambda/p\Lambda\mid\omega^E(w,v)=0\}/\langle v\rangle$. This $2n-2$-dimensional space carries a canonically induced symplectic structure and the Lagrangians $L$ in $V_v$ are in one-to-one correspondence with the Lagrangians $L+\langle v\rangle \subset \Lambda/p\Lambda$ containing $v$. Thus $\nu_1^v(n) = \nu_p(n-1)$; in particular, $\nu_1(n) := \nu_1^v(n)$ is independant of $v$. Now each of the $p^{2n}-1$ non-zero vectors in $\Lambda/p\Lambda$ occurs in $\nu_1(n)$ Lagrangians, and each Lagrangian subspace contains $p^n-1$ non-zero vectors. Hence

$$(p^n-1)\nu_p(n) = (p^{2n}-1)\nu_1(n) \tag{7.3}$$

or $\nu_p(n) = (p^n+1)\nu_1(n)$. Set for $\lambda \in \Lambda$

$$\alpha_\lambda(s) := \frac{\Gamma(2n-s-1)}{\Gamma(s)(n-1)!} \frac{\left\langle \lambda^{1,0}, \frac{-\Omega^E}{2\pi i}\lambda^{0,1}\right\rangle^{\wedge(n-1)}}{\left(\|\lambda^{1,0}\|^2\right)^{2n-1-s}} .$$

Neither $\nabla$ nor $J$ change when we replace $\Lambda$ by a sublattive $\Lambda'$. Hence $\theta$ does not change and we find

$$\begin{aligned}
T(p)\zeta(s) &= \nu_1(n)p^{n-s} \sum_{\substack{\lambda\in\Lambda\setminus p\Lambda\\ \lambda\neq 0}} \alpha_\lambda(s) + \nu_p(n)p^{n-s} \sum_{\substack{\lambda\in p\Lambda\\ \lambda\neq 0}} \alpha_\lambda(s)\\
&= \nu_1(n)p^{n-s} \sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \alpha_\lambda(s) + \nu_1(n)p^{s} \sum_{\substack{\lambda\in\Lambda\\ \lambda\neq 0}} \alpha_\lambda(s)\\
&= \nu_1(n)(p^{n-s}+p^s)\zeta(s) .
\end{aligned}$$

□

Note that the additional factor $\mathrm{Td}(E^{0,1})^{-1}$ of the torsion form commutes with the Hecke operators, since the curvature is invariant under passing to a sublattice and scaling the metric. Also the Hecke operators commute with $\partial$ and $\overline{\partial}$. Thus the Lemma implies that $T_{\pi,g^E} \in P/P'$ is an eigenfunction iff $c_{n-1}\left(\frac{-\Omega^E}{2\pi i}\right)$ vanishes in cohomology. As shall be explained at the end of this section, there are good reasons why $T_{\pi,g^E}$ is not an eigenfunction in general.

More general Hecke operators $T(D)$ are associated to a tuple $D = (d_1,\dots,d_n,e_1,\dots,e_n)$ of integers with $d_ie_i = \mu =$ const. for all $1\le i\le n$. One takes the sum over all maximal sublattices $\Lambda'\subset\Lambda$ with the property that there is a symplectic basis $(\lambda_1,\dots,\lambda_n,\eta_1,\dots,\eta_n)$ of $\Lambda$ such that $\Lambda'$ is generated by $(d_1\lambda_1,\dots,d_n\lambda_n,e_1\eta_1,\dots,e_n\eta_n)$. We denote this set of sublattices by $\mathcal{L}(D)$.

**Lemma 7.1.** *$\zeta(s)$ is an eigenfunctions for $T(D)$.*

*Proof.* Consider $\nu^\lambda(n) := \#\{\Lambda'\in\mathcal{L}(D) \mid \lambda\in\Lambda'\}$. $\nu^\lambda(n)$ is invariant under the action of $Sp(n,\mathbf{Z})$ on $\lambda$. The orbits of $Sp(n,\mathbf{Z})$ are the set of primitive lattice elements and its multiples. Hence for $d|\mu$ the multiplicity $\nu^d(n) := \nu^\lambda(n)$ is constant for $\lambda\in d\Lambda\setminus\bigcup\left\{d'\Lambda \,\middle|\, d'|\mu, d|d', d\neq d'\right\}$. Define recursively

$$\nu_d(n) := \nu^d(n) - \sum_{\substack{d'|d\\ d'\neq d}} \nu_{d'}(n) .$$

With the same argument as above one obtains

$$T(D)\zeta(s) = \mu^{n-s}\left(\sum_{d|\mu}\nu_d(n)d^{-2n+2s}\right)\zeta(s)\ . \tag{7.4}$$

□

The main application of torsion forms is the construction of a direct image in Arakelov geometry, which is used in the arithmetic Riemann-Roch theorem by Gillet and Soulé [GS3]. We shall explain this relation briefly without going into the details of arithmetic geometry. We use the concepts and the notation of [GS1],[GS2] and [S]. Assume that there is a projective flat map $f:\mathcal{M}\to\mathcal{B}$ between arithmetic varieties, i.e. regular quasi-projective flat schemes over Spec $\mathbf{Z}$, such that $M=\mathcal{M}(\mathbf{C})$, $B=\mathcal{B}(\mathbf{C})$ and $f$ induces the map $\pi: M\to B$. Let $a: P/P'\to\widehat{CH}(B)$ be the canonical map to Arakelov Chow groups and let $\widehat{\mathrm{ch}}$, $\widehat{\mathrm{Td}}$ denote the arithmetic Chern character and the Todd class. Let $\mathrm{Td}^A(T\mathcal{M}/\mathcal{B})$ denote the Gillet-Soulé Todd class (involving the $R$-genus) of the relative tangent sheaf of $f$. The arithmetic Grothendieck-Riemann-Roch theorem conjectured by Gillet-Soulé [GS2] states for the direct image of the sheaf $\mathcal{O}_\mathcal{M}$

$$\widehat{\mathrm{ch}}\big(\sum_{q=0}^{n}(-1)^qR^qf_*\mathcal{O}_\mathcal{M},g^E\big)+a(T_{\pi,g^E}) = f_*\mathrm{Td}^A(T\mathcal{M}/\mathcal{B})\ . \tag{7.5}$$

As the metric of the fibres is flat, the pushforward of the $R$-genus vanishes and the above formula gets

$$\widehat{\mathrm{ch}}\big(\sum_{q=0}^{n}(-1)^qR^qf_*\mathcal{O}_\mathcal{M},g^E\big)+a\big(\mathrm{Td}^{-1}\left(E^{0,1},\frac{-\Omega^{\overline{E}}}{2\pi i}\right)\gamma\big) = f_*\widehat{\mathrm{Td}}(T\mathcal{M}/\mathcal{B})\ . \tag{7.6}$$

Assume that $R^qf_*\mathcal{O}_\mathcal{M}=\Lambda^qR^1f_*\mathcal{O}_\mathcal{M}$; then one obtains

$$\widehat{\left(\frac{c_n}{\mathrm{Td}}\right)}\big((R^1f_*\mathcal{O}_\mathcal{M})^*,g^E\big)+a\big(\mathrm{Td}^{-1}\left(E^{0,1},\frac{-\Omega^{\overline{E}}}{2\pi i}\right)\gamma\big) = f_*\widehat{\mathrm{Td}}(T\mathcal{M}/\mathcal{B})\ , \tag{7.7}$$

in particular

$$\widehat{c_n}\big((R^1f_*\mathcal{O}_\mathcal{M})^*,g^E\big)+a(\gamma) = \big(f_*\widehat{\mathrm{Td}}(T\mathcal{M}/\mathcal{B})\big)^{(n)}\ . \tag{7.8}$$

Assume that the action of the Hecke operators is still well-defined. One cannot expect that the two arithmetic classes in (7.8) are $Sp(n,\mathbf{Z})$-invariant, as this does not even hold for $n=1$. But formula (7.8) implies that their difference is $Sp(n,\mathbf{Z})$-invariant, and using Lemma 7.0. one obtains

$$\begin{aligned}\left(T(p)-\prod_{i=1}^{n}(p^i+1)\right)\Big(\big(f_*\widehat{\mathrm{Td}}(T\mathcal{M}/\mathcal{B})\big)^{(n)}-\widehat{c_n}\big((R^1f_*\mathcal{O}_\mathcal{M})^*,g^E\big)\Big)&\\ =\prod_{i=1}^{n-1}(p^i+1)\cdot(1-p^n)\log p\cdot c_{n-1}\big(\frac{-\Omega^E}{2\pi i}\big)\ .&\end{aligned} \tag{7.9}$$

**VIII. The moduli spaces of abelian varieties and curves.**

We shall have a closer look at the torsion form for the universal bundle of abelian varieties. We shall use the description and notation of [LB]. Let $D := \mathrm{diag}(d_1, \dots, d_n)$ with $d_1, \dots, d_n \in \mathbf{N}$ being a polarization type and consider the Siegel upper half space

$$\mathfrak{H}_n := \{Z = X + iY \in \mathrm{End}(\mathbf{C}) |{}^t Z = Z, Y > 0\} \ ,$$

which is the universal covering of the moduli space for polarized abelian varieties of type $D$. Due to an unavoidable clash of notations, we are forced here to use the letters $Z$ and $Y$ again. Choose the trivial $\mathbf{C}^n$-bundle over $\mathfrak{H}_n$ as the holomorphic vector bundle $E^{1,0}$ and define the lattice $\Lambda^{1,0}$ over a point $Z \in \mathfrak{H}_n$ as

$$\Lambda^{1,0}_{|Z} := (Z, D)\mathbf{Z}^{2n}$$

where $(Z, D)$ denotes a $\mathbf{C}^{n\times 2n}$-matrix. The polarization defines a Kähler form on $E$; the associated metric is given by

$$\|Zr + Ds\|^2_{|Z} = {}^t(Zr + Ds)Y^{-1}\overline{(Zr + Ds)} \qquad \text{for } r, s \in \mathbf{Z}^n.$$

(one might scale the metric by a constant factor $(\det D)^{-1/n}/2$ to satisfy the condition $\mathrm{vol}(Z) = 1$. The torsion form is invariant under this scaling). The fibration $E^{1,0}/\Lambda^{1,0}$ is the universal family of polarized abelian varieties over the moduli space.

**Lemma 8.0.** *The $End(E)$-valued forms $\theta$ and $\Omega^E$ are given by*

$$\theta = \frac{1}{2i} d\overline{Z}\, Y^{-1} \qquad \text{on } E^{1,0},$$
$$\theta = -\frac{1}{2i} dZ\, Y^{-1} \qquad \text{on } E^{0,1}$$

*and*

$$\Omega^E = -\frac{1}{4} dZ\, Y^{-1} d\overline{Z}\, Y^{-1}.$$

*Proof.* For $\lambda, \eta \in \Lambda$,

$$\overline{\partial}\langle\lambda^{1,0}, \eta^{0,1}\rangle = \langle\lambda^{1,0}, \nabla^E\eta^{0,1}\rangle = \langle\lambda^{1,0}, \theta\eta^{1,0}\rangle \ .$$

Choose $\eta^{1,0} = Ds$, $s \in \mathbf{Z}^n$; then

$$\begin{aligned}\overline{\partial}\langle\lambda^{1,0}, \eta^{0,1}\rangle &= \lambda^{1,0}\overline{\partial}Y^{-1}Ds = \lambda^{1,0}(-Y^{-1}\overline{\partial}Y \cdot Y^{-1})Ds \\ &= \lambda^{1,0}Y^{-1}(\frac{1}{2i}d\overline{Z}\, Y^{-1})Ds \ ,\end{aligned}$$

thus $\theta\eta^{1,0} = \frac{1}{2i}d\overline{Z}\, Y^{-1}\eta^{1,0}$ and $\theta\eta^{0,1} = \overline{\theta\eta^{1,0}} = -\frac{1}{2i}dZ\, Y^{-1}\eta^{0,1}$. □

Now we investigate the torsion forms associated to the bundle of Jacobian varieties over the Teichmüller space $\mathfrak{T}_n$ of curves $C$ of genus $n$ [GH, Ch. 2.7], [N, Ch. 4.1]. Let $\sigma : \mathfrak{C} \to \mathfrak{T}_n$ denote the universal bundle of curves. By the Kodaira-Spencer isomorphism the tangent space of $\mathfrak{T}_n$ may be identified with the first derived image $R^1\sigma_* K^*$ with $K_{|s} = T^{*1,0}C_s$, $s \in \mathfrak{T}_n$. The homology group $H_1(C;\mathbf{Z})$ forms a lattice in $(R^0\sigma_* K)^*$ via the pairing

$$\lambda(\varphi) = \int_\lambda \varphi$$

for $\varphi \in H^0(C,K)$ and a smooth representative $\lambda \in H_1(C;\mathbf{Z})$. Then the Jacobian variety associated to the curve $C$ is defined as

$$H^0(C,K)^*/H_1(C;\mathbf{Z}) \ .$$

The intersection product in $H_1(C;\mathbf{Z})$ induces a canonical principal polarization on this torus, which induces on $E^{*1,0} := R^0\sigma_* K$ the metric

$$g^E(\varphi, \overline{\varphi'}) := \frac{i}{2}\int_C \varphi \wedge \overline{\varphi'} \ .$$

**Lemma 8.1.** *The form $\theta : T^{1,0}\mathfrak{T}_n \otimes E^{*1,0} \to E^{*0,1}$ is given by the composition of the cup product with the Hodge diamond isomorphism*

$$\theta : H^1(C,K^*) \otimes H^0(C,K) \to H^1(C,\mathcal{O}) \cong \overline{H^0(C,K)} \ . \tag{8.0}$$

*In particular, for $\lambda \in H_1(C;\mathbf{Z})$ and $X \in H^1(C,K^*)$,*

$$\theta_X \lambda^{0,1} = \int_\lambda X \in H^0(C,K)^* \tag{8.1}$$

*and*

$$\theta_{\overline{X}} \lambda^{1,0} = \int_\lambda \overline{X} \in \overline{H^0(C,K)}^* \ . \tag{8.2}$$

*Proof.* We shall apply Lemma 8.0 to a map $\Pi : \mathfrak{T}_n \to \mathfrak{H}_n$ such that the pullback of the universal bundle is the bundle of Jacobians. Let $(\lambda_1,\ldots,\lambda_n,\eta_1,\ldots,\eta_n)$ be a symplectic basis of $H_1(C;\mathbf{Z})$. Then there is a unique basis $(\varphi_1,\ldots,\varphi_n)$ of $H^0(C,K)$ such that $\int_{\lambda_i}\varphi_j = \delta_{ij}$. The map

$$\begin{aligned} \Pi : \mathfrak{T}_n &\to \mathfrak{H}_n \\ s &\to (\int_{\eta_i} \varphi_j)_{ij} \end{aligned}$$

is a holomorphic period map for the bundle of Jacobians. One can show that

$$\mathrm{Im}\,\Pi = \left(\frac{i}{2}\int_C \varphi_i \wedge \overline{\varphi_j}\right)_{ij}$$

and

$$\partial\Pi(X) = \left(\int_C (\varphi_i \otimes \varphi_j)(X)\right)_{ij} \qquad (X \in T^{1,0}\mathfrak{T}_n) \tag{8.3}$$

[N, Ch. 4.1]. By Lemma 8.0 we find for $\lambda^{1,0} \in \overline{H^0(C,K)}^*$, $X \in T^{1,0}\mathfrak{T}_n$

$$\begin{aligned}\varphi_i\big(\theta_X \lambda^{0,1}\big) &= {}^t\big(\langle\varphi_i(X),\varphi_j\rangle\big)_j \big(\langle\varphi_j,\overline{\varphi_k}\rangle\big)^{-1}_{jk} \big(\lambda^{0,1}(\overline{\varphi_k})\big)_k \\ &= \lambda^{0,1}\big(\varphi_i(X)\big)\ .\end{aligned} \tag{8.4}$$

□

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