# Holomorphic torsion on Hermitian symmetric spaces

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May 18, 2000

**Abstract**

We calculate explicitly the equivariant holomorphic Ray-Singer torsion for all equivariant Hermitian vector bundles over Hermitian symmetric spaces $G/K$ of the compact type with respect to any isometry $g \in G$. In particular, we obtain the value of the usual non-equivariant torsion. The result is shown to provide very strong support for Bismut's conjecture of an equivariant arithmetic Grothendieck-Riemann-Roch theorem.

## Contents

# 1 Introduction

The Ray-Singer analytic torsion is a positive real number associated to the spectrum of the Kodaira-Laplacian on Hermitian vector bundles over compact Hermitian manifolds [21]. It was shown by Quillen, Bismut, Gillet and Soulé that the torsion provides a metric with very beautiful properties on the determinant line bundle of direct images in $K$-theory over Kähler manifolds. This way the torsion can be regarded as a part of a direct image of Hermitian vector bundles.

The main application of this construction is related to arithmetic geometry. Extending ideas of Arakelov, Gillet and Soulé constructed for arithmetic varieties $\mathcal{X}$ (i.e. flat regular quasi-projective schemes over Spec $\mathbf{Z}$ with projectiv fibre $\mathcal{X}_{\mathbf{Q}}$ over the generic point) a Chow intersection ring and a $K$-theory by using differential geometric objects on the Kähler manifold $X := \mathcal{X} \otimes \mathbf{C}$ [22]. In particular, the $K$-theory consists of arithmetic vector bundles on $\mathcal{X}$ with Hermitian metric over $X$ and certain classes of differential forms. Using the torsion as part of a direct image, Bismut, Lebeau, Gillet and Soulé were able to prove an arithmetic Grothendieck-Riemann-Roch theorem relating the determinant of the direct image in the $K$-theory to the direct image in the arithmetic Chow ring. One of the most difficult steps was to show the compatibility of the conjectured theorem with immersions. For a generalization of these concepts to higher degrees, see Bismut-Köhler [7] and Faltings [12].

Another important step in the proof of the theorem was its explicit verification for the canonical projection of the projective spaces to Spec $\mathbf{Z}$ by Gillet, Soulé and Zagier [13]. In particular, the Gillet-Soulé $R$-genus, a rather complicated characteristic class occuring in the theorem was determined this way. The discovery of the same genus in a completely different calculation of secondary characteristic classes associated to short exact sequences by Bismut gave further evidence for the theorem.

In [17], an equivariant version of the analytic torsion was introduced and calculated for rotations with isolated fixed points of complex projective spaces. The result was remarkable for two reasons: First, it contained in any dimension (already for $\mathbf{P}^1\mathbf{C}$) a function $R^{\mathrm{rot}}$ with lots of functional properties, which resembles the power series defining the Gillet-Soulé $R$-genus found by far more extensive calculations. Second, it had the form of a Lefschetz fixed point formula where $R^{\mathrm{rot}}$ appeared as a factor in the contributions at the fixed points.

This second observation led Bismut to conjecture an equivariant arithmetic Grothendieck-Riemann-Roch formula [6]. Redoing his calculations concerning short exact sequences, he found an equivariant characteristic class $R$ which equals the Gillet-Soulé $R$-genus in the non-equivariant case and the function $R^{\mathrm{rot}}$ in the case of isolated fixed points. He conjectured that this class should replace the Gillet-Soulé $R$-genus in an equivariant formula. In [5], he was able to show the compatibility of his conjecture with immersions. Nevertheless, there is still no definition of equivariant arithmetic Chow rings or $K$-theories.

In this paper, we calculate the equivariant torsion for all compact Hermitian symmetric spaces $G/K$ with respect to the action of any $g \in G$ (Theorem 9). In section 9, we show that the result fits perfectly well with Bismut's conjecture. In particular, one gets for any dimension of the fixed point set the most significant part of the Bismut $R$-genus. For isolated fixed points, one reobtains the function $R^{\mathrm{rot}}$. In the first sections we only consider the trivial line bundle on $G/K$ because of the relative simplicity of the result in this case. In the last section, we calculate the torsion for any equivariant vector bundle. The result is of interest also in the non-equivariant case: The torsion was known only for very few manifolds; the projective spaces, the elliptic curves and the tori of dimension $> 2$ (for which it is zero for elementary reasons). Also, Wirsching [23] found a complicated algorithm for the determination of the torsion of complex Grassmannians $G(p, n)$, which allowed him to calculate it for $G(2, 4)$, $G(2, 5)$ and $G(2, 6)$. Thus, our results extend largely the known examples for the torsion. A similar calculation of the real analytic torsion of odd-dimensional symmetric spaces leads to the diffeomorphy classification of some locally symmetric spaces [18].

# 2 Definition of the torsion

We repeat here the definition of an equivariant torsion which we gave in [17]. Let $M$ be a compact $n$-dimensional Kähler manifold with Kähler form $\omega$. Consider a Hermitian holomorphic vector bundle $E$ on $M$ and let

$$\bar\partial : \Gamma(\Lambda^q T^{*0,1}M \otimes E) \to \Gamma(\Lambda^{q+1} T^{*0,1}M \otimes E) \tag{1}$$

be the Dolbeault operator. As in [13], we equip $\Gamma(\Lambda^q T^{*0,1}M \otimes E)$ with a Hermitian metric by setting

$$(\eta, \eta') := \int_M \langle \eta(x), \eta'(x)\rangle \frac{\omega^{\wedge n}}{(2\pi)^n n!}. \tag{2}$$

Let $\bar\partial^*$ be the adjoint of $\bar\partial$ relative to this metric and let $\Box_q := (\bar\partial + \bar\partial^*)^2$ be the Kodaira-Laplacian acting on $\Gamma(\Lambda^q T^{*0,1}M \otimes E)$. We denote by $\mathrm{Eig}_\lambda(\Box_q)$ the eigenspace of $\Box_q$ corresponding to an eigenvalue $\lambda$. Consider a holomorphic isometry $g$ of $M$ which induces a holomorphic isometry $g^*$ of $E$. Then the equivariant analytic torsion is defined via the zeta function

$$Z_g(s) := \sum_{q>0} (-1)^q q \sum_{\substack{\lambda \in \mathrm{Spec}\,\Box_q \\ \lambda \neq 0}} \lambda^{-s} \operatorname{Tr} g^*_{|\mathrm{Eig}_\lambda(\Box_q)} \tag{3}$$

for $\mathfrak{Re}\, s \gg 0$. Classically, this zeta function has a meromorphic continuation to the complex plane which is holomorphic at zero.

**Definition 1** *The equivariant analytic torsion is defined as*

$$\tau_g := e^{-\frac{1}{2}Z_g'(0)}. \tag{4}$$

This gives for $g = \mathrm{Id}_M$ the ordinary analytic torsion $\tau$ of Ray and Singer [21]. Ray showed in [20, Ch. 2] the following lemma:

**Lemma 1** *Let $\Gamma$ be a finite group acting on $M$ by holomorphic fixed point free isometries. Assume that this action lifts holomorphically to isometries of $E$. Then the analytic torsion of $E/\Gamma$ over the quotient space $M/\Gamma$ is given by*

$$\log \tau(M/\Gamma, E/\Gamma) = \frac{1}{\#\Gamma} \sum_{g\in\Gamma} \log \tau_g(M, E). \tag{5}$$

Let $E_1 \to M_1$, $E_2 \to M_2$ be two holomorphic Hermitian vector bundles over compact Kähler manifolds. Let $g$ be a holomorphic isometry of $M_1$, $M_2$ whose action lifts to $E_1$, $E_2$. Let for $\nu = 1, 2$

$$L_g(M_\nu, E_\nu) := \sum_{q\geq 0} (-1)^q \ \mathrm{Tr}\ g^*_{|H^{0,q}(M_\nu,E_\nu)} \tag{6}$$

denote the holomorphic Lefschetz number of $g$. The following lemma follows by an immediate generalization of the proof of [21, Th. 3.3]

**Lemma 2** *The equivariant torsion of the product bundle $E_1 \times E_2$ over $M_1 \times M_2$ is given by*

$$\begin{aligned} \log \tau_g(M_1 \times M_2, E_1 \times E_2) \quad &= \quad L_g(M_1, E_1) \log \tau_g(M_2, E_2) \\ &\quad\ + L_g(M_2, E_2) \log \tau_g(M_1, E_1). \end{aligned} \tag{7}$$

# 3 Complex homogeneous spaces

Let $G$ be a connected compact Lie group and let $K$ be a connected subgroup of maximal rank. Assume that $(G, K)$ is a complex homogeneous pair of the compact type. According to [8, 10.1] we may assume $G$ to be semi-simple and simply connected. Let $T \subseteq K$ denote a fixed maximal torus.We denote the Lie algebras of $G$, $K$, $T$ by $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{t}$. Let $\Theta$ be a system of positive roots of $K$ (with respect to some ordering) and let $\Psi$ denote the set of roots of an invariant complex structure in the sense of [8]. Then $\Theta \cup \Psi =: \Delta^+$ is a system of positive roots of $G$ for a suitable ordering, which we fix [8, 13.7]. The holomorphic tangent space at the class of $\{1\} \in G$ in the coset space $G/K$ may be identified with $(\mathfrak{g}/\mathfrak{k} \otimes \mathbf{C})^{1,0}$. The canonical action of $K$ on $G/K$ induces a representation $\mathrm{Ad}^{1,0}_{G/K}$ of $K$ on $(\mathfrak{g}/\mathfrak{k} \otimes \mathbf{C})^{1,0}$ which is called the isotropy representation. Its weights are given by $\Psi$. The negative Killing form induces a metric on $G/K$.

The space of forms $\Gamma(\Lambda^q T^{*0,1}G/K)$ is an infinite dimensional $G$-representation which contains the space of its irreducible subrepresentations $(V_\pi, \pi)$ as a $L^2$-dense subspace. Thus,

$$\Gamma(\Lambda^q T^{*0,1}G/K) \overset{\text{dense}}{\supset} \bigoplus_\pi \mathrm{Hom}_G\left(V_\pi, \Gamma(\Lambda^q T^{*0,1}G/K)\right) \otimes V_\pi. \tag{8}$$

In this imbedding, the tensor product $\mathrm{Hom}_G\left(V_\pi, \Gamma(\Lambda^q T^{*0,1}G/K)\right) \otimes V_\pi$ is the direct sum of $\dim \mathrm{Hom}_G\left(V_\pi, \Gamma(\Lambda^q T^{*0,1}G/K)\right)$ copies of the representations $(V_\pi, \pi)$. By a Frobenious law due to Bott [11], there is a canonical isomorphism

$$\mathrm{Hom}_G\left(V_\pi, \Gamma(\Lambda^q T^{*0,1}G/K)\right) \cong \mathrm{Hom}_K\left(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K}\right) \tag{9}$$

(Note that $(\mathfrak{g}/\mathfrak{k} \otimes \mathbf{C})^{*0,1} \cong (\mathfrak{g}/\mathfrak{k} \otimes \mathbf{C})^{1,0}$ via the metric). In particular, the occuring representations $(V_\pi, \pi)$ are finite dimensional.

# 4 The zeta function

Let $(X_1, \ldots, X_N)$ be an orthonormal basis of $\mathfrak{g}$ with respect to the negativ Killing form. The Casimir operator of $\mathfrak{g}$ is defined as the following element of the universal enveloping algebra of $\mathfrak{g}$

$$\mathrm{Cas} := -\sum X_j.X_j. \tag{10}$$

Ikeda and Taniguchi proved the following beautiful result [16]:

**Theorem 3 (Ikeda, Taniguchi)** *Assume that $G/K$ is a Hermitian symmetric space equipped with the metric induced by the Killing form. Then the Laplacian $\Box_q$ acts on the $V_\pi$'s as $-\frac{1}{2}\mathrm{Cas}$ with respect to the imbedding (8).*

The Casimir is known to act by multiplication with a constant on irreducible representations. Thus, the eigenspaces of the Laplacian correspond to the irreducible representations $\pi$ with multiplicity $\dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K})$.

Let $\rho_G := \frac{1}{2}\sum_{\alpha\in\Delta^+} \alpha$ be the half sum of the positive roots of $G$ and let $W_G$ be its Weyl group. Let $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote the metric and the norm on $\mathfrak{t}^*$ induced by the Killing form. We denote the sign of an element $w \in W_G$ by $(-1)^w$. As usual, we define

$$(\alpha, \rho_G) := \frac{2\langle\alpha, \rho_G\rangle}{\|\alpha\|^2} \tag{11}$$

for any weight $\alpha$. For an irreducible representation $\pi$ we denote by $b_\pi$ the sum of its highest weight and $\rho_G$. Then, classically, the action of the Casimir is given by

$$\pi(\mathrm{Cas}) = \|\rho_G\|^2 - \|b_\pi\|^2. \tag{12}$$

To abbreviate we set

$$\mathrm{Alt}_G\{b\} := \sum_{w\in W_G} (-1)^w e^{2\pi i w b}. \tag{13}$$

Then the Weyl character formula for the character $\chi_{b_\pi}$ of the representation evaluated at $t \in T$ may be written as

$$\chi_{b_\pi}(t) = \frac{\mathrm{Alt}_G\{b_\pi\}(t)}{\mathrm{Alt}_G\{\rho_G\}(t)}. \tag{14}$$

This formula provides the definition of the so-called virtual (or formal) character $\chi_b$ for general $b = \rho_G$+any weight. This extends to all of $G$ by setting $\chi_b$ to be invariant under the adjoint action. The corresponding virtual representation shall be denoted by $V_b$. Occasionally we shall use the notation $\chi_b^G$, $\chi_b^K$, $V_b^G$, $V_b^K$ to distinguish $G$- and $K$-representations. From now on we shall consider the symmetric space $G/K$ to be equipped with any $G$-invariant metric $\langle\cdot,\cdot\rangle_\diamond$. These metrics are classified as follows:

Classically, a Hermitian symmetric space $G/K$ decomposes as a finite product of irreducible Hermitian spaces [15]

$$G/K = G_1/K_1 \times \cdots \times G_m/K_m. \tag{15}$$

On each $G_\nu/K_\nu$ every $G_\nu$-invariant metric is a multiple of the metric induced by the Killing form [4, Th. 7.44]. Thus, the metric on $G/K$ is induced by the direct sum of some (negative) factors times the Killing forms of the $G_\nu$. We shall denote the dual metric and norm on $\mathfrak{t}^*$ by $\langle\cdot,\cdot\rangle_\diamond$, $\|\cdot\|_\diamond$, too. These are given by the direct sum of the Killing forms divided by the corresponding factors. In particular, with respect to $\langle\cdot,\cdot\rangle_\diamond$ the Laplacian $\Box$ acts on $V_\pi$ as

$$\frac{1}{2}\left(\|b_\pi\|_\diamond^2 - \|\rho_G\|_\diamond^2\right) \tag{16}$$

by theorem 3. Thus, we may write the equivariant zeta function $Z(s)$ for $G/K$ Hermitian symmetric as

$$Z(s) = \sum_{q=1}^{n} (-1)^q q \sum_\pi \dim\, \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}^{1,0}) \left(\frac{2}{\|b_\pi\|_\diamond^2 - \|\rho_G\|_\diamond^2}\right)^s \chi_{b_\pi}. \tag{17}$$

Presumably, the expression $\dim\, \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}_{G/K}^{1,0})$ is very difficult to evaluate. One had to decompose $V_\pi$ and $\Lambda^q \mathrm{Ad}_{G/K}^{1,0}$ in their irreducible $K$-representations and to compare the occuring representations. It is possible to do this explicitly for the $\mathbf{P}^n\mathbf{C}$ [16], because in this case $\Lambda^q \mathrm{Ad}_{G/K}^{1,0}$ turns out to be irreducible itself. Nevertheless, this is not the case in general (see e.g. [23] for the case SU(4)/S(U(2) × U(2))). For this reason, we do not try to determine all of the eigenspaces and eigenvalues but to calculate the sum

$$\sum_{q=1}^{n}(-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K}) \tag{18}$$

which will turn out to be something relatively simple. In the next section, we shall prove the following lemma

**Lemma 4** *For any irreducible representation $\pi$, the sum*

$$\sum_{q=1}^{n}(-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K})\chi_{b_\pi} \tag{19}$$

*is non zero only if $b_\pi$ lies in the $W_G$-orbit of some $\rho_G + k\alpha$, $k \in \mathbf{N}$, $\alpha \in \Psi$. In this case, it equals the sum over $-\chi_{\rho_G+k\alpha}$ for all such pairs $k, \alpha$.*

As a corollary of lemma 4 and theorem 3 we obtain

**Theorem 5** *Assume that $G/K$ is symmetric. Then the zeta function $Z$ is given by*

$$Z(s) = -2^s \sum_{\substack{\alpha\in\Psi \\ k>0}} \langle k\alpha, k\alpha + 2\rho_G\rangle_\diamond^{-s} \chi_{\rho_G+k\alpha}. \tag{20}$$

## 5 Determination of the occuring representations

In this section we do not assume that $G/K$ is symmetric. Let $\rho_K := \frac{1}{2}\sum_{\alpha\in\Theta}\alpha$, $W_K$ and $\mathrm{Alt}_K$ be defined analogously to $\rho_G$, $W_G$ and $\mathrm{Alt}_K$. Let $\chi^K$ denote the virtual character given by

$$\chi^K := \sum_{q=1}^{n}(-1)^q q\chi\left(\Lambda^q \mathrm{Ad}^{1,0}_{G/K}\right) \tag{21}$$

where $\chi\left(\Lambda^q \mathrm{Ad}^{1,0}_{G/K}\right)$ is the character of the $K$-representation $\Lambda^q \mathrm{Ad}^{1,0}_{G/K}$. Then one knows

$$\sum_{q=1}^{n}(-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K}) = \int_K \overline{\chi^K} \cdot \chi_\pi \mathrm{dvol}_K. \tag{22}$$

Using the Weyl integral formula, this transforms to

$$\frac{1}{\#W_K}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}}\mathrm{Alt}_K\{\rho_K\}\overline{\chi^K} \cdot \chi_\pi \mathrm{dvol}_T \tag{23}$$

(Here we identify $T$ with the quotient of $\mathfrak{t}$ by the integral lattice). This integral may seem complicated at a first sight, but notice that for each integral form $\beta$

$$\int_T e^{2\pi i\beta}\mathrm{dvol}_T = \begin{cases} 1 & \text{if } \beta = 0 \\ 0 & \text{if } \beta \neq 0. \end{cases} \tag{24}$$

By classical representation theory, the restriction of $\mathrm{Alt}_G\{\rho_G\}$, $\mathrm{Alt}_K\{\rho_K\}$ to $T$ is given by

$$\mathrm{Alt}_G\{\rho_G\}_{|T} = \prod_{\alpha\in\Theta\cup\Psi} \left(e^{\pi i\alpha} - e^{-\pi i\alpha}\right) \tag{25}$$

and

$$\mathrm{Alt}_K\{\rho_K\}_{|T} = \prod_{\alpha\in\Theta} \left(e^{\pi i\alpha} - e^{-\pi i\alpha}\right). \tag{26}$$

By definition,

$$\begin{aligned} \chi^K &= \frac{\partial}{\partial s}_{|s=1} \det(1 - s\mathrm{Ad}^{1,0}_{G/K}) \\ &= \det(1-\mathrm{Ad}^{1,0}_{G/K})\mathrm{Tr}\left(-\mathrm{Ad}^{1,0}_{G/K}(1-\mathrm{Ad}^{1,0}_{G/K})^{-1}\right) \\ &= \det(1-\mathrm{Ad}^{1,0}_{G/K})\mathrm{Tr}(1-(\mathrm{Ad}^{1,0}_{G/K})^{-1})^{-1}. \end{aligned} \tag{27}$$

Hence we find for the restriction to the maximal Torus

$$\begin{aligned} \overline{\chi^K{}_{|T}} &= \prod_{\alpha\in\Psi}\left(1-e^{-2\pi i\alpha}\right)\cdot\sum_{\alpha\in\Psi}\left(1-e^{2\pi i\alpha}\right)^{-1} \\ &= e^{2\pi i(\rho_K-\rho_G)}\frac{\mathrm{Alt}_G\{\rho_G\}}{\mathrm{Alt}_K\{\rho_K\}}\sum_{\alpha\in\Psi}\left(1-e^{2\pi i\alpha}\right)^{-1}. \end{aligned} \tag{28}$$

Thus we get by the formulas (23), (28) and by the Weyl character formula

$$\begin{aligned} &\sum_{q>0}(-1)^q q\dim \mathrm{Hom}_K(V_\pi, \Lambda^q\mathrm{Ad}^{1,0}_{G/K}) \\ &= \frac{1}{\#W_K}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}}e^{2\pi i(\rho_K-\rho_G)}\sum_{\alpha\in\Psi}\frac{-e^{-2\pi i\alpha}}{1-e^{-2\pi i\alpha}}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T \\ &= \frac{1}{\#W_K}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}}e^{2\pi i(\rho_K-\rho_G)}\sum_{\alpha\in\Psi}\frac{e^{-2\pi iN\alpha}-e^{-2\pi i\alpha}}{1-e^{-2\pi i\alpha}}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T \\ &\text{(for } N\in\mathbf{N} \text{ sufficiently large)} \\ &= \frac{-1}{\#W_K}\sum_{k=1}^{N-1}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}}e^{2\pi i(\rho_K-\rho_G)}\sum_{\alpha\in\Psi}e^{-2\pi ik\alpha}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T \\ &= \frac{-1}{\#W_K}\sum_{k=1}^{\infty}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}}e^{2\pi i(\rho_K-\rho_G)}\sum_{\alpha\in\Psi}e^{-2\pi ik\alpha}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T. \end{aligned} \tag{29}$$

We observe that any $w \in W_K \subseteq W_G$ acts on the set $\Psi$ as a permutation, just because $\mathrm{Ad}^{1,0}_{G/K}$ is a $K$-representation. Thus for any $w \in W_K$

$$\sum_{\substack{\alpha\in\Psi\\ k>0}} e^{-2\pi i k\alpha} = \sum_{\substack{\alpha\in\Psi\\ k>0}} e^{-2\pi i k w(\alpha)} \tag{30}$$

and

$$e^{2\pi i(\rho_K-\rho_G)} = e^{2\pi i w(\rho_K-\rho_G)}. \tag{31}$$

Thus the right hand side in (29) becomes

$$\frac{-1}{\#W_K}\sum_{\substack{\alpha\in\Psi\\ k>0}}\int_T \overline{\mathrm{Alt}_K\{\rho_G+k\alpha\}}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T. \tag{32}$$

The integral is non-zero if and only if

$$w(b_\pi) = \rho_G + k\alpha \tag{33}$$

for an appropriate $w \in W_G$ and for such a $b_\pi$ it is equal to $(-1)^w \#W_K$. Notice that in general $\rho_G + k\alpha$ does not lie in the positive Weyl chamber. Anyhow, the expression

$$\frac{-1}{\#W_K}\int_T \overline{\mathrm{Alt}_K\{\rho_G+k\alpha\}}\mathrm{Alt}_G\{b\}\mathrm{dvol}_T\cdot\frac{\mathrm{Alt}_G\{b\}}{\mathrm{Alt}_G\{\rho_G\}} \tag{34}$$

is invariant under the action of $W_G$ on $b$, thus lemma 4 is proven. □

# 6 Remarks on zeta functions

Define $Z_\alpha$ for $\alpha \in \Psi$ as

$$Z_\alpha(s) := -\sum_{k>0} k^{-s}\,(k+(\alpha,\rho_G))^{-s}\,\chi^G_{\rho_G+k\alpha} \tag{35}$$

for $\mathfrak{Re}\, s > 2$. Thus $Z$ decomposes at

$$Z(s) = \sum_{\alpha\in\Psi}\left(\frac{2}{\|\alpha\|^2_\diamond}\right)^s Z_\alpha(s) \tag{36}$$

and its derivative $Z'$ at zero is given by

$$Z'(0) = \sum_{\alpha\in\Psi} Z'_\alpha(0) - \sum_{\alpha\in\Psi}\log\frac{\|\alpha\|^2_\diamond}{2} Z_\alpha(0). \tag{37}$$

Now we shall prove a fundamental symmetry property of the zeta function $Z$:

**Lemma 6** *For all $\alpha \in \Psi$, $k \in \mathbf{N}$ and $k' := k + (\alpha, \rho_G)$, the following equation holds*

$$\langle k\alpha, k\alpha + 2\rho_G \rangle^{-s} \chi^G_{\rho_G + k\alpha} = -\langle k'\alpha, k'\alpha - 2\rho_G \rangle^{-s} \chi^G_{\rho_G - k'\alpha}. \tag{38}$$

*For $0 < k < (\alpha, \rho_G)$, the character $\chi^G_{\rho_G - k\alpha}$ is equal to zero.*

*Proof* Let $S_\alpha$ denote the reflection of the weights by the hyperplane orthogonal to $\alpha$. Then $S_\alpha \rho_G = \rho_G - (\alpha, \rho_G)\alpha$, $S_\alpha \alpha = -\alpha$ and

$$\mathrm{Alt}_G\{\rho_G - (k + (\alpha, \rho_G))\,\alpha\} = \mathrm{Alt}_G\{S_\alpha(\rho_G + k\alpha)\} = -\mathrm{Alt}_G\{\rho_G + k\alpha\}, \tag{39}$$

because $S_\alpha$ has sign $-1$. The weights $\rho_G - k\alpha$ are singular for $0 < k < (\alpha, \rho_G)$ because they are contained in the convex hull of $W_G \cdot \rho_G$ (see Fig. 1). $\square$

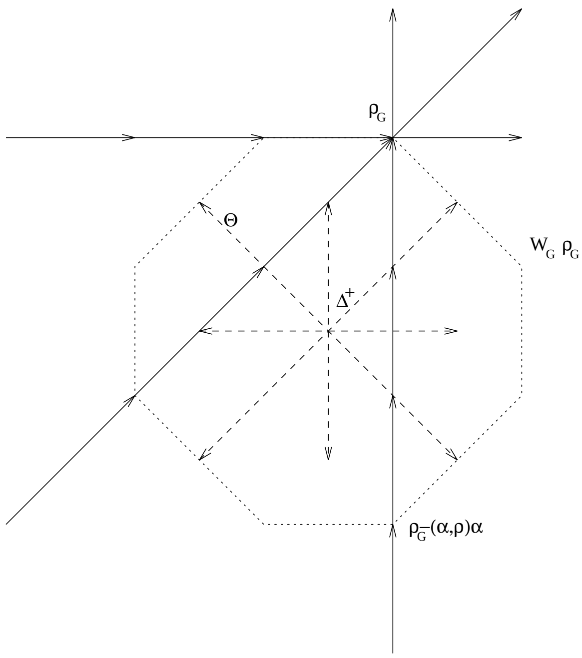


Figure 1: The lines $\rho_G + k\alpha$ in the case $\mathbf{SO}(5)/\mathbf{SO}(3) \times \mathbf{SO}(2)$

In particular, the derivatives of the $Z_\alpha$'s are given by

$$\begin{aligned} Z'_\alpha(s) &= \sum_{k>0} \frac{\log k + \log\left(k + (\alpha, \rho_G)\right)}{k^s\left(k + (\alpha, \rho_G)\right)^s} \chi^G_{\rho_G + k\alpha} \\ &= \sum_{k>0} \frac{\log k}{k^s\left(k + (\alpha, \rho_G)\right)^s} \chi^G_{\rho_G + k\alpha} \end{aligned}$$

$$- \sum_{k>(\alpha,\rho_G)} \frac{\log k}{k^s\,(k-(\alpha,\rho_G))^s} \chi^G_{\rho_G-k\alpha}. \tag{40}$$

Now we give a general formula for values at zero of zeta functions of this kind. Let for $\phi \in \mathbf{R}$ and $\mathfrak{Re}\, s > 2$

$$\zeta_L(s,\phi) = \sum_{k>0} \frac{e^{ik\phi}}{k^s} \tag{41}$$

denote the Lerch zeta function and let $\zeta_L'$ denote its derivative with respect to $s$. Let $P : \mathbf{Z} \to \mathbf{C}$ be a function of the form

$$P(k) = \sum_{j=0}^{m} c_j k^{n_j} e^{ik\phi_j} \tag{42}$$

with $m \in \mathbf{N}_0$, $n_j \in \mathbf{N}_0$, $c_j \in \mathbf{C}$, $\phi_j \in \mathbf{R}$ for all $j$. Set for $p \in \mathbf{R}$ and $\mathfrak{Re}\, s > (\max_j\{n_j\}+1)/2$

$$\tilde{\zeta}_{P,p}(s) := \sum_{\substack{k\in\mathbf{N}\\ k>p}} \frac{P(k)}{k^s(k-p)^s} \tag{43}$$

and

$$D\tilde{\zeta}_{P,p}(s) := \sum_{\substack{k\in\mathbf{N}\\ k>p}} \frac{-P(k)\log k}{k^s(k-p)^s}. \tag{44}$$

Then $\tilde{\zeta}$ and $D\tilde{\zeta}$ have meromorphic extensions to the complex plane. To express their values at zero, we define analogously to [13, 2.3.4]

$$\boldsymbol{\zeta} P := \sum_{j=0}^{m} c_j \zeta_L(-n_j,\phi_j), \tag{45}$$

$$\boldsymbol{\zeta}' P := \sum_{j=0}^{m} c_j \zeta_L'(-n_j,\phi_j), \tag{46}$$

$$\mathrm{Res}\; P(p) := \sum_{\substack{j=0\\ \phi_j\equiv 0 \mod 2\pi}}^{m} c_j \frac{p^{n_j+1}}{2(n_j+1)} \tag{47}$$

and $$P^*(p) := -\sum_{\substack{j=0\\ \phi_j\equiv 0 \mod 2\pi}}^{m} c_j \frac{p^{n_j+1}}{4(n_j+1)} \sum_{\ell=1}^{n_j} \frac{1}{\ell} \tag{48}$$

Let $[p] \in \mathbf{Z}$ denote the greatest integer less or equal than $p$. We define a sum $\sum_n^m$ with $n > m$ to be zero.

**Lemma 7** *$\tilde{\zeta}$ is holomorphic at zero and its value there is given by*

$$\tilde{\zeta}_{P,p}(0) = \boldsymbol{\zeta} P + \mathrm{Res}\; P(p) - \sum_{k=1}^{[p]} P(k) \tag{49}$$

*For $s \to 0$, $D\tilde{\zeta}$ has the Laurent expansion at zero*

$$D\tilde{\zeta}_{P,p}(s) = -\frac{\mathrm{Res}\; P(p)}{2s} + \boldsymbol{\zeta'} P + P^*(p) + \sum_{k=1}^{[p]} P(k)\log k + \mathcal{O}(s). \tag{50}$$

*Proof* It suffices to prove the lemma only for $P(k) = k^n e^{ik\phi}$, $n \in \mathbf{N}_0$, $\phi \in \mathbf{R}$. We get by a Taylor expansion of $(1 + p/k)^{-s}$ (for $k > |p|$) for $\mathfrak{Re}\, s > (\max_j\{n_j\} + 1)/2$

$$\begin{aligned} D\tilde{\zeta}(s) &= -\sum_{k=1}^{[-p]} \frac{k^n e^{ik\phi}\log k}{k^s(k-p)^s} - \sum_{k>|p|} \frac{e^{ik\phi}\log k}{k^{2s-n}}(1-\frac{p}{k})^{-s} \\ &= -\sum_{k=1}^{[-p]} \frac{k^n e^{ik\phi}\log k}{k^s(k-p)^s} - \sum_{k>|p|}\sum_{\ell=0}^{\infty} \binom{-s}{\ell}(-p)^\ell k^{n-2s-\ell} e^{ik\phi}\log k \\ &= -\sum_{k=1}^{[-p]} \frac{k^n e^{ik\phi}\log k}{k^s(k-p)^s} \\ &\quad + \sum_{\ell=0}^{\infty}\binom{-s}{\ell}(-p)^\ell \left(\zeta_L'(2s-n+\ell,\phi) + \sum_{1}^{[|p|]} \frac{e^{ik\phi}\log k}{k^{2s+\ell-n}}\right). \end{aligned} \tag{51}$$

The Lerch zeta function $\zeta_L(\cdot,\phi)$ is holomorphic for $\phi \not\equiv 0 \bmod 2\pi$. For $\phi \equiv 0$ it equals the Riemann zeta function which has a unique pole at $s = 1$,

$$\zeta_L(s+1, 0) = \frac{1}{s} + \mathcal{O}(1). \tag{52}$$

Thus for $s \to 0$

$$\begin{aligned} D\tilde{\zeta}(s) &= -\sum_{k=1}^{[-p]} k^n e^{ik\phi}\log k + \left(\zeta_L'(-n,\phi) + \sum_{k=1}^{[|p|]} k^n e^{ik\phi}\log k\right) \\ &\quad + \binom{-s}{n+1}(-p)^{n+1}\zeta_L'(2s+1,\phi) + \mathcal{O}(s) \\ &= \zeta_L'(-n,\phi) + \sum_{k=1}^{[p]} k^n e^{ik\phi}\log k \\ &\quad + \binom{-s}{n+1}(-p)^{n+1}\begin{cases} -\frac{1}{4s^2} & \text{if } \phi \equiv 0 \\ 0 & \text{otherwise} \end{cases} + \mathcal{O}(s). \end{aligned} \tag{53}$$

Using the fact that

$$\binom{-s}{n+1} = (-1)^{n+1}\frac{s(s+1)\cdots(s+n)}{(n+1)!} = \frac{(-1)^{n+1}s}{n+1}\left(1+s\sum_1^n \frac{1}{\ell}\right) + \mathcal{O}(s) \tag{54}$$

for $s \to 0$, equation (50) is proven. The identity for $\tilde{\zeta}$ is deduced the same way. □

Now we specialize a bit more to our situation. We denote the function $k \mapsto P(-k)$ by $P^-$. Set $P^{\mathrm{odd}} := (P - P^-)/2$.

**Lemma 8** *Assume that $p \in \mathbf{N}_0$ and that $P$ verifies the symmetry condition*

$$P(p-k) = -P(k). \tag{55}$$

*Then the values at zero of $\tilde{\zeta}$ and its derivative are given by*

$$\tilde{\zeta}(0) = \boldsymbol{\zeta} P^{\mathrm{odd}} + P(0)/2 = \boldsymbol{\zeta}(P^{\mathrm{odd}} - P(0)) \tag{56}$$

*and*

$$\tilde{\zeta}'(0) = 2\boldsymbol{\zeta'} P^{\mathrm{odd}} + 2P^*(p) - P(0)\log p - \sum_{k=1}^{[p/2]} P(k)\log(\frac{p}{k} - 1). \tag{57}$$

*Proof* Differentiating $\tilde{\zeta}$ leads to

$$\begin{aligned}
\tilde{\zeta}'(s) &= -\sum_{k>p} \frac{P(k)\,(\log k + \log(k-p))}{k^s(k-p)^s} \\
&= -\sum_{k>p} \frac{P(k)\log k}{k^s(k-p)^s} - \sum_{k>0} \frac{P(k+p)\log k}{k^s(k+p)^s} \\
&= -\sum_{k>p} \frac{P(k)\log k}{k^s(k-p)^s} + \sum_{k>0} \frac{P(-k)\log k}{k^s(k+p)^s} \\
&= D\tilde{\zeta}_{P,p}(s) - D\tilde{\zeta}_{P^-,-p}(s).
\end{aligned} \tag{58}$$

Applying lemma 4 gives for $s \to 0$

$$\begin{aligned}
\tilde{\zeta}'(s) &= -\frac{1}{s}\mathrm{Res}\ P(p) + \boldsymbol{\zeta'} P - \boldsymbol{\zeta'} P^- + 2P^*(p) \\
&\quad + \sum_{k=1}^{p} P(k)\log k + \mathcal{O}(s).
\end{aligned} \tag{59}$$

But we know that $\tilde{\zeta}$ is holomorphic at zero, thus

$$\mathrm{Res}\ P(p) = -\mathrm{Res}\ P^-(-p) = 0 \tag{60}$$

in this case. In particular, the expression for $\tilde{\zeta}(0)$ in lemma 4 becomes

$$\tilde{\zeta}(0) = \boldsymbol{\zeta} P - \sum_{k=1}^{p} P(k). \tag{61}$$

On the other hand,

$$\tilde{\zeta}_{P,p}(s) = \sum_{k>0} \frac{P(k+p)}{k^s(k+p)^s} = -\sum_{k>0} \frac{P(-k)}{k^s(k+p)^s} = -\tilde{\zeta}_{P^-,-p}(s) \tag{62}$$

by i), hence lemma 4 gives

$$\tilde{\zeta}(0) = -\boldsymbol{\zeta} P^- \tag{63}$$

or

$$\tilde{\zeta}(0) = \boldsymbol{\zeta} P^{\mathrm{odd}} - \sum_{k=1}^{p} P(k)/2 \tag{64}$$

by adding (61),(63). The lemma follows from the equations (59),(64) by applying (55) to the last terms. □

In particular, the functions $D\tilde{\zeta}_{P,p}$ and $D\tilde{\zeta}_{P^-,-p}$ are holomorphic at zero in this case. Now we shall apply lemma 8 to the zeta functions $Z_\alpha$. First we have to prove that the character map $k \mapsto \chi_{\rho_G - k\alpha}$ is indeed a function of the type (42). Set for $X_0 \in \mathfrak{t}$

$$\Theta_{X_0} := \left\{\beta \in \Delta^+ \middle| \beta(X_0) = 0\right\}. \tag{65}$$

Choose $X \in \mathfrak{t}$ so that $\Theta_{X_0+\epsilon X} = \emptyset$ for $\epsilon$ small. Then the (virtual) character $\chi_{\rho_G - k\alpha}$, $\alpha \in \Psi$, evaluated at $e^{X_0}$ is determined by

$$\begin{aligned}
&\chi_{\rho_G-k\alpha}(e^{X_0}) \cdot \prod_{\beta\in\Delta^+\setminus\Theta_{X_0}} 2i\sin\pi\beta(X_0) \\
&= \lim_{\epsilon\searrow 0} \frac{\mathrm{Alt}\{\rho_G - k\alpha\}(X_0+\epsilon X)}{\prod_{\beta\in\Theta_{X_0}} 2i\sin\pi\beta(X_0+\epsilon X)} \\
&= \lim_{\epsilon\searrow 0} \frac{\sum_{W_G}(-1)^w e^{2\pi i w(\rho_G+k\alpha)(X_0)} e^{2\pi i\epsilon w(\rho_G+k\alpha)(X)}}{(2\pi i\epsilon)^{\#\Theta_{X_0}} \prod_{\beta\in\Theta_{X_0}}\beta(X)} \\
&= \sum_{W_G}(-1)^w e^{2\pi i w(\rho_G+k\alpha)(X_0)} \frac{(w(\rho_G+k\alpha)(X))^{\#\Theta_{X_0}}}{\#\Theta_{X_0}!\prod_{\beta\in\Theta_{X_0}}\beta(X)}.
\end{aligned} \tag{66}$$

Hence one obtains a function of the type (42). Using $\chi_{\rho_G} = 1$, we get by lemma 6 and lemma 8

$$Z_\alpha(0) = \boldsymbol{\zeta}\chi^{\mathrm{odd}}_{\rho_G-k\alpha} + 1/2 \tag{67}$$

and

$$Z'_\alpha(0) = 2\boldsymbol{\zeta}'\chi^{\mathrm{odd}}_{\rho_G-k\alpha} + 2\chi^*_{\rho-k\alpha}\left((\alpha,\rho_G)\right) - \log(\alpha,\rho_G). \tag{68}$$

Here the application of $\boldsymbol{\zeta}$, $\boldsymbol{\zeta'}$ etc. to characters depending on $k$ is meant in the sense that one fixes first an element $g \in G$ acting on $G/K$, then one evaluates the characters at $g$ and then one applies the operators $\boldsymbol{\zeta}$ etc. By formula (37) and by the linearity of $\boldsymbol{\zeta}$, $\boldsymbol{\zeta'}$, we get the following theorem

**Theorem 9** *The logarithm of the equivariant torsion of a symmetric space $G/K$ is given by*

$$\begin{aligned} -\frac{1}{2}Z'(0) &= \boldsymbol{\zeta'}\sum_{\Psi}\chi^{\mathrm{odd}}_{\rho_G+k\alpha} - \sum_{\Psi}\chi^{*}_{\rho_G-k\alpha}\left((\alpha,\rho_G)\right) \\ &\quad + \frac{1}{2}\log\prod_{\Psi}(\alpha,\rho_G) + \frac{1}{2}\sum_{\Psi}\left(\frac{1}{2} - \boldsymbol{\zeta}\chi^{\mathrm{odd}}_{\rho_G+k\alpha}\right)\log\frac{\|\alpha\|^2_\diamond}{2} \qquad (69) \\ &= \boldsymbol{\zeta'}\sum_{\Psi}\chi^{\mathrm{odd}}_{\rho_G+k\alpha} - \sum_{\Psi}\chi^{*}_{\rho_G-k\alpha}\left((\alpha,\rho_G)\right) \\ &\quad + \frac{1}{2}\log\prod_{\Psi}\langle\alpha,\rho_G\rangle_\diamond - \frac{1}{2}\sum_{\Psi}\left(\frac{1}{2} + \boldsymbol{\zeta}\chi^{\mathrm{odd}}_{\rho_G+k\alpha}\right)\log\frac{\|\alpha\|^2_\diamond}{2}. \qquad (70) \end{aligned}$$

We shall prove in section 9 that the polynomial degree in $k$ of $\sum_\Psi \chi_{\rho_G+k\alpha}(g)$ for $g \in G$ is at most the dimension of the fixed point set of the action of $g$ on $G/K$. In particular, it is less or equal $\#\Psi$.
*Remarks:* Note that the term $\prod_\Psi\langle\alpha,\rho_G\rangle_\diamond^{-1}$ equals the volume $\mathrm{vol}_\diamond(\mathrm{G/K})$ by [3, Cor. 7.27]:

$$\mathrm{vol(G/K)} = \frac{\mathrm{vol(G/T)}}{\mathrm{vol(K/T)}} = \frac{\prod_\Theta\langle\alpha,\rho_{\mathrm{K}}\rangle}{\prod_{\Delta^+}\langle\alpha,\rho_{\mathrm{G}}\rangle} = \prod_\Psi\langle\alpha,\rho_{\mathrm{G}}\rangle^{-1} \qquad (71)$$

because of $\langle\alpha,\rho_G-\rho_K\rangle = 0$ for $\alpha\in\Theta$. This term would cancel with the $L^2$-norm if we considered equivariant Quillen metrics as defined by Bismut in [5] instead of the analytic torsion.

Assume that, for each $\nu$, all complementary roots of the space $G_\nu/K_\nu$ in the decomposition (15) have the same length (this is the case iff the decomposition does not contain one of the spaces $\mathbf{SO}(p+2)/\mathbf{SO}(p)\times\mathbf{SO}(2)$ ($p\geq 3$ odd) or $\mathbf{Sp}(n)/\mathbf{U}(n)$ ($n\geq 2$), see [15]). Then one may choose the metric $\langle\cdot,\cdot\rangle_\diamond$ in such a way that

$$\log\frac{\|\alpha\|^2_\diamond}{2} = 0 \qquad (72)$$

for all $\alpha\in\Psi$. Thus the corresponding term in theorem 9 vanishes. On the complex Grassmannians $G(p,q)$, this metric is just the usual Fubini-Study metric.

# 7 The case of isolated fixed points

Assume that $X\in\mathfrak{t}$ acts on $G/K$ with isolated fixed points, i.e. the set $\Theta_X$ is empty. Then one may calculate the values $\chi_{\rho_G-k\alpha}(e^X)$ using the Weyl character formula. One verifies easily that the fixed point set is given by the quotient of the Weyl groups $W(G,K) := W_G/W_K$. Set $R^{\mathrm{rot}}(\phi) := \left(\zeta_L'(0,\phi) - \zeta_L'(0,-\phi)\right)/2i$ for $0 < \phi < 2\pi$. The real valued function $R^{\mathrm{rot}}$ was already introduced and investigated in [17]. In particular, it was shown to be given by the formula

$$R^{\mathrm{rot}}(\phi) = \frac{C + \log\phi}{\phi} - i \sum_{\substack{\ell\geq 1\\ \ell \mathrm{odd}}} \zeta'(-\ell)\frac{(i\phi)^\ell}{\ell!} \tag{73}$$

and for $\phi = 2\pi\frac{p}{q}$, $p,q \in \mathbf{N}$, $0 < p < q$ it is given by

$$R^{\mathrm{rot}}(\phi) = -\frac{1}{2}\log q \cdot \cot\frac{\phi}{2} + \sum_{\ell=1}^{q-1} \log\Gamma\left(\frac{j}{q}\right) \cdot \sin j\phi \tag{74}$$

where $C$ denotes the Euler constant and $\Gamma$ is the gamma function. We shall prove the following fixed point formula

**Theorem 10** *Let $X \in \mathfrak{t}$ act with isolated fixed points. Then the logarithm of the equivariant torsion with respect to $e^X$ is given by*

$$\begin{aligned}
-\frac{1}{2}Z'(0)(e^X) \quad &= \sum_{[w]\in W(G,K)} \frac{\sum_{\alpha\in\Psi} iR^{\mathrm{rot}}\left(2\pi w\alpha(X)\right)}{\prod_{\alpha\in\Psi}\left(1-e^{-2\pi i w\alpha(X)}\right)} \\
&\quad + \frac{1}{2}\sum_{[w]\in W(G,K)} \frac{\sum_{\alpha\in\Psi}\left(1-e^{-2\pi i w\alpha(X)}\right)^{-1}\log\frac{\|\alpha\|_\diamond^2}{2}}{\prod_{\alpha\in\Psi}\left(1-e^{-2\pi i w\alpha(X)}\right)} \\
&\quad + \frac{1}{2}\log\prod_{\alpha\in\Psi}(\alpha,\rho_G).
\end{aligned} \tag{75}$$

Application of $[w] \in W(G,K)$ to weights means here that the entire expression does not depend on the choice of $w$.

*Remark* The term $\prod_{\alpha\in\Psi}\left(1-\exp\left(-2\pi i w\alpha(X)\right)\right)^{-1}$ is exactly equal to the factor from the fixed point $[w]$ in the Atiyah-Bott fixed point formula. This similarity will be explained in the next two chapters.

*Proof* Using the Weyl character formula, one obtains

$$\begin{aligned}
\sum_{\Psi}\chi_{\rho_G+k\alpha} \quad &= \quad \frac{\sum_{\alpha\in\Psi}\sum_{w\in W_G}(-1)^w e^{2\pi i w(\rho_G+k\alpha)}}{\prod_{\alpha\in\Delta^+} 2i\sin\pi\alpha} \\
&= \sum_{[w]\in W(G,K)} \frac{\sum_{\alpha\in\Psi}\sum_{w'\in W_K}(-1)^w(-1)^{w'}e^{2\pi i w w'(\rho_G+k\alpha)}}{\prod_{\alpha\in\Delta^+}2i\sin\pi\alpha} \\
&= \sum_{[w]\in W(G,K)} \frac{e^{2\pi i(\rho_G-\rho_K)}\sum_{w'\in W_K}(-1)^{w'}e^{2\pi i w w'\rho_K}}{(-1)^w\prod_{\alpha\in\Delta^+}2i\sin\pi\alpha} \sum_{\alpha\in\Psi} e^{2\pi i k w\alpha}
\end{aligned}$$

(because each $w' \in W_K$ permutes the set $\Psi$)

$$= \sum_{[w]\in W(G,K)} \frac{e^{2\pi i(\rho_G-\rho_K)} \prod_{\alpha\in\Theta} 2i\sin\pi w\alpha}{\prod_{\alpha\in\Delta^+} 2i\sin\pi w\alpha} \sum_{\alpha\in\Psi} e^{2\pi i k w\alpha}$$
$$= \sum_{[w]\in W(G,K)} \prod_{\alpha\in\Psi} \left(1-e^{-2\pi i w\alpha}\right)^{-1} \sum_{\alpha\in\Psi} e^{2\pi i k w\alpha}. \tag{76}$$

The same procedure applies to the last term in equation (69). Using the equation $\zeta_L(0,\phi) = -1/2 + i/2\cot\frac{\phi}{2} = (e^{-i\phi}-1)^{-1}$ (see [17]), one obtains the theorem by applying theorem 9. □

*Remark* Assume that $\phi \neq 0$. Because of $\frac{\partial}{\partial\phi}\zeta_L(s,\phi) = i\zeta_L(s-1,\phi)$, the values of $\zeta_L$ and $\zeta_L'$ at the negative integers are determined by the values at zero via the Taylor expansions

$$\sum_{n\in\mathbf{N}} \zeta_L(-n,\phi)\frac{(ix)^n}{n!} = \zeta_L(0,\phi+x) = (e^{-i\phi-ix}-1)^{-1} \tag{77}$$

and

$$\sum_{n\in\mathbf{N}} \zeta_L'(-n,\phi)\frac{(ix)^n}{n!} = \zeta_L'(0,\phi+x) \tag{78}$$

for small x. In particular, $\zeta_L(-n,\phi)$ is a rational function (with rational coefficients) of $e^{i\phi}$. Also, the results of [17] show that for $\phi = 0$, one still has the formula

$$\sum_{n\in\mathbf{N}} \zeta'(-n)\frac{(ix)^n}{n!} = \zeta_L'(0,x) + \frac{1}{x}(\frac{\pi}{2} + iC + i\log x). \tag{79}$$

Thus, when $\boldsymbol{\zeta'}$ is applied to an odd function, the occuring coefficients $\zeta_L'(-n,\phi) - (-1)^n\zeta_L'(-n,-\phi)$ are mainly the Taylor coefficients of $R^{\mathrm{rot}}$, e.g. for $\phi \neq 0$

$$\sum_{n\in\mathbf{N}} \left(\zeta_L'(-n,\phi) - (-1)^n\zeta_L'(-n,-\phi)\right)\frac{(ix)^n}{n!} = 2iR^{\mathrm{rot}}(0,\phi+x). \tag{80}$$

# 8 Application of equivariant K-theory

In this section, we shall recall some concepts of equivariant K-theory and apply them to theorem 9. We shall give short proofs to illustrate the similarity with section 5. For more information see Bott's article [10] and [1]. Let $\pi : K \hookrightarrow G$ be the inclusion of a compact subgroup of maximal rank and assume $G/K$ to be a complex homogeneous space. Let $R(G)$, $R(K)$ denote the representation rings of $G$ and $K$. We define the map $\pi^!$ as the restriction of representations. For any $K$-representation $V^K$ let

$$E_{V^K} := G \times V^K/K \tag{81}$$

denote the associated holomorphic homogeneous vector bundle on $G/K$.

**Definition 2** *The direct image map* $\pi_! : R(K) \to R(G)$ *is defined as*

$$\pi_! V^K := \sum_{q\geq 0} (-1)^q H^{0,q}(G/K, E_{V^K}). \tag{82}$$

We shall use the following properties of $\pi_!$:

**Theorem 11 (Bott)** *The image under* $\pi_!$ *of an irreducible* $K$*-representation* $V^K_{\rho_K+\lambda}$ *of highest weight* $\lambda$ *is given by*

$$\pi_! V^K_{\rho_K+\lambda} = V^G_{\rho_G+\lambda}. \tag{83}$$

*For any* $K$*-representation* $V^K$ *with character* $\chi(V^K)$*, the character of* $\pi_! V^K$ *is given by*

$$\chi(\pi_! V^K) = \sum_{[w]\in W(G,K)} \frac{\chi(V^K)\circ w}{\prod_{\alpha\in\Psi}(1-e^{-2\pi i w\alpha})}. \tag{84}$$

*If* $K = T$ *then* $\pi_!$ *is a left inverse of* $\pi^!$*:*

$$\pi_! \pi^! V^G = V^G \tag{85}$$

*for any* $G$*-representation* $V^G$*.*

*Remark* $\pi_!$ and $\pi^!$ may be regarded as the direct image and the pullback in equivariant K-theory, which are maps between $K_G(G/K) \cong R(K)$ and $K_G(\text{point}) \cong R(G)$. Bott showed the above theorem for more general mappings.
*Proof* As the Euler characteristic of a complex is equal to the Euler characteristic of the homology of the complex, we get by the equations (8) and (9) (i.e. their analogues for nontrivial coefficients)

$$\pi_! V^K_{\rho_K+\lambda} = \sum_{q\geq 0} (-1)^q \sum_{\pi \text{irr.}} \dim \operatorname{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K} \otimes V^K_{\rho_K+\lambda}) V_\pi. \tag{86}$$

Similar to (27), the character of the $K$-representation $\sum_q (-1)^q \Lambda^q \mathrm{Ad}^{1,0}_{G/K} \otimes V^K_{\rho_K+\lambda}$ is given by $\det(1-\mathrm{Ad}^{1,0}_{G/K})\chi^K_{\rho_K+\lambda}$. By the same reasoning as in section 5 one finds

$$\begin{aligned}
&\sum_{q\geq 0} (-1)^q \dim \operatorname{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K} \otimes V^K_{\rho_K+\lambda}) \\
&= \frac{1}{\#W_K} \int_T \overline{\mathrm{Alt}_K\{\rho_K\} e^{2\pi i(\rho_G-\rho_K)} \chi^K_{\rho_K+\lambda}} \mathrm{Alt}_G\{b_\pi\} \mathrm{dvol}_T \\
&= \frac{1}{\#W_K} \int_T \overline{\mathrm{Alt}_K\{\rho_G+\lambda\}} \mathrm{Alt}_G\{b_\pi\} \mathrm{dvol}_T.
\end{aligned} \tag{87}$$

Equation (83) follows the same way as theorem 5 follows by equation (32). To prove the relation (84), we assume $V^K$ to be irreducible, $V^K = V^K_{\rho_K+\lambda}$. Then, similar to the proof of theorem 10,

$$\begin{aligned}
\chi^G_{\rho_G+\lambda} &= \frac{\mathrm{Alt}_G\{\rho_G+\lambda\}}{\mathrm{Alt}_G\{\rho_G\}} \\
&= \sum_{[w]\in W(G,K)} (-1)^w e^{2\pi i(\rho_G-\rho_K)} \frac{\mathrm{Alt}_K\{w(\rho_G+\lambda)\}}{\prod_{\alpha\in\Psi} 2i\sin\pi\alpha\cdot\mathrm{Alt}_K\{\rho_K\}} \\
&= \sum_{[w]\in W(G,K)} \frac{\mathrm{Alt}_K\{w(\rho_G+\lambda)\}}{\mathrm{Alt}_K\{w\rho_K\}} \prod_{\alpha\in\Psi}(1-e^{-2\pi i w\alpha})^{-1}. \qquad (88)
\end{aligned}$$

Equation (85) is obtained by applying formula (84) to the character $\chi(\pi_!\pi^! V^G_{\rho_G+\lambda})$ of $\pi_!\pi^! V^G_{\rho_G+\lambda}$, thus

$$\chi(\pi_!\pi^! V^G_{\rho_G+\lambda}) = \sum_{w\in W_G} \frac{\chi^G_{\rho_G+\lambda}}{e^{-2\pi i w\rho_G}\mathrm{Alt}_G\{\rho_G\}} \qquad (89)$$

and using the Weyl character formula for $\chi^G_{\rho_G+\lambda}$. □

**Definition 3** *For any Lie group $G$ the Adams operator $\psi^k : R(G)\to R(G)$ of order $k\in\mathbf{Z}$ is defined as*

$$\psi^k\rho(g) = \rho(g^k) \qquad (90)$$

*for a virtual representation $\rho$ and $g\in G$.*

The Adams operators provide the following formula for the $\boldsymbol{\zeta'}$-term in theorem 9:

**Lemma 12** *For any $k\in\mathbf{Z}$,*

$$\bigoplus_{\alpha\in\Psi} V^G_{\rho_G+k\alpha} = \pi_!\psi^k \mathrm{Ad}^{1,0}_{G/K}. \qquad (91)$$

*Proof* Let $\tilde\pi_!$ and $\pi_!$ denote the direct image maps

$$R(T) \xrightarrow{\tilde\pi_!} R(K) \xrightarrow{\pi_!} R(G) \qquad (92)$$

and let $\tilde\pi^!$ be the restriction map

$$\tilde\pi^! : R(K)\to R(T). \qquad (93)$$

Then one gets by theorem 11

$$\begin{aligned}
\bigoplus_{\alpha\in\Psi} V^G_{\rho_G+k\alpha} &= \pi_!\tilde\pi_! \bigoplus_{\Psi} V^T_{k\alpha} = \pi_!\tilde\pi_!\psi^k \bigoplus_{\Psi} V^T_\alpha \\
&= \pi_!\tilde\pi_!\psi^k\tilde\pi^!\mathrm{Ad}^{1,0}_{G/K} = \pi_!\tilde\pi_!\tilde\pi^!\psi^k\mathrm{Ad}^{1,0}_{G/K} \\
&= \pi_!\psi^k\mathrm{Ad}^{1,0}_{G/K}. \qquad (94)
\end{aligned}$$

$\square$

In particular, equation (76) may be reproven using lemma 12 and equation (84). Geometrically, the maps $\tilde{\pi}_!$ and $\pi_!$ correspond to the direct images associated to the double fibration

$$G/T \xrightarrow{\tilde{\pi}} G/K \xrightarrow{\pi} \text{point} \cong G/G. \tag{95}$$

# 9 Fixed point formulas

In this section, we shall combine lemma 12 and theorem 9 with the equivariant index theorem of Atiyah-Segal-Singer [1],[2]. We shall employ this theorem only for the case of complex homogeneous spaces. In this case, it has already been shown by Borel and Hirzebruch [9]. We shall compare the result with Bismut's conjecture of an equivariant Riemann-Roch formula.

Let $g$ be a holomorphic isometry of a compact Kähler manifold $M$ with holomorphic tangent bundle $TM$ and let $M_g$ denote the fixed point set. Let $E$ be a Hermitian holomorphic vector bundle over $M$ acted on by $g$. Let $N$ be the normal bundle of the imbedding $M_g \hookrightarrow M$. Let $\gamma^N_{|x}$ (resp. $\gamma^E_{|x}$) denote the isometry of $N_{|x}$ (resp. $E_{|x}$) which is the infinitesimal action of $g$ at $x \in M_g$. Let $\Omega^{TM}$, $\Omega^{TM_g}$, $\Omega^N$ and $\Omega^E$ denote the curvatures of the corresponding bundles with respect to the Hermitian holomorphic connection. Define the function Td on square matrices $A$ as

$$\mathrm{Td}(A) := \det \frac{A}{1 - e^{-A}}. \tag{96}$$

**Definition 4** *Let* $\mathrm{Td}_g(TM)$ *and* $\mathrm{ch}_g(TM)$ *denote the following differential forms on* $M_g$*:*

$$\mathrm{Td}_g(TM) := \mathrm{Td}\left(\frac{-\Omega^{TM_g}}{2\pi i}\right) \det\left(1 - (\gamma^N)^{-1} \exp \frac{\Omega^N}{2\pi i}\right)^{-1} \tag{97}$$

*and*

$$\mathrm{ch}_g(E) := \mathrm{Tr}\, \gamma^E \exp \frac{-\Omega^E_{|M_g}}{2\pi i}. \tag{98}$$

Then the Atiyah-Segal-Singer index formula states in this case

**Theorem 13 (Atiyah-Segal-Singer)** *The holomorphic Lefschetz number of* $g$ *equals*

$$\sum_q (-1)^q \mathrm{Tr}\, g^*_{|H^{0,q}(M,E)} = \int_{M_g} \mathrm{Td}_g(TM)\mathrm{ch}_g(E). \tag{99}$$

This theorem, combined with lemma 12, gives the formula

$$\sum_{\alpha \in \Psi} \chi^G_{\rho_G + k\alpha}(g) = \int_{(G/K)_g} \mathrm{Td}_g(T(G/K))\mathrm{ch}_g(\psi^k T(G/K)). \tag{100}$$

Hence the first term on the right hand side in theorem 9 is given by

$$\boldsymbol{\zeta'} \sum_{\Psi} \chi^{\text{odd}}_{\rho_G+k\alpha}(g) = \int_{(G/K)_g} \text{Td}_g(T(G/K))\boldsymbol{\zeta'}\text{ch}_g(\psi^k T(G/K))^{\text{odd}}. \tag{101}$$

In particular one gets the following corollary of equation (100) and theorem 13

**Corollar 14** *The polynomial degree in $k$ of $\sum_{\alpha\in\Psi} \chi^G_{\rho_G+k\alpha}(g)$ is less than or equal to dim $(G/K)_g$.*

This fact may be deduced also by lemma 12 and equation (84). Define for $P$ as in (42) the complex number $\overline{\boldsymbol{\zeta}}P$ as

$$\overline{\boldsymbol{\zeta}}P := \sum_{j=0}^{m} c_j \zeta_L(-n_j, \phi_j) \sum_{\ell=1}^{n_j} \frac{1}{\ell}. \tag{102}$$

In [6], Bismut introduced the following characteristic class called the equivariant $R$-genus:

$$R_g(TM) := (2\boldsymbol{\zeta'} + \overline{\boldsymbol{\zeta}})\text{ch}_g(\psi^k TM_g)^{\text{odd}}. \tag{103}$$

Note that the $\boldsymbol{\zeta'}$-part of this genus is given by the series $R^{\text{rot}}$ as an equivariant characteristic class in the sense of [3] by equation (78), up to a singular part in directions where the action of $g$ is trivial. Using the $R$-genus we may reformulate theorem 9 as follows:

**Theorem 15** *The logarithm of the torsion is given by the equation*

$$\begin{aligned}
&2\log\tau_g(G/K) - \log\text{vol}_\diamond(\text{G/K}) + \sum_{\Psi}\big(\frac{1}{2} + \boldsymbol{\zeta}\chi^{\text{odd}}_{\rho_\text{G}+\text{k}\alpha}(\text{g})\big)\log\frac{\|\alpha\|^2_\diamond}{2}\\
&= \int_{(G/K)_g} \text{Td}_g(T(G/K))R_g(T(G/K))\\
&\quad - \overline{\boldsymbol{\zeta}}\sum_{\Psi}\chi^{\text{odd}}_{\rho_G+k\alpha}(g) - 2\sum_{\Psi}\chi_{\rho_G-k\alpha}(g)^*\left((\alpha,\rho_G)\right).
\end{aligned} \tag{104}$$

Using the $R$-genus, Bismut formulated a conjectural equivariant arithmetic Grothendieck-Riemann-Roch theorem [6]. We shall show now that this conjecture fits perfectly well with theorem 15. For any details of the following discussion, see [22] or [14]. Suppose that $M$ is given by $\mathcal{M}\otimes\mathbf{C}$ for a flat regular scheme $\pi : \mathcal{M} \to \text{Spec}\ \mathbf{Z}$ and that $E$ stems from an algebraic vector bundle $\mathcal{E}$ over $\mathcal{M}$. Let $\sum(-1)^q R^q\pi_*\mathcal{E}$ denote the direct image of $\mathcal{E}$ under $\pi$. We equip the associated complex vector space with a Hermitian metric induced by (2) via Hodge theory. Bismut's conjecture implies that the equivariant torsion verifies the equation

$$2\log\tau_g(M,E) + \hat{c}^1_g\left(\sum_{q\geq 0}(-1)^q R^q\pi_*\mathcal{E}\right) = \pi_*\left(\widehat{\text{Td}}_g(T\mathcal{M})\widehat{\text{ch}}_g(\mathcal{E})\right)^{(1)}$$

$$+\int_{M_g} \mathrm{Td}_g(TM) R_g(TM) \mathrm{ch}(E) \tag{105}$$

(We identify the first arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathrm{Spec}\,\mathbf{Z})$ with $\mathbf{R}$). Here $\hat{c}_g^1$, $\widehat{\mathrm{Td}}_g$ and $\widehat{\mathrm{ch}}_g$ denote certain equivariant arithmetic characteristic classes which are only defined in a non-equivariant situation up to now. In [5] Bismut has proven that formula (105) is compatible with the behaviour of the equivariant torsion under immersions and changes of the occuring metrics. In the non-equivariant case, equation (105) has been conjectured by Gillet and Soulé in [13] and it has been proven by Gillet, Soulé, Bismut and Lebeau [14]. An important step in this proof was the calculation of the non-equivariant torsion for the $\mathbf{P}^n\mathbf{C}$.

In our case, the cohomology of the trivial line bundle $\mathcal{O}$ over $G/K$ is given by

$$H^q(G/K, \mathcal{O}) = \begin{cases} \mathbf{Z} & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases} \tag{106}$$

and the action of $g \in G$ on $H^0(G/K)$ is trivial. Thus the $\hat{c}_g^1$ term in (105) should be independent of $g$. By the definition of $\hat{c}^1$, it should equal minus the logarithm of the norm of the element $1 \in H^0(G/K)$, thus $-\log \mathrm{vol}_\diamond(\mathrm{G/K})$. Hence, theorem 15 fits very well with Bismut's conjecture. If one assumes the characteristic classes in (105) to be defined, theorem 15 and the conjecture imply for a model $\mathcal{M}$ of $G/K$ e.g. in the case (72)

$$\left(\pi_* \widehat{\mathrm{Td}}_g(T\mathcal{M})\right)^{(1)} = \overline{\boldsymbol{\zeta}} \sum_{\Psi} \chi^{\mathrm{odd}}_{\rho_G + k\alpha}(g) + 2 \sum_{\Psi} \chi_{\rho_G - k\alpha}(g)^* \left((\alpha, \rho_G)\right). \tag{107}$$

Consider now the map $\tilde{\alpha} : T \to S^1 \subset \mathbf{C}$ associated to a root $\alpha \in \Delta^+$. Let $\mathbf{K}$ be a number field and assume that $g$ is conjugate to some $t \in T$ such that $\tilde{\alpha}(t) \in \mathbf{K}$ for all $\alpha \in \Delta^+$. Using eq. (66) it is easily verified that $\chi_{\rho_G - k\alpha}$, $\alpha \in \Delta^+$, is a function of the form

$$\sum_j c_j \gamma_j^k k^{n_j} \tag{108}$$

with $c_j, \gamma_j \in \mathbf{K}$ and $n_j \in \mathbf{N}$. By the last remark on $\zeta_L(-n, \phi)$ in section 7, this implies

$$\left(\pi_* \widehat{\mathrm{Td}}_g(T\mathcal{M})\right)^{(1)} \in \mathbf{K}. \tag{109}$$

In particular, this applies for $g = 0$ and $\mathbf{K} = \mathbf{Q}$. If assumption (72) does not hold, one may still adjust the metric $\langle \cdot, \cdot \rangle_\diamond$ in such a way that

$$\left(\pi_* \widehat{\mathrm{Td}}_g(T\mathcal{M})\right)^{(1)} \in \mathbf{K} \oplus \mathbf{K} \log 2. \tag{110}$$

# 10 The non-equivariant case

We consider now the case $g = 0$, i.e. the action of the identity map. For this action, the equivariant torsion equals the original Ray-Singer torsion. The values of the characters $\chi_{\rho_G+k\alpha}$ at zero are given by the Weyl dimension formula

$$\chi_{\rho_G+k\alpha}(0) = \dim V_{\rho_G+k\alpha} = \prod_{\beta\in\Delta^+} \left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right). \tag{111}$$

In particular, the first term in equation (69) is given by $\boldsymbol{\zeta'}$ applied to the odd part of the polynomial

$$\sum_{\alpha\in\Psi} \chi_{\rho_G+k\alpha}(0) = \sum_{\alpha\in\Psi}\prod_{\beta\in\Delta^+} \left(1 + k\frac{\langle\beta,\alpha\rangle}{\langle\beta,\rho_G\rangle}\right). \tag{112}$$

At a first sight, this looks like a polynomial of degree $\#\Delta^+$, but we know by corollary 14 that it has in fact degree $\leq \#\Psi$, thus all higher degree terms cancel. By combining theorem 15 with the arithmetic Riemann-Roch theorem of Gillet and Soulé (i.e. equation (105) for $g = 0$), we get the following formula:

**Theorem 16** *The direct image of the arithmetic Todd class is given by*

$$\begin{aligned}\left(\pi_*\widehat{\mathrm{Td}}(T\mathcal{M})\right)^{(1)} &= -\sum_{\Psi}\boldsymbol{\zeta}\left(\dim V_{\rho_G-k\alpha}\right)\log\frac{\|\alpha\|_\diamond^2}{2} \\ &+ \overline{\boldsymbol{\zeta}}\sum_{\Psi}\left(\dim V_{\rho_G+k\alpha}\right)^{\mathrm{odd}} + 2\sum_{\Psi}\left(\dim V_{\rho_G+k\alpha}\right)^*\left((\alpha,\rho_G)\right).\end{aligned}$$

# 11 Nontrivial coefficients

In this section we calculate the torsion of a homogeneous vector bundle $E$ over the Hermitian symmetric space $G/K$. Let $V^K_{\rho_K+\Lambda}$ be an irreducible $K$-representation and let $E_{\rho_K+\Lambda}$ denote the associated $G$-invariant holomorphic vector bundle on $G/K$ as in (81). $V^K_{\rho_K+\Lambda}$ carries a $K$-invariant hermitian metric which is unique up to a factor. This metric induces a metric on $E_{\rho_K+\Lambda}$. Of course, the Laplacian does not depend on the factor. By just the same proof as in [19], one shows the following generalization of theorem 3:

**Lemma 17** *The Laplacian $\square_q$ with coefficients in $E_{\rho_K+\Lambda}$ acts on a finite dimensional $G$-representation $V_\pi \subset \Gamma(\Lambda^q T^{*0,1}G/K \otimes E_{\rho_K+\Lambda})$ as*

$$\frac{1}{2}\left(\|b_\pi\|_\diamond^2 - \|\rho_G + \Lambda\|_\diamond^2\right). \tag{113}$$

An analogue of lemma 4 follows by the observation

$$\sum_{q>0}(-1)^q q \dim \mathrm{Hom}_K(V_\pi, \Lambda^q \mathrm{Ad}^{1,0}_{G/K} \otimes V^K_{\rho_K+\Lambda})$$
$$= \frac{1}{\#W_K}\int_T \overline{\mathrm{Alt}_K\{\rho_K\}\chi_{\rho_K+\Lambda}} e^{2\pi i(\rho_K-\rho_G)} \sum_{\alpha\in\Psi} \frac{-e^{-2\pi i\alpha}}{1-e^{-2\pi i\alpha}} \mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T$$
$$= \frac{-1}{\#W_K}\sum_{\substack{\alpha\in\Psi\\ k>0}}\int_T \overline{\mathrm{Alt}_K\{\rho_G+\Lambda+k\alpha\}}\mathrm{Alt}_G\{b_\pi\}\mathrm{dvol}_T. \tag{114}$$

Hence, the associated zeta function $Z_{\rho_K+\Lambda}$ is given by

$$Z_{\rho_K+\Lambda}(s) = -2^s \sum_{\substack{\alpha\in\Psi\\ k>0}} \langle k\alpha, k\alpha+2\rho_G+2\Lambda\rangle_\diamond^{-s}\chi_{\rho_G+\Lambda+k\alpha}. \tag{115}$$

By lemma 8 one gets the result

**Theorem 18** *The logarithm of the equivariant torsion of $E_{\rho_K+\Lambda}$ on a symmetric space $G/K$ is given by*

$$-\frac{1}{2}Z'_{\rho_K+\Lambda}(0) = \boldsymbol{\zeta}'\sum_{\Psi}\chi^{\mathrm{odd}}_{\rho_G+\Lambda+k\alpha} - \sum_{\Psi}\chi^*_{\rho_G+\Lambda-k\alpha}\left((\alpha,\rho_G+\Lambda)\right)$$
$$+\frac{1}{2}\sum_{\Psi}\sum_{k=1}^{(\alpha,\rho_G+\Lambda)}\chi_{\rho_G+\Lambda-k\alpha}\log k - \frac{1}{2}\sum_{\Psi}\boldsymbol{\zeta}\chi_{\rho_G+\Lambda+k\alpha}\log\frac{\|\alpha\|^2_\diamond}{2}$$

The Atiyah-Segal-Singer index formula implies again

$$\sum_{\alpha\in\Psi}\chi^G_{\rho_G+\Lambda+k\alpha}(g) = \int_{(G/K)_g}\mathrm{Td}_g(T(G/K))\mathrm{ch}_g(\psi^k T(G/K))\mathrm{ch}_g(E_{\rho_K+\Lambda}). \tag{116}$$

Hence one finds for the first term in theorem 18

$$\boldsymbol{\zeta}'\sum_{\Psi}\chi^{\mathrm{odd}}_{\rho_G+\Lambda+k\alpha}(g) = \int_{(G/K)_g}\mathrm{Td}_g(T(G/K))R_g(T(G/K))\mathrm{ch}_g(E_{\rho_K+\Lambda})$$
$$-\overline{\boldsymbol{\zeta}}\sum_{\Psi}\chi^{\mathrm{odd}}_{\rho_G+\Lambda+k\alpha}(g) \tag{117}$$

which fits again with Bismut's conjecture (105). Using the fact that the logarithm of the torsion behaves additively under direct sum of vector bundles, one obtains similar results for any homogeneous vector bundle.

**Acknowledgement** *I would like to thank Michael Schröder and Gerald Höhn for helpful discussions. Also, I am indepted to the referee for helpful comments. During the preparation of this paper I have been supported by the Max-Planck-Institut in Bonn.*

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