Prime Numbers and the Riemann Hypothesis

Benjamin Klopsch

Mathematisches Institut
Heinrich-Heine-Universität zu Düsseldorf

Tag der Forschung ○ November 2005

"Investigation of the Distribution of Prime Numbers"

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch die Interesse, welches *Gauss* und *Dirichlet* demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von *Euler* gemachte Bemerkung, dass das Product

$$\Pi \frac{1}{1-\frac{1}{p^s}} = \Sigma \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1}\, dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}\, dx}{e^x - 1}.$$

What will we encounter in the next 45 minutes?

The 'Building Blocks of Numbers'
- What are prime numbers?
- How many prime numbers are there?

Riemann – a Pioneer in Complex Analysis
- Infinite Series and Complex Functions
- Riemann and the Zeta Function $\zeta(s)$

A 'Wonderful Formula'
- Riemann's Formula
- The Riemann Hypothesis

What will we encounter in the next 45 minutes?

The 'Building Blocks of Numbers'
- What are prime numbers?
- How many prime numbers are there?

Riemann – a Pioneer in Complex Analysis
- Infinite Series and Complex Functions
- Riemann and the Zeta Function $\zeta(s)$

A 'Wonderful Formula'
- Riemann's Formula
- The Riemann Hypothesis

What will we encounter in the next 45 minutes?

The 'Building Blocks of Numbers'

- What are prime numbers?
- How many prime numbers are there?

Riemann – a Pioneer in Complex Analysis

- Infinite Series and Complex Functions
- Riemann and the Zeta Function $\zeta(s)$

A 'Wonderful Formula'

- Riemann's Formula
- The Riemann Hypothesis

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001 = 7 \cdot 143.$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001 = 7 \cdot 11 \cdot 13$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001 = 7 \cdot 11 \cdot 13$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

What are prime numbers?

Definition

A prime number is a natural number p admitting precisely two distinct divisors, namely 1 and p.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)

Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001 = 7 \cdot 11 \cdot 13$$

and similarly

$$112005 = 3^2 \cdot 5 \cdot 19 \cdot 131.$$

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99									

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

1	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99									

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99									

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	4	5	6	7	8	9	10	11	12
13	14	15	16	17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80	81	82	83	84
85	86	87	88	89	90	91	92	93	94	95	96
97	98	99									

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	5	7	9	11
13		15	17	19	21	23
25		27	29	31	33	35
37		39	41	43	45	47
49		51	53	55	57	59
61		63	65	67	69	71
73		75	77	79	81	83
85		87	89	91	93	95
97		99				

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	**3**	5	7	9	11
13		15	17	19	21	23
25		27	29	31	33	35
37		39	41	43	45	47
49		51	53	55	57	59
61		63	65	67	69	71
73		75	77	79	81	83
85		87	89	91	93	95
97		99				

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	**3**	5	7	9	11
13		15	17	19	21	23
25		27	29	31	33	35
37		39	41	43	45	47
49		51	53	55	57	59
61		63	65	67	69	71
73		75	77	79	81	83
85		87	89	91	93	95
97		99				

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	**3**	5	7	11
13			17	19	23
25			29	31	35
37			41	43	47
49			53	55	59
61			65	67	71
73			77	79	83
85			89	91	95
97					

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	**5**	7	11
13			17	19	23
25			29	31	35
37			41	43	47
49			53	55	59
61			65	67	71
73			77	79	83
85			89	91	95
97					

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	**5**	7	11
13			17	19	23
			29	31	
37			41	43	47
49			53		59
61				67	71
73			77	79	83
			89	91	
97					

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	5	**7**	11
13			17	19	23
			29	31	
37			41	43	47
49			53		59
61				67	71
73			77	79	83
			89	91	
97					

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	5	**7**	11
13			17	19	23
			29	31	
37			41	43	47
			53		59
61				67	71
73				79	83
			89		
97					

The Sieve of Erathostenes

Among the first 100 numbers there are 25 prime numbers.

	2	3	5	7	11
13			17	19	23
			29	31	
37			41	43	47
			53		59
61				67	71
73				79	83
			89		
97					

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

Question: How many is "infinitely many"?

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

Question: How many is "infinitely many"?

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

Question: How many is "infinitely many"?

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

Question: How many is "infinitely many"?

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

Question: How many is "infinitely many"?

There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, $p_1 = 2, p_2 = 3, \ldots, p_r$ are all the prime numbers.
- Set $N := p_1 \cdot p_2 \cdots p_r + 1$.
- Then $N \geq 2$, but N is not divisible by any prime number.
- Contradiction!

$\square$

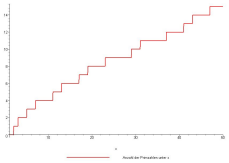
Question: How many is "infinitely many"?

Number of Primes up to a Given Size

Definition

For every real number x let $\pi(x)$ denote the number of primes between 1 and x.

x	$\pi(x)$	$x/\pi(x)$
100	25	4.0
1000	168	≈ 6.0
10,000	1,229	≈ 8.1
100,000	9,592	≈ 10.4
1,000,000	78,498	≈ 12.7
10,000,000	664,579	≈ 15.0
100,000,000	5,761,455	≈ 17.4

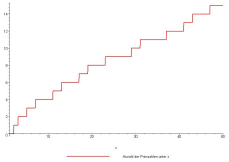


Number of Primes up to a Given Size

Definition

For every real number x let $\pi(x)$ denote the number of primes between 1 and x.

x	$\pi(x)$	$x/\pi(x)$
100	25	4.0
1000	168	≈ 6.0
10,000	1,229	≈ 8.1
100,000	9,592	≈ 10.4
1,000,000	78,498	≈ 12.7
10,000,000	664,579	≈ 15.0
100,000,000	5,761,455	≈ 17.4

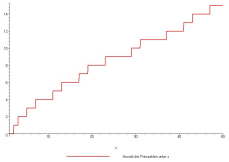


Number of Primes up to a Given Size

Definition

For every real number x let $\pi(x)$ denote the number of primes between 1 and x.

x	$\pi(x)$	$x/\pi(x)$
100	25	4.0
1000	168	≈ 6.0
10,000	1,229	≈ 8.1
100,000	9,592	≈ 10.4
1,000,000	78,498	≈ 12.7
10,000,000	664,579	≈ 15.0
100,000,000	5,761,455	≈ 17.4



The Logarithmic Integral Function

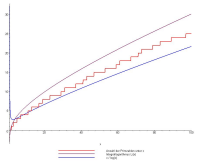
Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The 'density' of primes around a large number x is about $1/log(x)$.

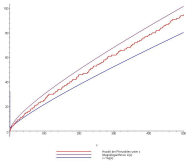
Remark: $\log(x) \approx 2.3 \cdot$ (# digits before the decimal point of x).

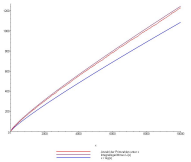
Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\mathrm{Li}(x)$ as a more precise approximation:

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log\lfloor x \rfloor}.$$







The Logarithmic Integral Function

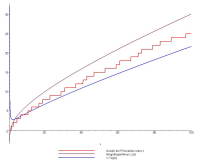
Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The 'density' of primes around a large number x is about $1/log(x)$.

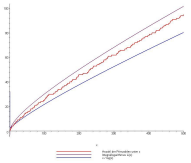
Remark: $\log(x) \approx 2.3 \cdot$ (# digits before the decimal point of x).

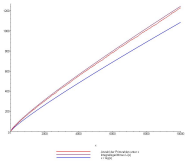
Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\mathrm{Li}(x)$ as a more precise approximation:

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log\lfloor x \rfloor}.$$







The Logarithmic Integral Function

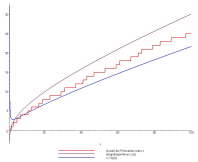
Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The 'density' of primes around a large number x is about $1/log(x)$.

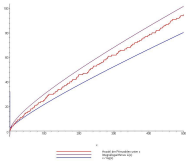
Remark: $\log(x) \approx 2.3 \cdot$ (# digits before the decimal point of x).

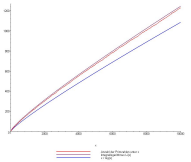
Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\mathrm{Li}(x)$ as a more precise approximation:

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log\lfloor x \rfloor}.$$







The Logarithmic Integral Function

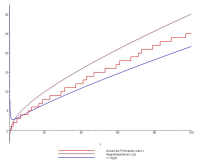
Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The 'density' of primes around a large number x is about $1/log(x)$.

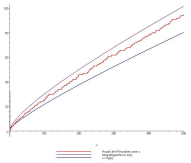
Remark: $\log(x) \approx 2.3 \cdot$ (# digits before the decimal point of x).

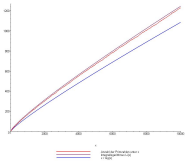
Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\mathrm{Li}(x)$ as a more precise approximation:

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log\lfloor x \rfloor}.$$







The Logarithmic Integral Function

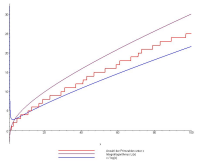
Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The 'density' of primes around a large number x is about $1/log(x)$.

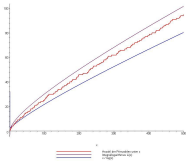
Remark: $\log(x) \approx 2.3 \cdot$ (# digits before the decimal point of x).

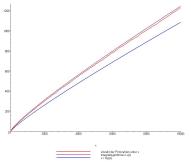
Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\mathrm{Li}(x)$ as a more precise approximation:

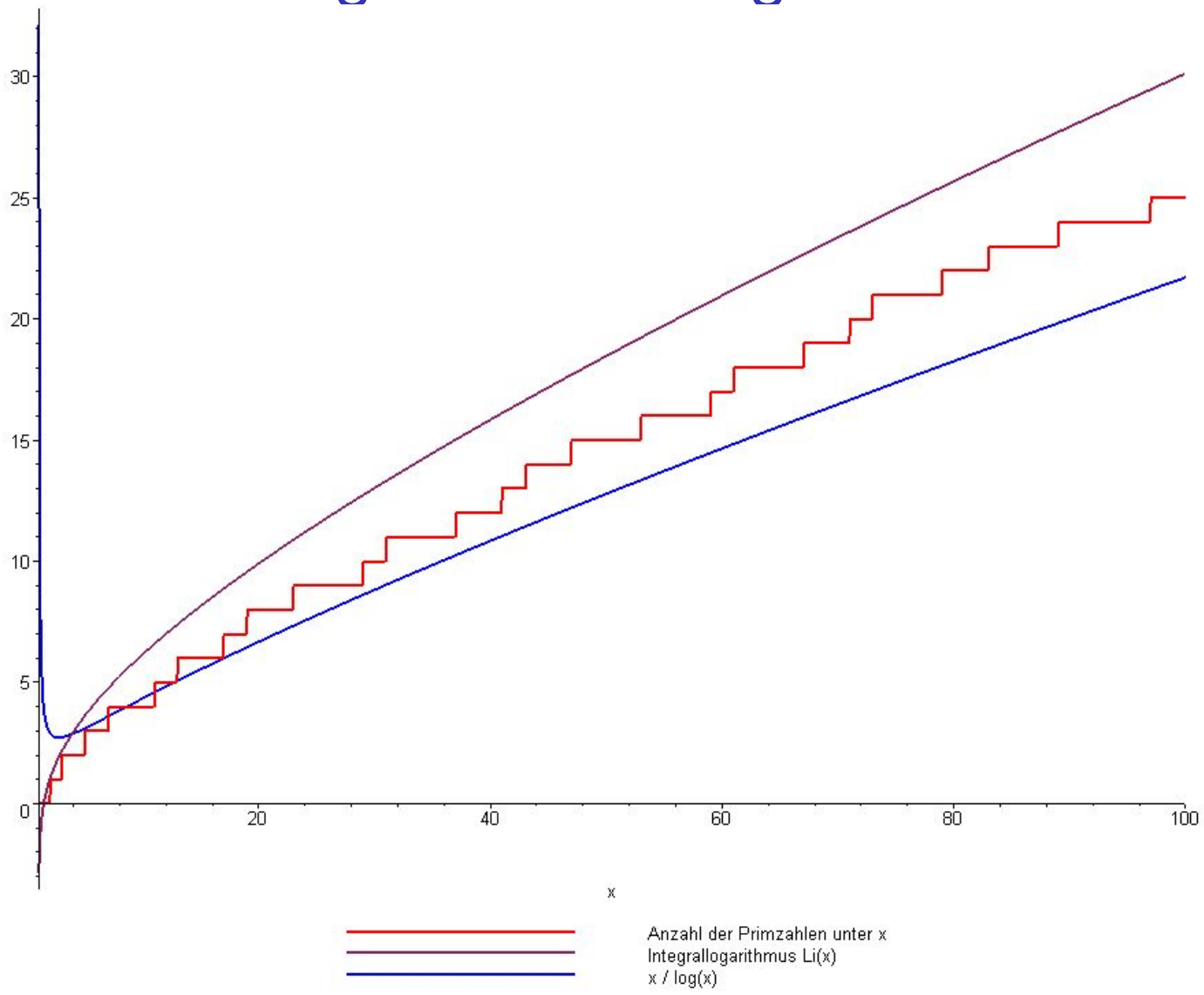
$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log\lfloor x \rfloor}.$$



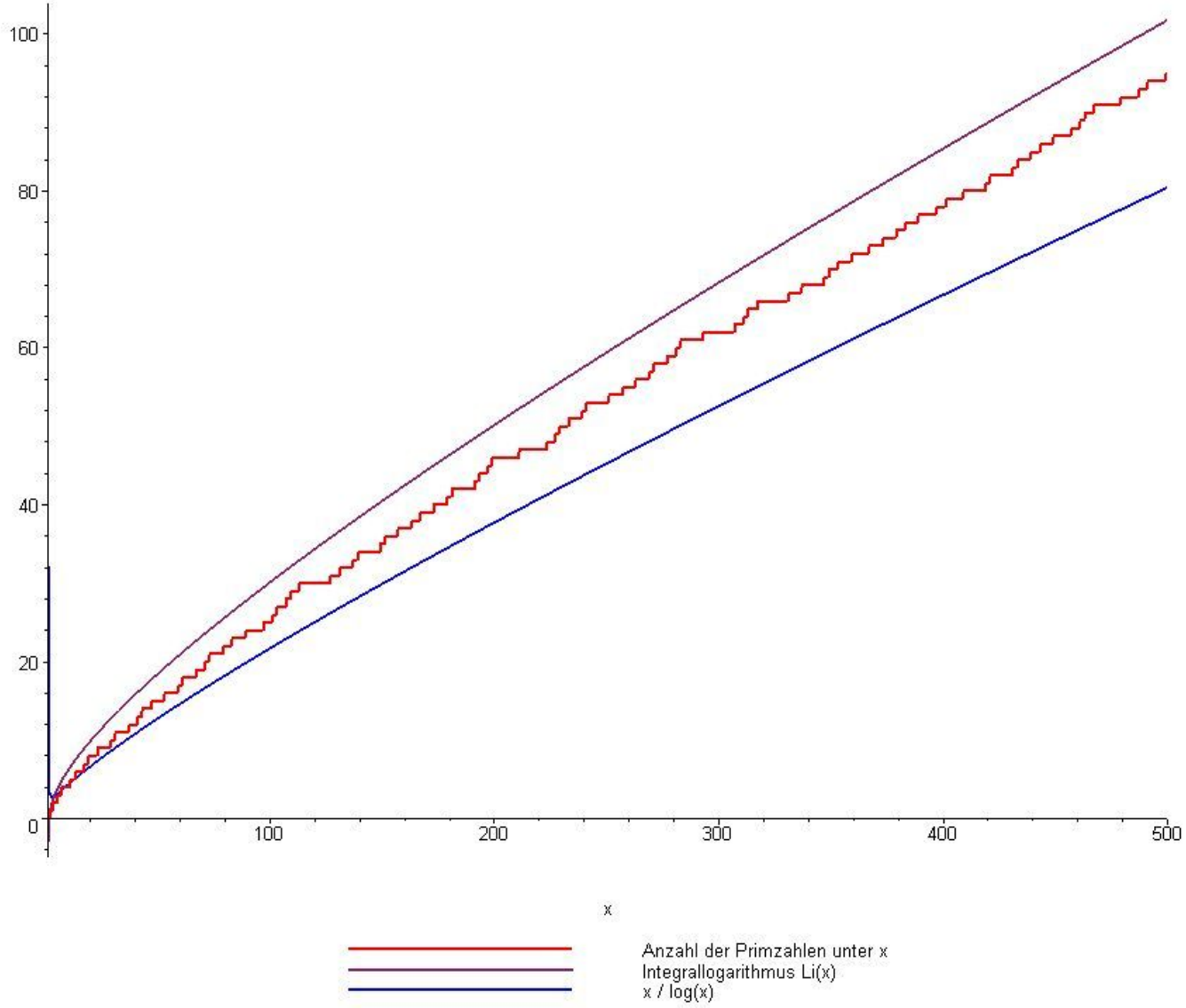




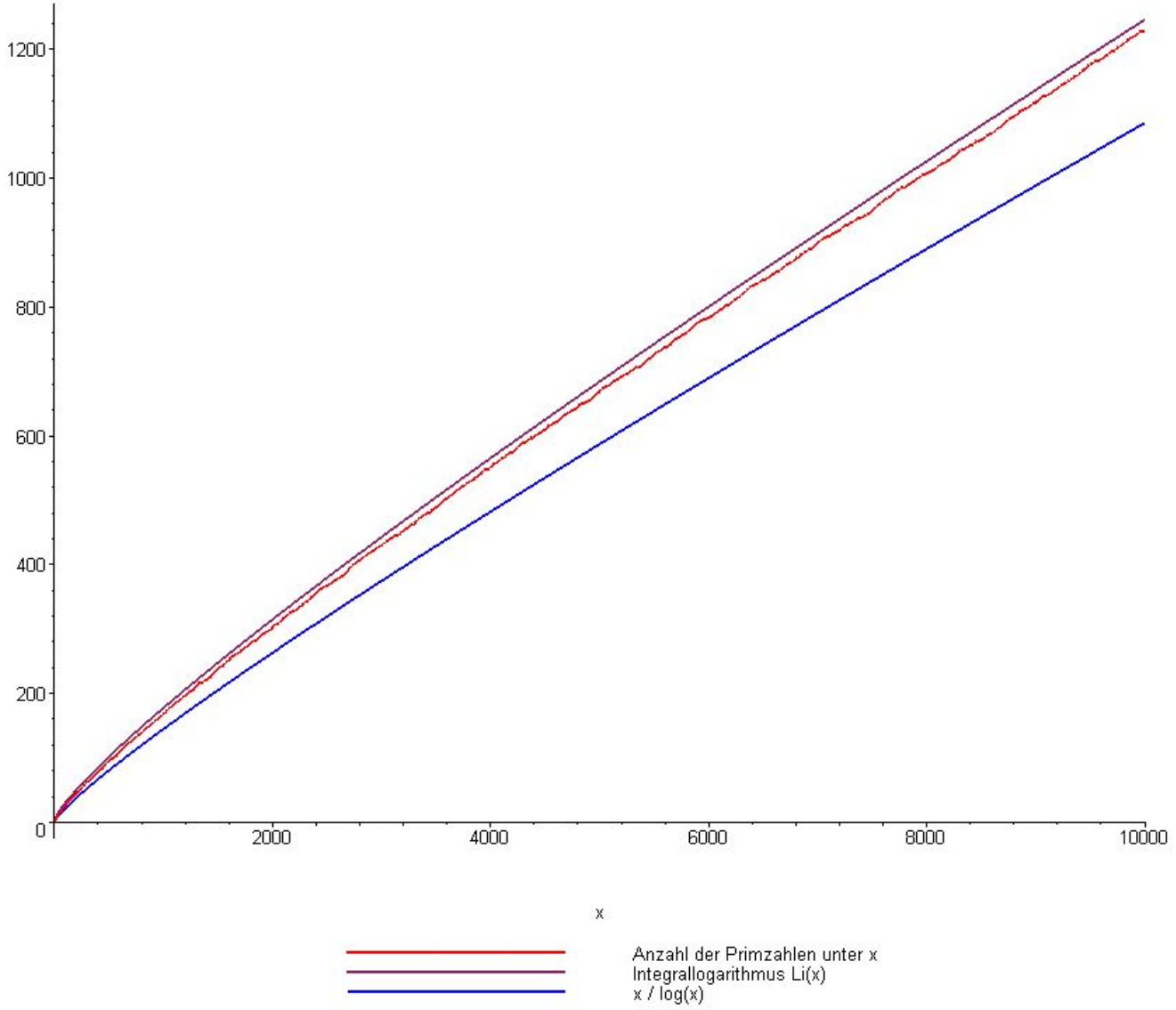
The Logarithmic Integral Function



The Logarithmic Integral Function



The Logarithmic Integral Function



We build infinite series

We build infinite series

- The harmonic series diverges to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty.$$

to the proof

We build infinite series

For

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

We build infinite series

For

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots$$

$$\geq 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots$$

We build infinite series

For

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots\\ &\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots\\ &= 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\dots\end{aligned}$$

We build infinite series

For

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n} &= 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\dots\\ &\geq 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots\\ &= 1+\frac{1}{2}+\frac{2}{4}+\frac{4}{8}+\dots\\ &= 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\dots \quad = \quad \infty.\end{aligned}$$

We build infinite series

- The harmonic series diverges to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty.$$

to the proof

We build infinite series

- The harmonic series diverges to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \infty.$$

- In contrast, the sum of quadratic reciprocals

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

is bounded above and converges.

to the proof

We build infinite series

For

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}$$

We build infinite series

For

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \quad \leq \quad 1 + \sum_{n=2}^{\infty} \frac{1}{(n-1)n}$$

We build infinite series

For

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\sum_{n=2}^{\infty}\frac{1}{n^2} \quad \leq \quad 1+\sum_{n=2}^{\infty}\frac{1}{(n-1)n}$$

$$= 1+\sum_{n=2}^{\infty}\frac{n-(n-1)}{(n-1)n} \quad = \quad 1+\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

We build infinite series

For

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\sum_{n=2}^{\infty}\frac{1}{n^2} \quad \leq \quad 1+\sum_{n=2}^{\infty}\frac{1}{(n-1)n}$$

$$= 1+\sum_{n=2}^{\infty}\frac{n-(n-1)}{(n-1)n} \quad = \quad 1+\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

We build infinite series

For

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = 1+\sum_{n=2}^{\infty}\frac{1}{n^2} \quad \leq \quad 1+\sum_{n=2}^{\infty}\frac{1}{(n-1)n}$$

$$= 1+\sum_{n=2}^{\infty}\frac{n-(n-1)}{(n-1)n} \quad = \quad 1+\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right)$$

$$= 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

$$= 1+1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\dots$$

We build infinite series

For

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{n^2} &= 1+\sum_{n=2}^{\infty}\frac{1}{n^2} \quad \leq \quad 1+\sum_{n=2}^{\infty}\frac{1}{(n-1)n} \\
&= 1+\sum_{n=2}^{\infty}\frac{n-(n-1)}{(n-1)n} \quad = \quad 1+\sum_{n=2}^{\infty}\left(\frac{1}{n-1}-\frac{1}{n}\right) \\
&= 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots \\
&= 1+1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\dots \\
&= 1+1+0+0+0+\dots \quad = \quad 2.
\end{aligned}$$

We build infinite series

- The harmonic series diverges to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \infty.$$

- In contrast, the sum of quadratic reciprocals

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

is bounded above and converges.

to the proof

We build infinite series

- The harmonic series diverges to infinity:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots = \infty.$$

- In contrast, the sum of quadratic reciprocals

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = 1 + \frac{1}{4} + \frac{1}{9} + \dots$$

is bounded above and converges.

Question: What is the sum of all the quadratic reciprocals?

Euler and the 'Real Zeta Function'

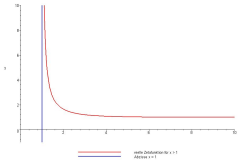
One has

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449,$$

where $\pi = 3.1415....$

Euler considered more generally the real function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s > 1$$



and he computed $\zeta(2m)$ for all even numbers 2, 4, 6, 8, ...

Euler and the 'Real Zeta Function'

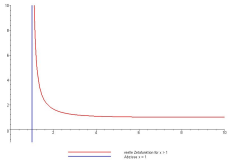
One has

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449,$$

where $\pi = 3.1415....$

Euler considered more generally the real function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s > 1$$



and he computed $\zeta(2m)$ for all even numbers 2, 4, 6, 8, ...

Euler and the 'Real Zeta Function'

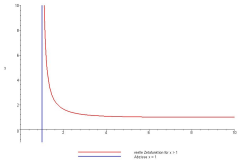
One has

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449,$$

where $\pi = 3.1415....$

Euler considered more generally the real function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \qquad \text{for } s > 1$$



and he computed $\zeta(2m)$ for all even numbers 2, 4, 6, 8, ...

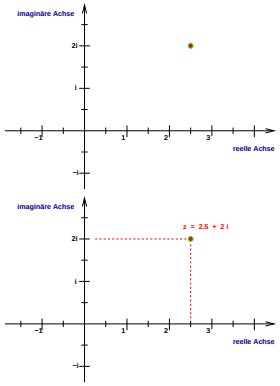
The Euler Product Formula

The Euler Product Formula yields a interesting connection between the function $\zeta(s)$ and prime numbers:

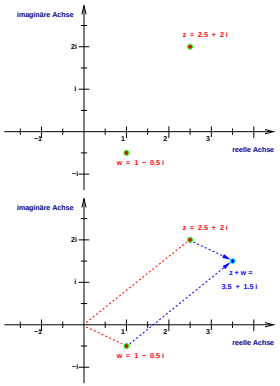
$$\begin{aligned}
\zeta(s) &= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \dots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \dots\right) \cdot \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \dots\right) \\
&\quad \cdot \left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \dots\right) \cdots \\
&= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \dots\right) \\
&= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.
\end{aligned}$$

The Complex Number Plane

The familiar real number line $\mathbb{R}$ extends to the so-called complex number plane $\mathbb{C} = \mathbb{R} + i\mathbb{R}$.



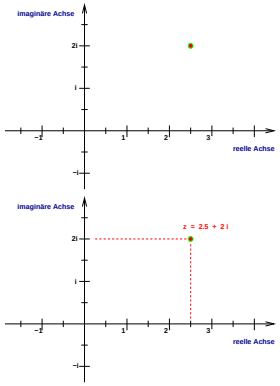
The complex nmber i, called the imaginary unit, has the property that its square equals -1.



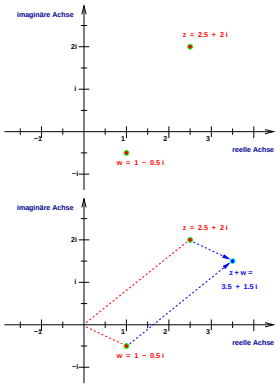
The basic arithmetic operations addition, subtraction, multiplication and division extend from the real numbers to the complex numbers.

The Complex Number Plane

The familiar real number line $\mathbb{R}$ extends to the so-called complex number plane $\mathbb{C} = \mathbb{R} + i\mathbb{R}$.



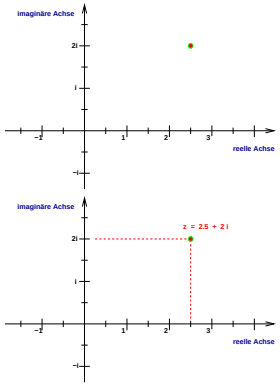
The complex nmber i, called the imaginary unit, has the property that its square equals -1.



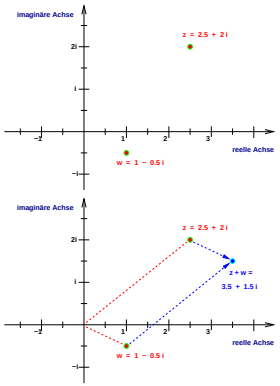
The basic arithmetic operations addition, subtraction, multiplication and division extend from the real numbers to the complex numbers.

The Complex Number Plane

The familiar real number line $\mathbb{R}$ extends to the so-called complex number plane $\mathbb{C} = \mathbb{R} + i\mathbb{R}$.



The complex nmber i, called the imaginary unit, has the property that its square equals -1.



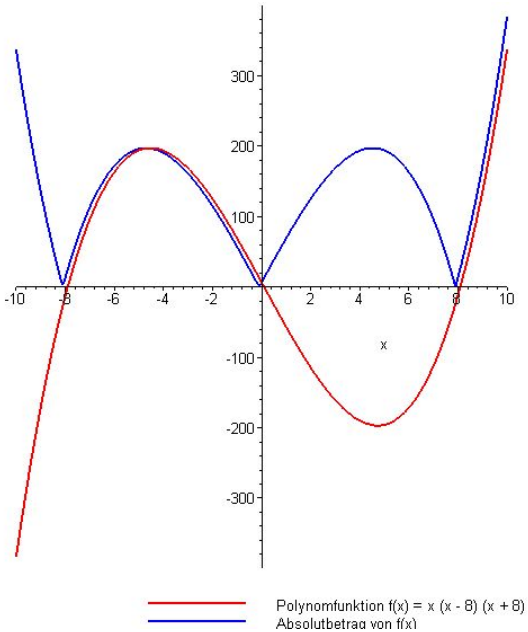
The basic arithmetic operations addition, subtraction, multiplication and division extend from the real numbers to the complex numbers.

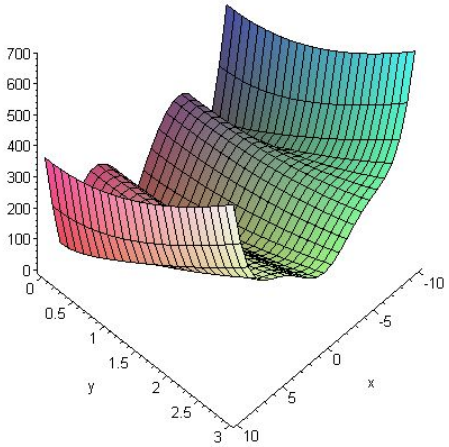
Complex Functions

Simple transformations of the complex number plane to itself, such as the polynomial function

$$f(z) = f(x + iy) := z^3 - 64z,$$

can be visualised as landscapes.



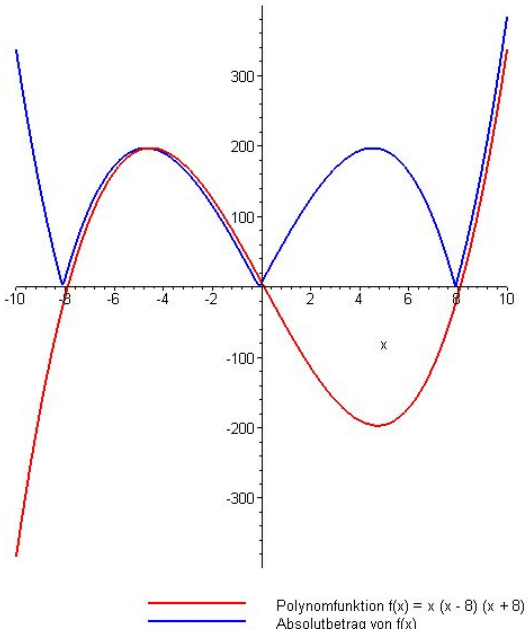


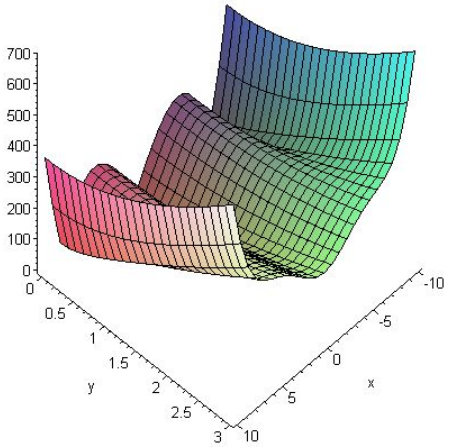
Complex Functions

Simple transformations of the complex number plane to itself, such as the polynomial function

$$f(z) = f(x + iy) := z^3 - 64z,$$

can be visualised as landscapes.





Georg Friedrich Bernhard Riemann

Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

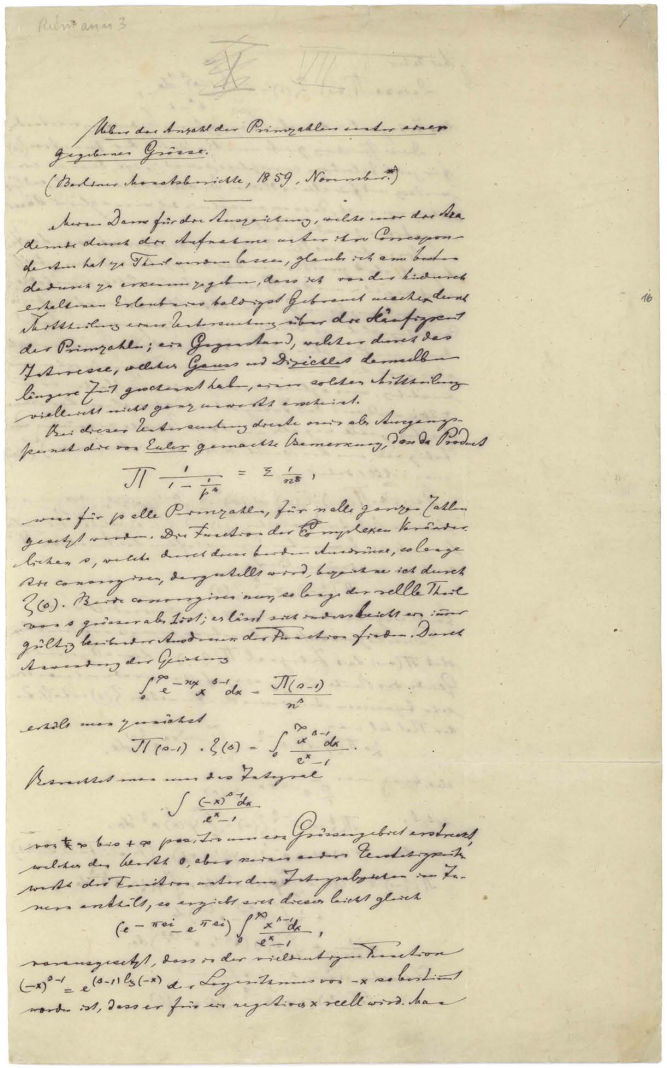
Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

Georg Friedrich Bernhard Riemann

- born 1826 in Breselenz, Lüchow-Dannenberg
- student in Göttingen (initially in Theology) and Berlin
- 1851 PhD with the seminal Inauguraldissertation "Grundlagen für eine allgemeine Theorie der Functionen einer veränderlichen complexen Grösse"
- 1854 Habilitation lecture "Über die Hypothesen, welche der Geometrie zu Grunde liegen" - a cornerstone of modern Differential Geometry
- 1859 successor of Dirichlet to the Gauß-Chair, publication of the article "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse"
- deceased 1866 from tuberculosis in Selasca, Lago Maggiore, Italy

Georg Friedrich Bernhard Riemann



Über die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November)

Wenn Dank für die Auszeichnung, welche man der Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaubt ich am besten dadurch zu erzeugen zu geben, dass ich von der kürzlich erhaltenen Erlaubniss, welche mir die Freude gewährt, eine Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit gewidmet haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Variablen s, welche durch den linken Ausdruck, so lange er convergirt, dargestellt wird, bezeichne ich durch $\zeta(s)$. Diese convergirt nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indessen leicht ein ihr gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1) \cdot \zeta(s) = \int_0^\infty \frac{x^{s-1} dx}{e^x - 1}.$$

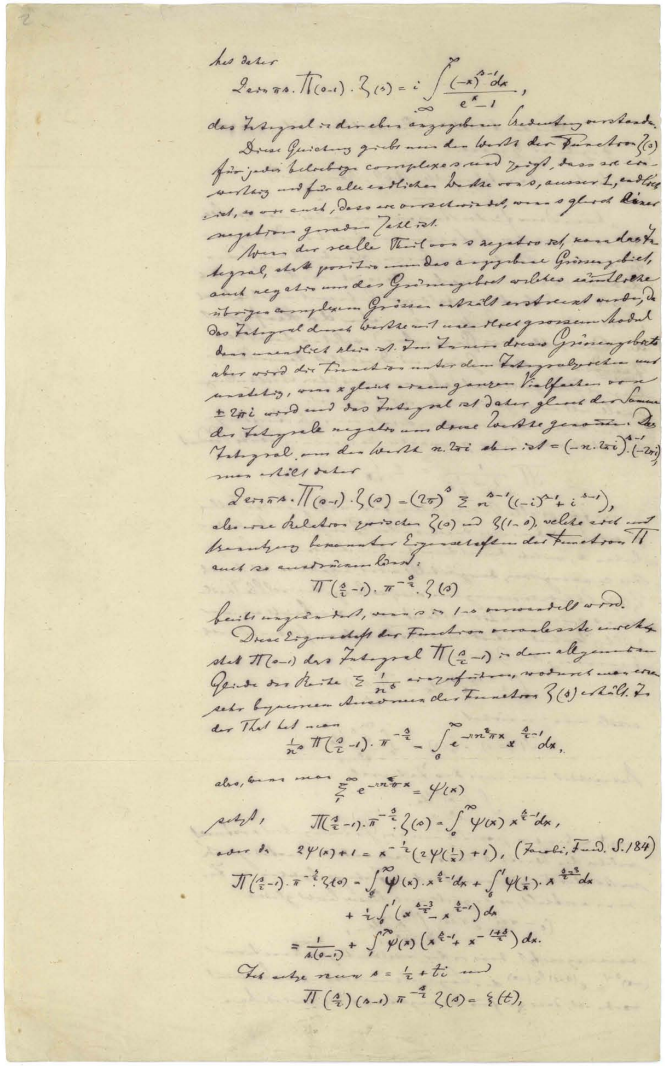
Nimmt man nun das Integral

$$\int \frac{(-x)^{s-1} dx}{e^x - 1}$$

von $x = \infty$ bis $x = +\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Verzweigungspunkt der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich aus diesem leicht gleich

$$(e^{-\pi i} - e^{\pi i}) \int_0^\infty \frac{x^{s-1} dx}{e^x - 1},$$

vorausgesetzt, dass in der vielwerthigen Function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ der Logarithmus von $-x$ so bestimmt wird, dass er für ein negatives x reell wird. Hat



hat daher

$$2\sin\pi s \cdot \Pi(s-1) \cdot \zeta(s) = i \int_\infty^{\infty} \frac{(-x)^{s-1} dx}{e^x - 1},$$

das Integral in der eben angegebenen Bedeutung verstanden.

Diese Gleichung gilt nun der Lehre der Functionen zufolge für jeden beliebigen complexen Werth, dass sie eine Bestimmung und für alle endlichen Werthe von s, ausser 1, endlich ist, so wie auch, dass sie unverändert bleibt, wenn s gleich $1-n$ negativen geraden Zahl ist.

Wenn der reelle Theil von s negativ ist, kann das Integral, statt positiv um das angegebene Grössengebiet, auch negativ um das Grössengebiet erstreckt werden, welches ausserhalb der Grössen $\pm 2\pi i$ ist. Das Integral ist dann gleich der Summe des Integrals negativ um diese beiden Werthe genommen. Das Integral um den Werth $n \cdot 2\pi i$ aber ist $= (-n \cdot 2\pi i)^{s-1} \cdot (-2\pi i)$ und erhält daher

$$2\sin\pi s \cdot \Pi(s-1) \cdot \zeta(s) = (2\pi)^s \sum n^{s-1}((-i)^{s-1} + i^{s-1}),$$

also eine Relation zwischen $\zeta(s)$ und $\zeta(1-s)$, welche auch durch bekannte Eigenschaften der Function Π auch so ausgedrückt werden kann:

$$\Pi\left(\tfrac{s}{2}-1\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s)$$

bleibt ungeändert, wenn s in $1-s$ verwandelt wird.

Diese Eigenschaft der Function veranlasste mich, statt $\Pi(s-1)$ das Integral $\Pi\left(\frac{s}{2}-1\right)$ in den allgemeinen Gliede der Reihe $\sum \frac{1}{n^s}$ einzuführen, wodurch man eine andere bequemere Ausdrucksweise der Function $\zeta(s)$ erhält. Zu dem Theil hat man

$$\frac{1}{n^s}\Pi\left(\tfrac{s}{2}-1\right) \cdot \pi^{-\frac{s}{2}} = \int_0^\infty e^{-n^2\pi x} x^{\frac{s}{2}-1} dx,$$

also, wenn man

$$\sum_1^\infty e^{-n^2\pi x} = \psi(x)$$

setzt,

$$\Pi\left(\tfrac{s}{2}-1\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s) = \int_0^\infty \psi(x) x^{\frac{s}{2}-1} dx,$$

oder da $2\psi(x) + 1 = x^{-\frac{1}{2}}(2\psi(\frac{1}{x}) + 1)$, (Jacobi, F.-D. S. 184)

$$\Pi\left(\tfrac{s}{2}-1\right) \cdot \pi^{-\frac{s}{2}} \cdot \zeta(s) = \int_1^\infty \psi(x) \cdot x^{\frac{s}{2}-1} dx + \int_0^1 \psi(\tfrac{1}{x}) \cdot x^{\frac{s}{2}-1} dx$$

$$+ \tfrac{1}{2}\int_0^1 \left(x^{\frac{s-3}{2}} - x^{\frac{s}{2}-1}\right) dx$$

$$= \frac{1}{s(s-1)} + \int_1^\infty \psi(x)\left(x^{\frac{s}{2}-1} + x^{-\frac{1+s}{2}}\right) dx.$$

Setzt man nun $s = \frac{1}{2} + ti$ und

$$\Pi\left(\tfrac{s}{2}\right)(s-1)\pi^{-\frac{s}{2}} \zeta(s) = \xi(t),$$

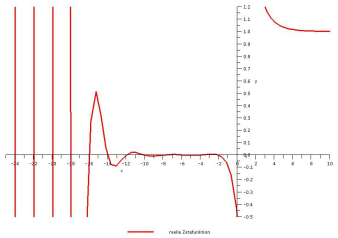
The Riemann Zeta Function

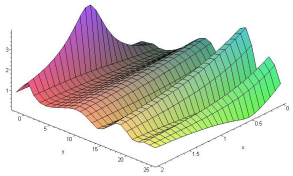
The infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

converges for all complex numbers $s = x + yi$ with real part $x > 1$.

Moreover, the function $\zeta(s)$, extends uniquely to the entire complex number plane, apart from a pole at $s = 1$.





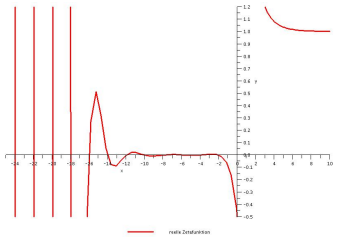
The Riemann Zeta Function

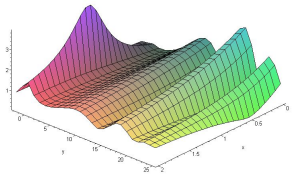
The infinite series

$$\zeta(s) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^s} \quad = \quad 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

converges for all complex numbers $s = x + yi$ with real part $x > 1$.

Moreover, the function $\zeta(s)$, extends uniquely to the entire complex number plane, apart from a pole at $s = 1$.





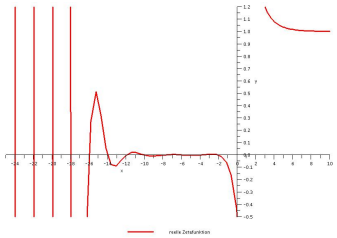
The Riemann Zeta Function

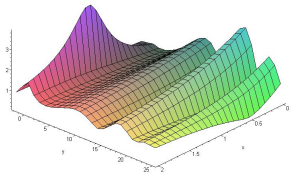
The infinite series

$$\zeta(s) \quad = \quad \sum_{n=1}^{\infty} \frac{1}{n^s} \quad = \quad 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \dots$$

converges for all complex numbers $s = x + yi$ with real part $x > 1$.

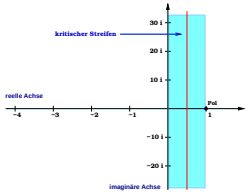
Moreover, the function $\zeta(s)$, extends uniquely to the entire complex number plane, apart from a pole at $s = 1$.



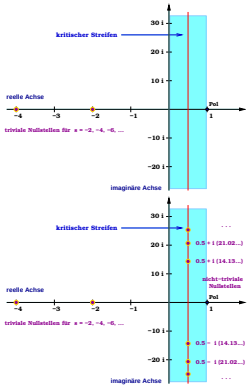


The Critical Strip

The 'real' function $\zeta(s)$ can be extended to the entire complex number plane by means of the Functional Equation

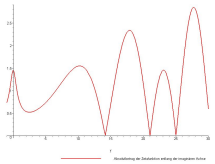


$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s).$$



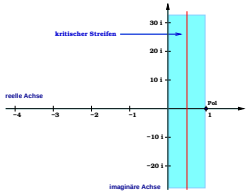
Of particular importance are the zeros of $\Lambda(s)$. They correspond to the non-trivial zeros of $\zeta(s)$ and lie in the critical strip $\mathcal{S} := \{x + yi \mid 0 \leq x \leq 1\}$.

All *known* zeros of $\Lambda(s)$ lie on the line $\mathcal{G} := \{x + yi \mid x = 1/2\}$.

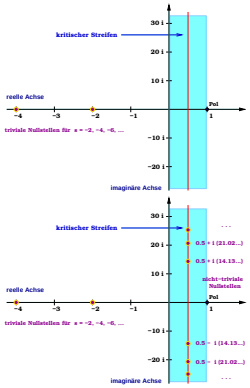


The Critical Strip

The 'real' function $\zeta(s)$ can be extended to the entire complex number plane by means of the Functional Equation

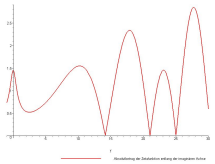


$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s).$$



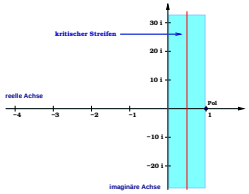
Of particular importance are the zeros of $\Lambda(s)$. They correspond to the non-trivial zeros of $\zeta(s)$ and lie in the critical strip $\mathcal{S} := \{x + yi \mid 0 \leq x \leq 1\}$.

All *known* zeros of $\Lambda(s)$ lie on the line $\mathcal{G} := \{x + yi \mid x = 1/2\}$.

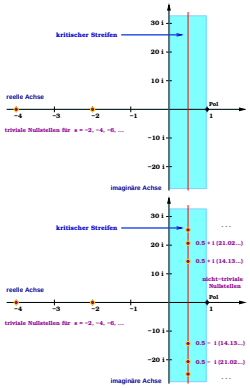


The Critical Strip

The 'real' function $\zeta(s)$ can be extended to the entire complex number plane by means of the Functional Equation

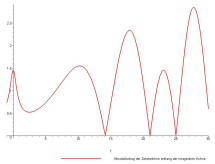


$$\Lambda(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \Lambda(1-s).$$



Of particular importance are the zeros of $\Lambda(s)$. They correspond to the non-trivial zeros of $\zeta(s)$ and lie in the critical strip $\mathcal{S} := \{x + yi \mid 0 \leq x \leq 1\}$.

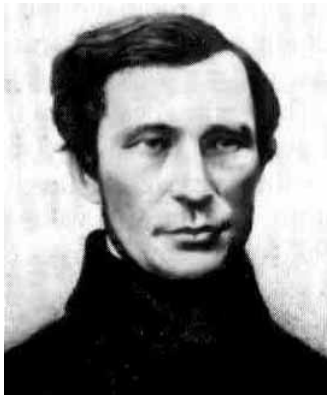
All *known* zeros of $\Lambda(s)$ lie on the line $\mathcal{G} := \{x + yi \mid x = 1/2\}$.



The Chebyshev ψ-Function

Recall: We are to show that, for large x, the number $\pi(x)$ of prime numbers less than or equal to x is closely approximated by the logarithmic integral $\mathrm{Li}(x)$.

Grafik



More handy than the prime counting function $\pi(x) = \sum_{p \le x} 1$ is a variant, introduced by Chebyshev,

$$\psi(x) := \sum_{p^k \le x} \log(p),$$

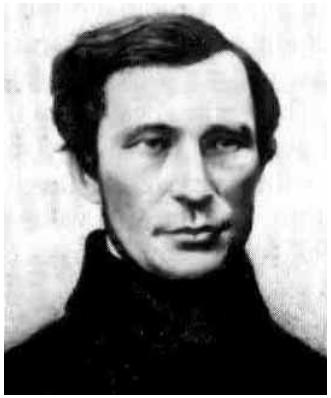
which counts prime numbers with a logarithmic weight.

Loosely speaking, the approximation $\pi(x) \approx \mathrm{Li}(x)$ is equivalent to $\psi(x) \approx x$.

The Chebyshev ψ-Function

Recall: We are to show that, for large x, the number $\pi(x)$ of prime numbers less than or equal to x is closely approximated by the logarithmic integral $\mathrm{Li}(x)$.

Grafik



More handy than the prime counting function $\pi(x) = \sum_{p \le x} 1$ is a variant, introduced by Chebyshev,

$$\psi(x) := \sum_{p^k \le x} \log(p),$$

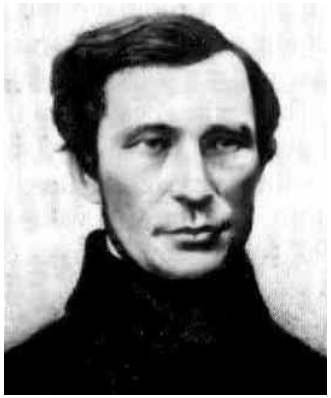
which counts prime numbers with a logarithmic weight.

Loosely speaking, the approximation $\pi(x) \approx \mathrm{Li}(x)$ is equivalent to $\psi(x) \approx x$.

The Chebyshev ψ-Function

Recall: We are to show that, for large x, the number $\pi(x)$ of prime numbers less than or equal to x is closely approximated by the logarithmic integral $\mathrm{Li}(x)$.

Grafik



More handy than the prime counting function $\pi(x) = \sum_{p \le x} 1$ is a variant, introduced by Chebyshev,

$$\psi(x) := \sum_{p^k \le x} \log(p),$$

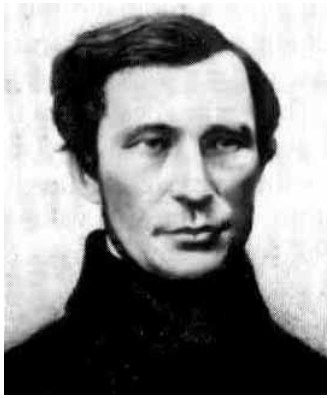
which counts prime numbers with a logarithmic weight.

Loosely speaking, the approximation $\pi(x) \approx \mathrm{Li}(x)$ is equivalent to $\psi(x) \approx x$.

The Chebyshev ψ-Function

Recall: We are to show that, for large x, the number $\pi(x)$ of prime numbers less than or equal to x is closely approximated by the logarithmic integral $\mathrm{Li}(x)$.

Grafik



More handy than the prime counting function $\pi(x) = \sum_{p \le x} 1$ is a variant, introduced by Chebyshev,

$$\psi(x) := \sum_{p^k \le x} \log(p),$$

which counts prime numbers with a logarithmic weight.

Loosely speaking, the approximation $\pi(x) \approx \mathrm{Li}(x)$ is equivalent to $\psi(x) \approx x$.

Riemann's Formula for the Prime Counting Function

The step function $\psi(x)$ can be expressed precisely in terms of the complex zeros of the Riemann zeta function:

$$\psi.(x) \quad = \quad x - \log(2\pi) - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

The following graphics visualise how the prime counting function $\psi(x)$ is stepwise approximated using the first 300 pairs of zeros of the Riemann zeta function.

Graphics

Riemann's Formula for the Prime Counting Function

The step function $\psi(x)$ can be expressed precisely in terms of the complex zeros of the Riemann zeta function:

$$\psi.(x) \quad = \quad x - \log(2\pi) - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

The following graphics visualise how the prime counting function $\psi(x)$ is stepwise approximated using the first 300 pairs of zeros of the Riemann zeta function.

Graphics

Riemann's Formula for the Prime Counting Function

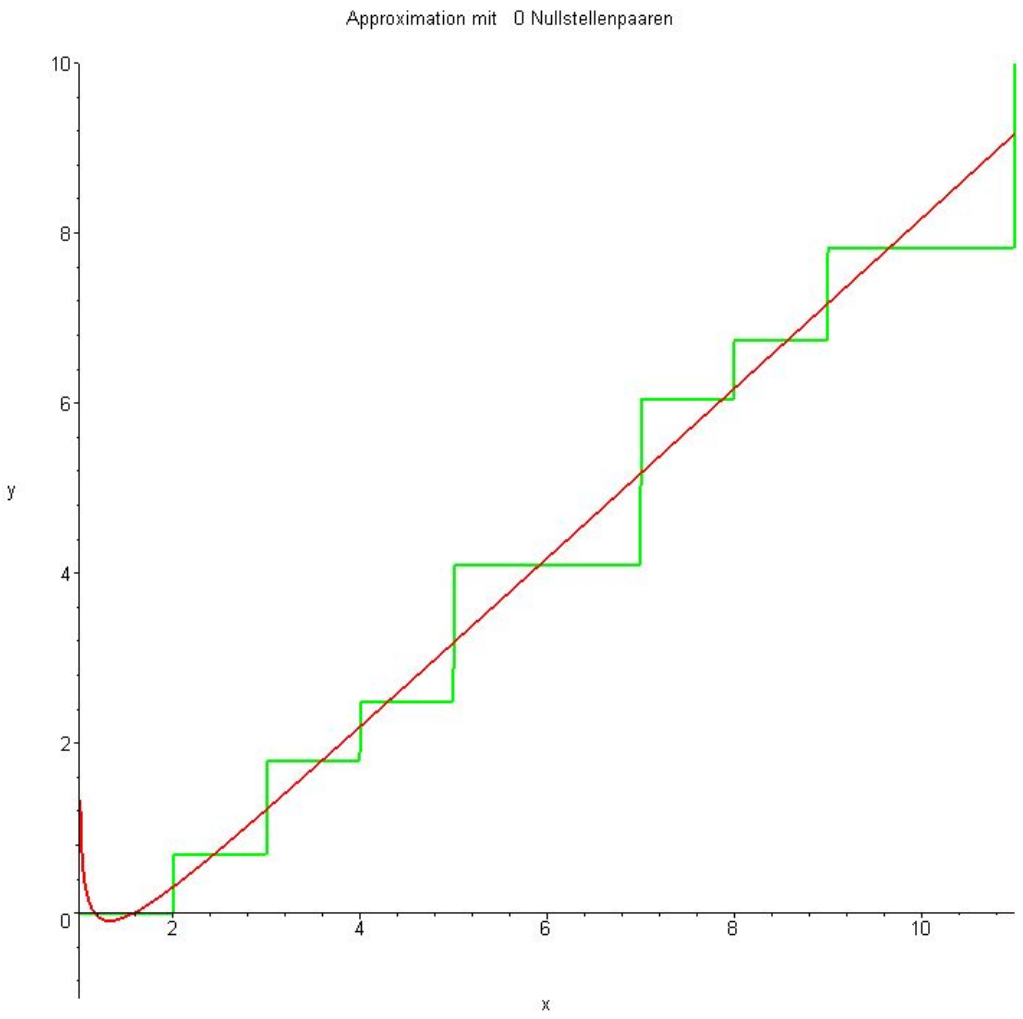
The step function $\psi(x)$ can be expressed precisely in terms of the complex zeros of the Riemann zeta function:

$$\psi.(x) \quad = \quad x - \log(2\pi) - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log\left(1 - \frac{1}{x^2}\right).$$

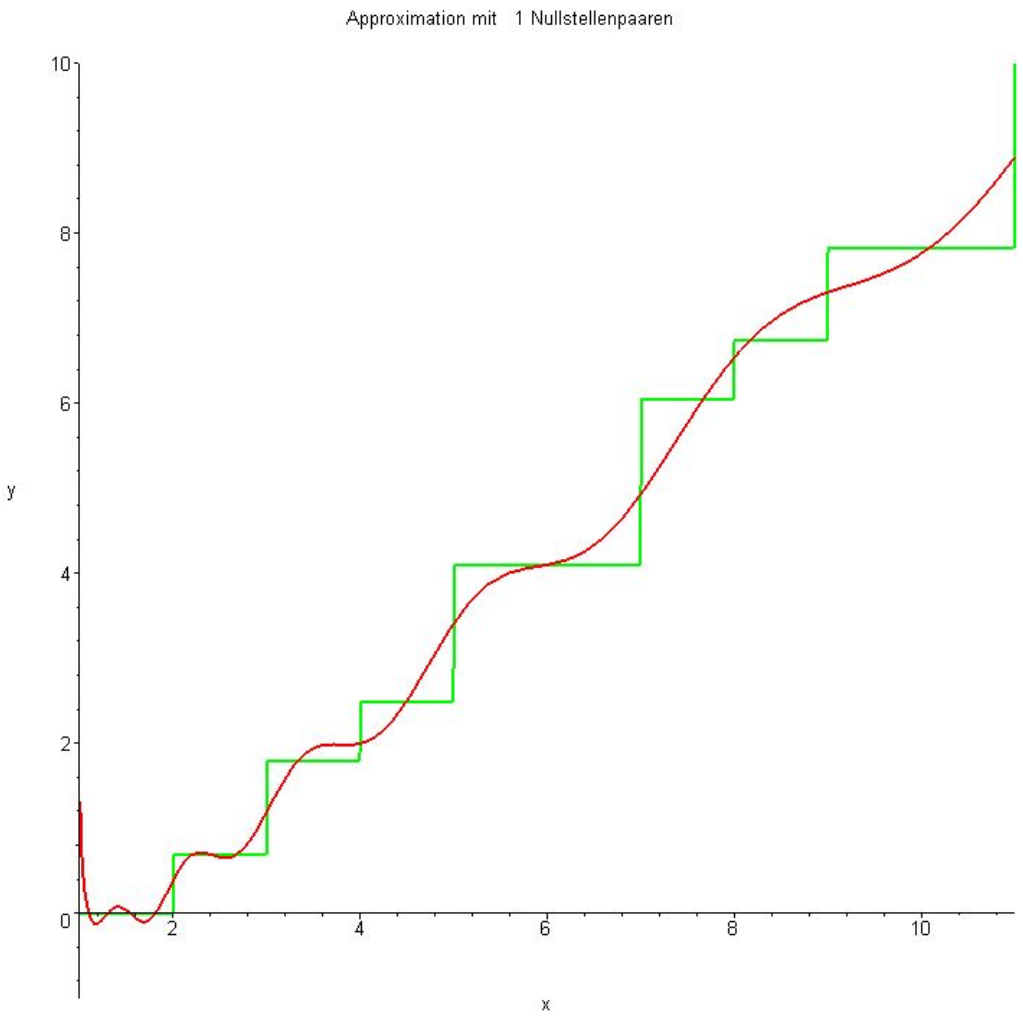
The following graphics visualise how the prime counting function $\psi(x)$ is stepwise approximated using the first 300 pairs of zeros of the Riemann zeta function.

Graphics

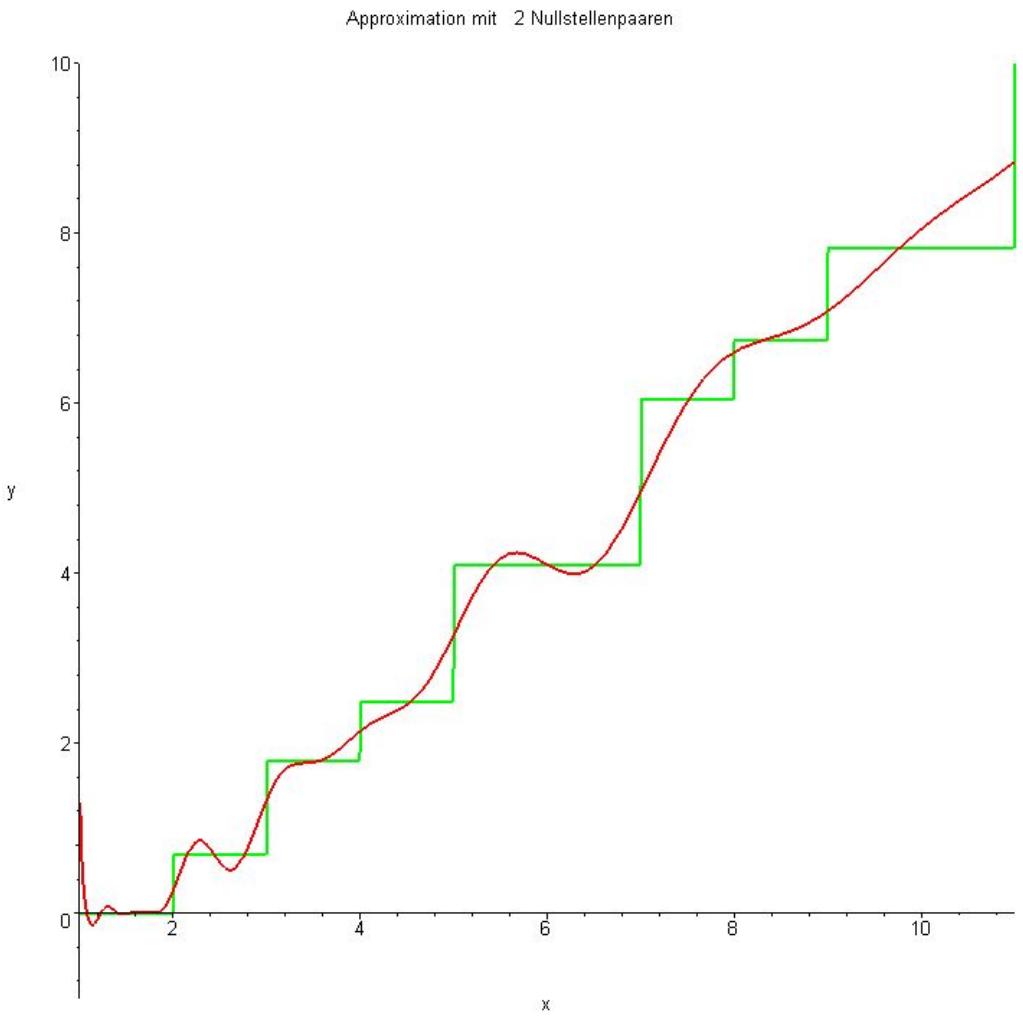
Stepwise Approximation of $\psi(x)$



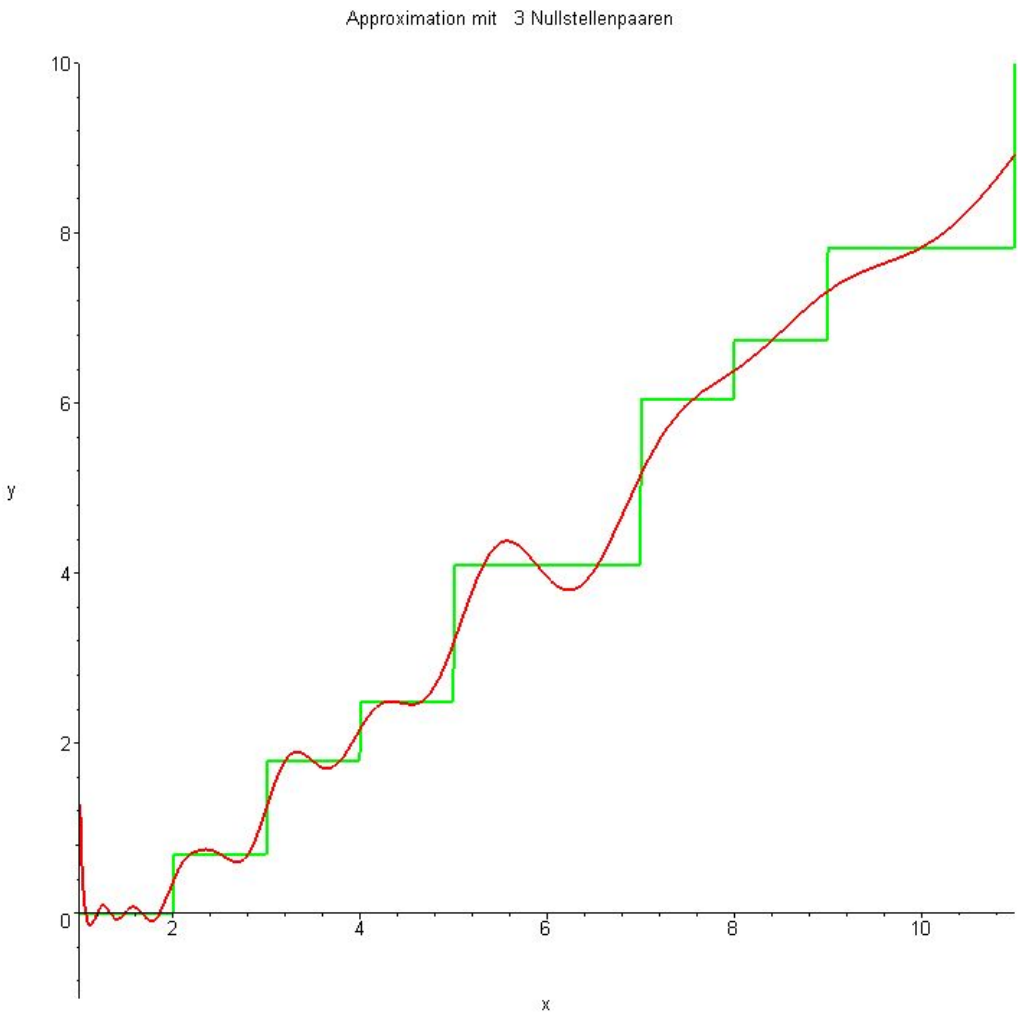
Stepwise Approximation of $\psi(x)$



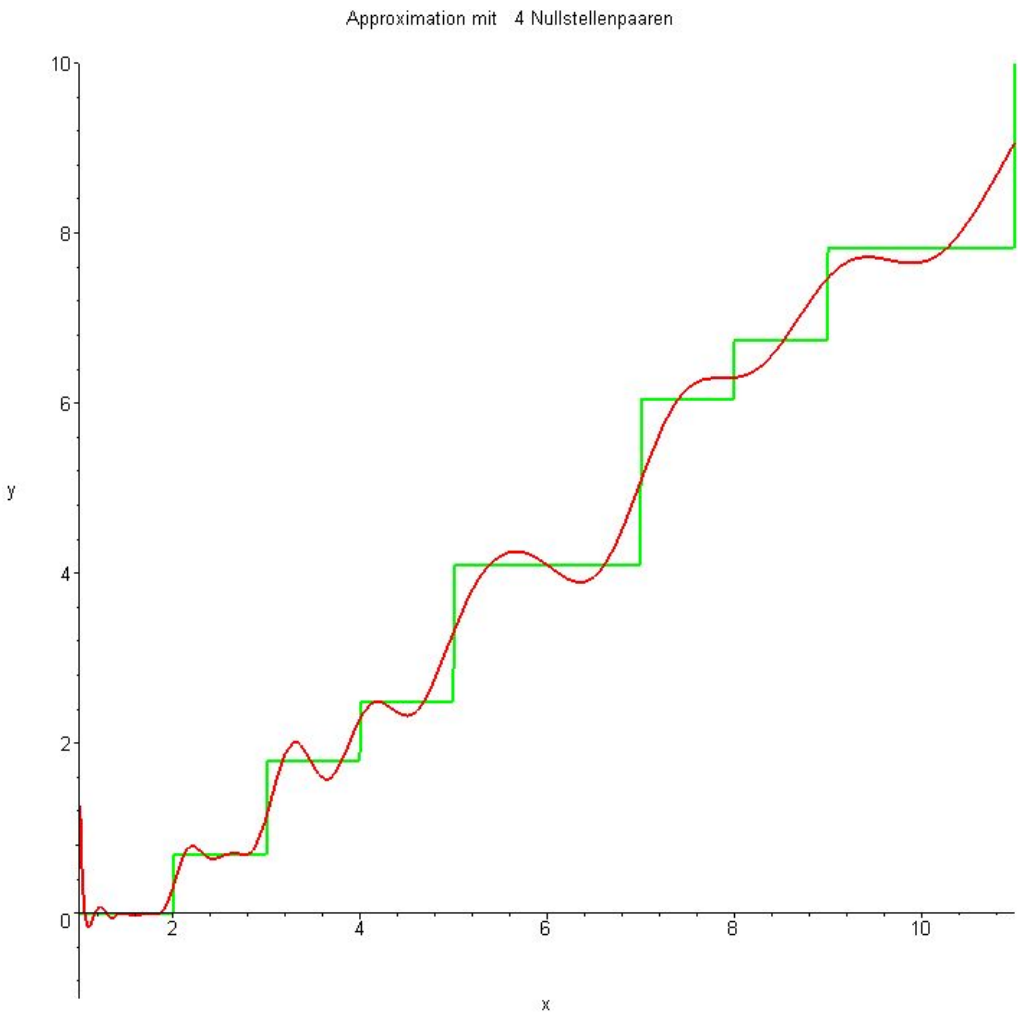
Stepwise Approximation of $\psi(x)$



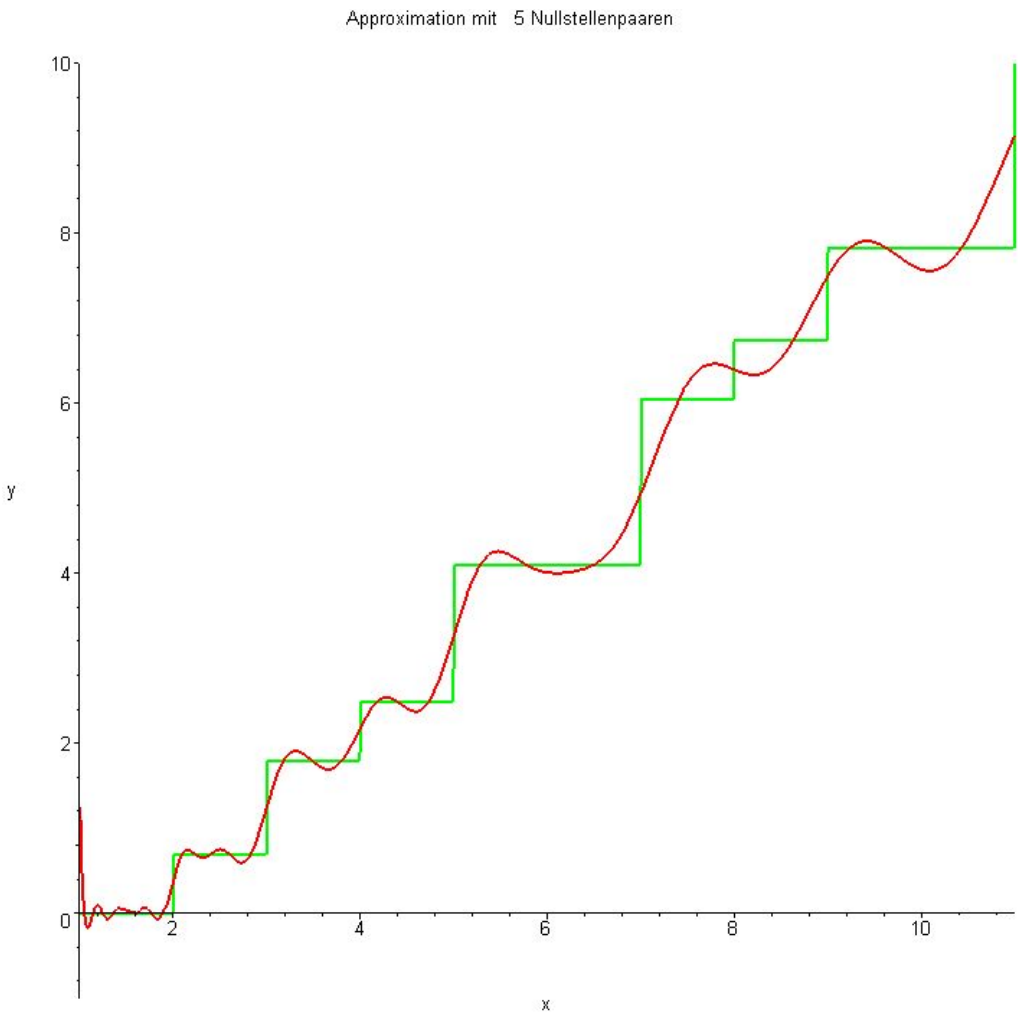
Stepwise Approximation of $\psi(x)$



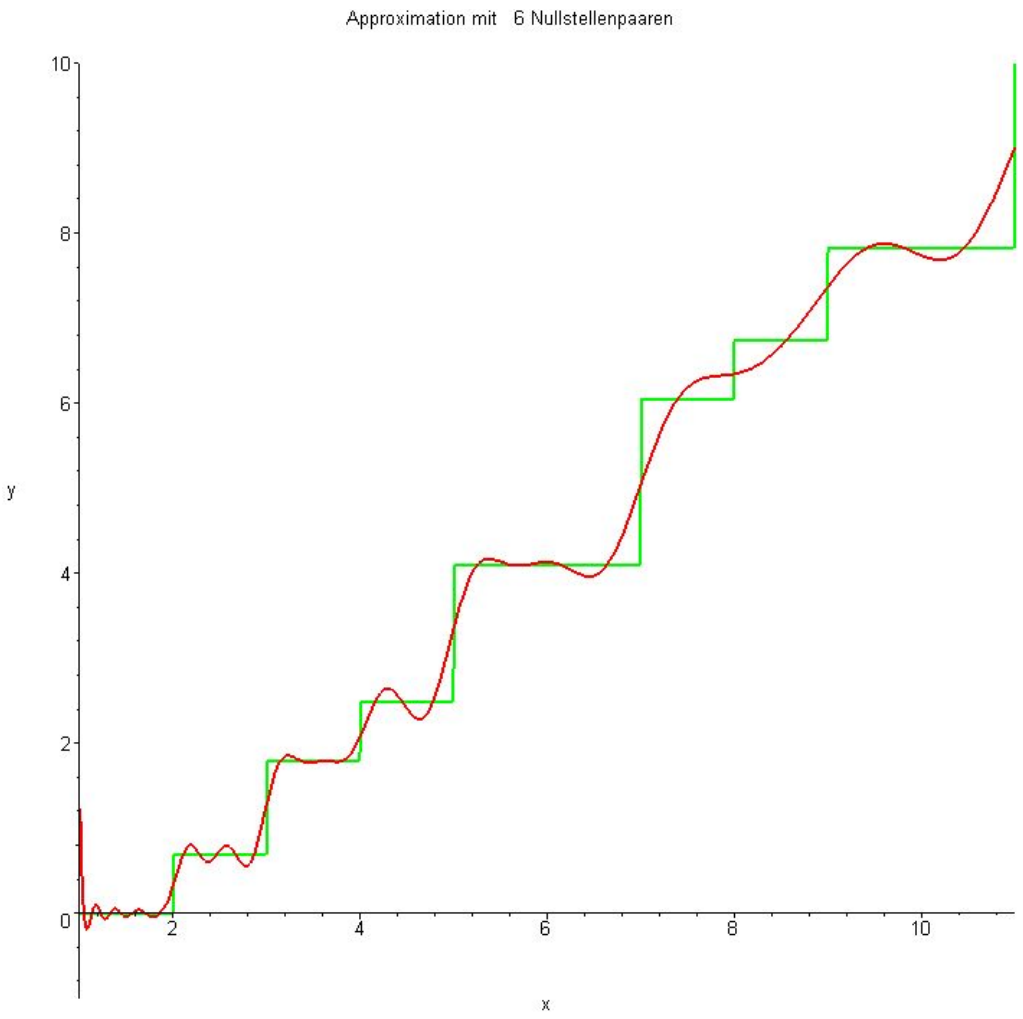
Stepwise Approximation of $\psi(x)$



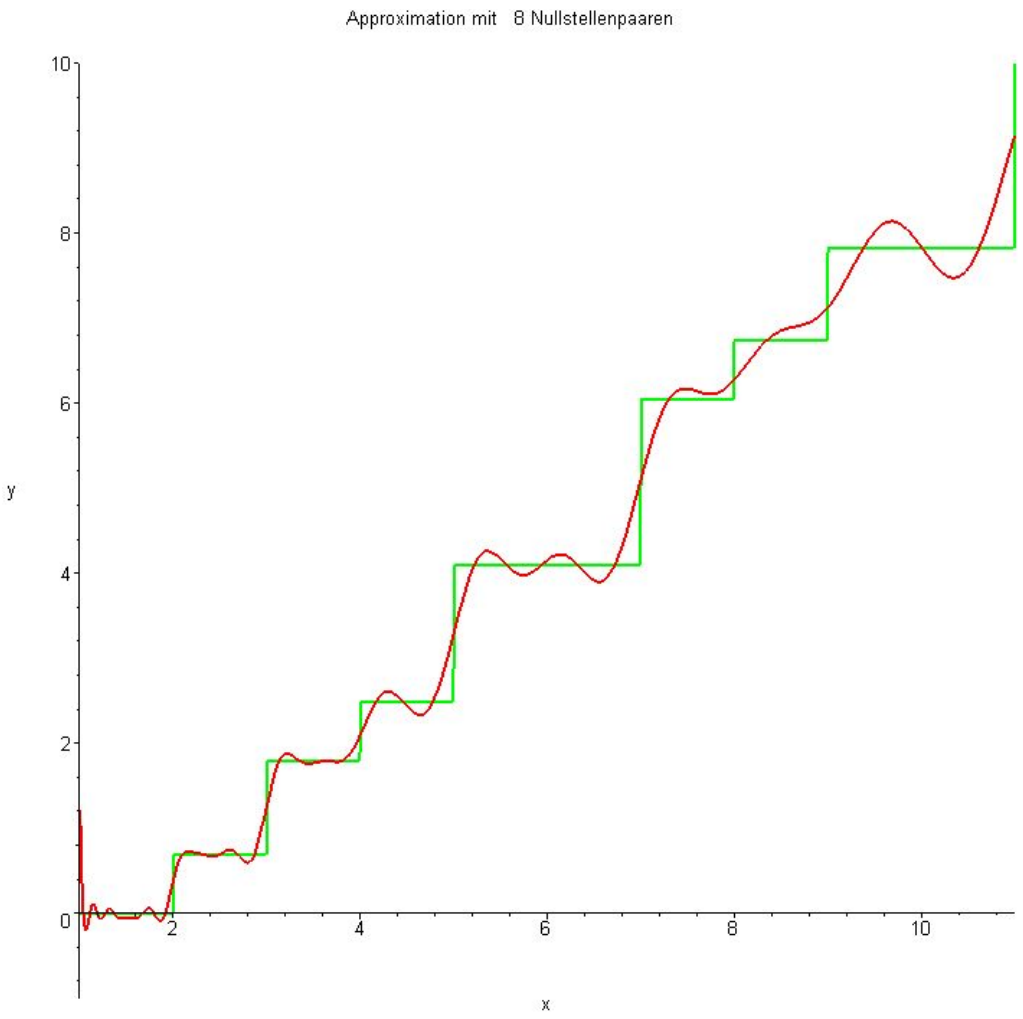
Stepwise Approximation of $\psi(x)$



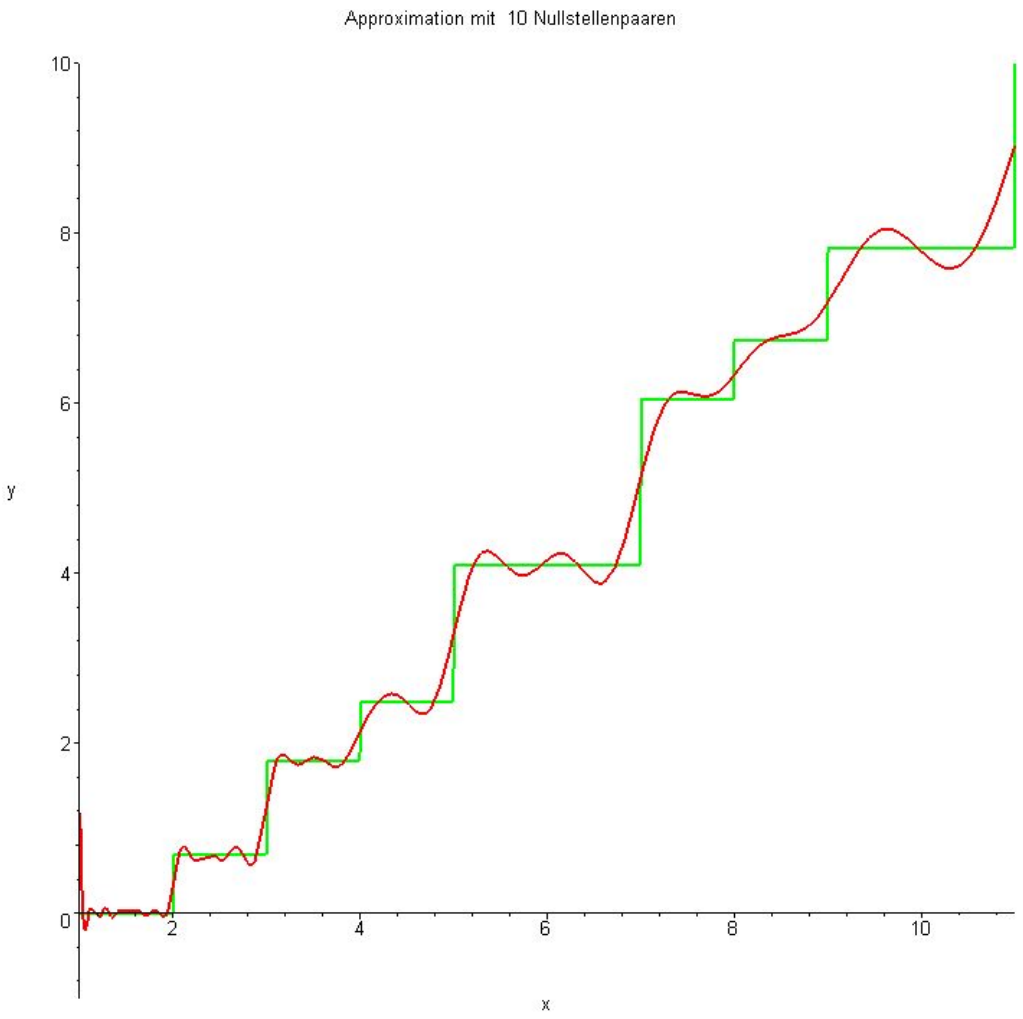
Stepwise Approximation of $\psi(x)$



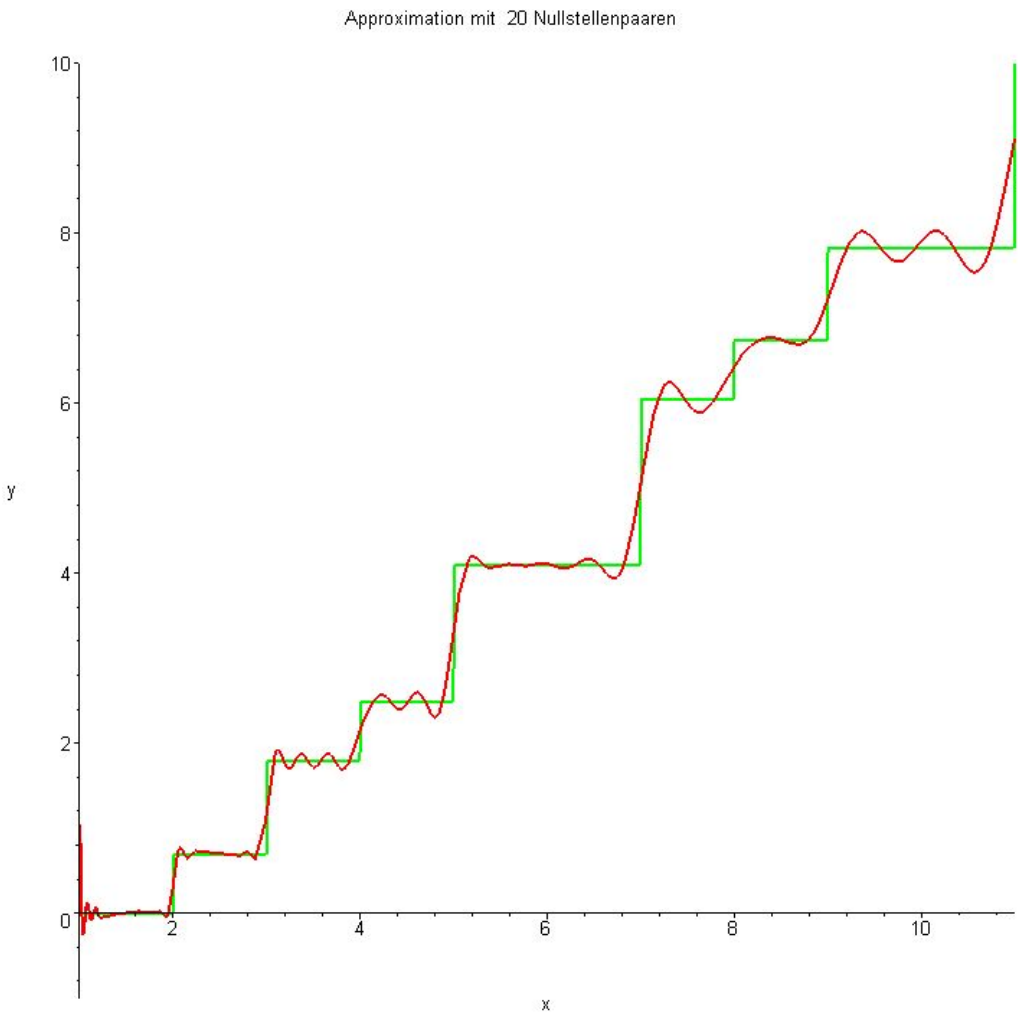
Stepwise Approximation of $\psi(x)$



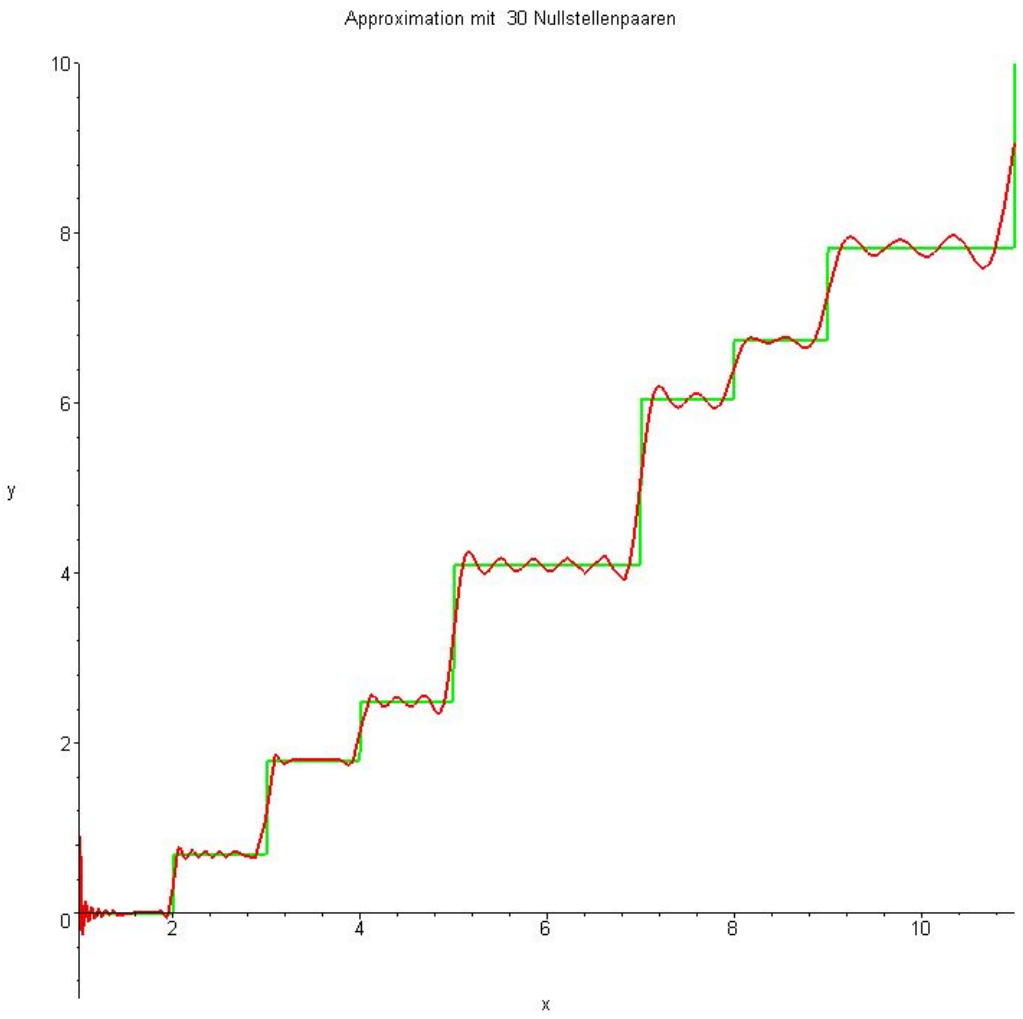
Stepwise Approximation of $\psi(x)$



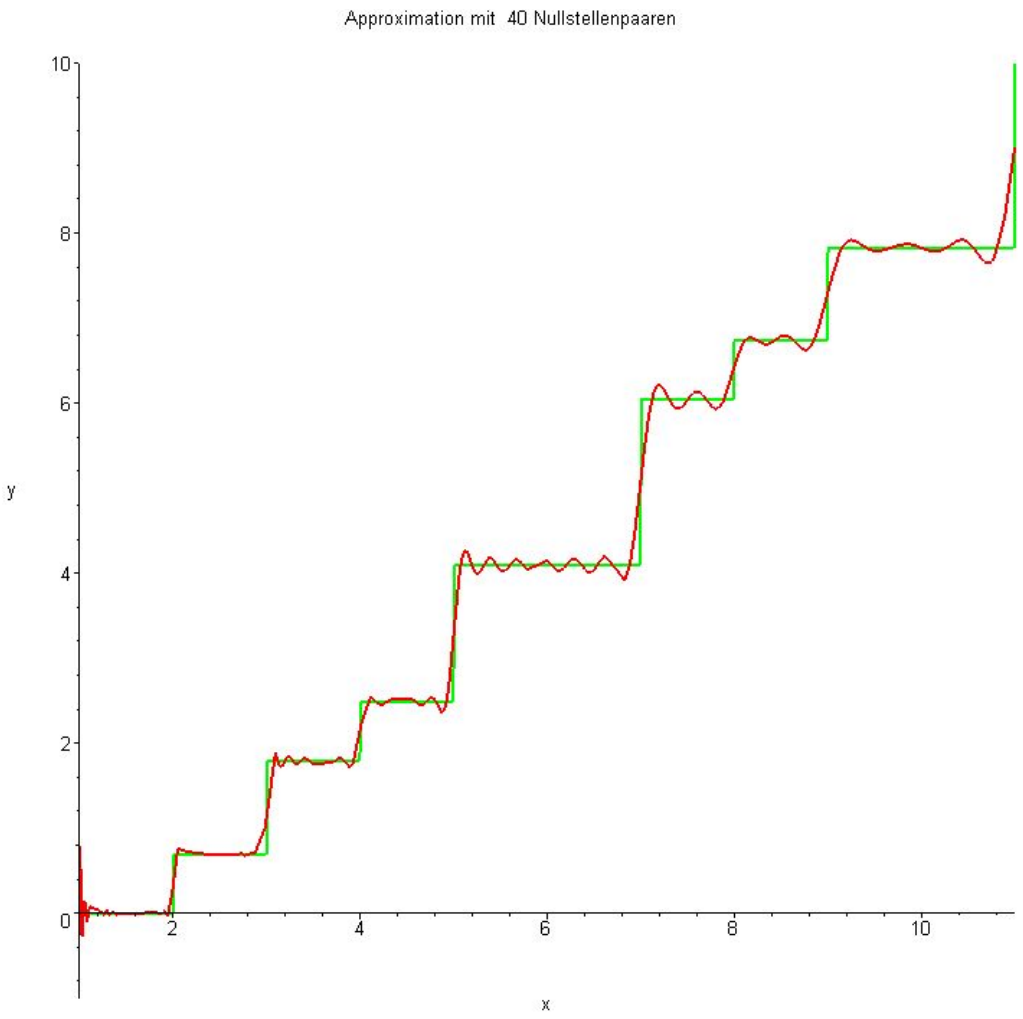
Stepwise Approximation of $\psi(x)$



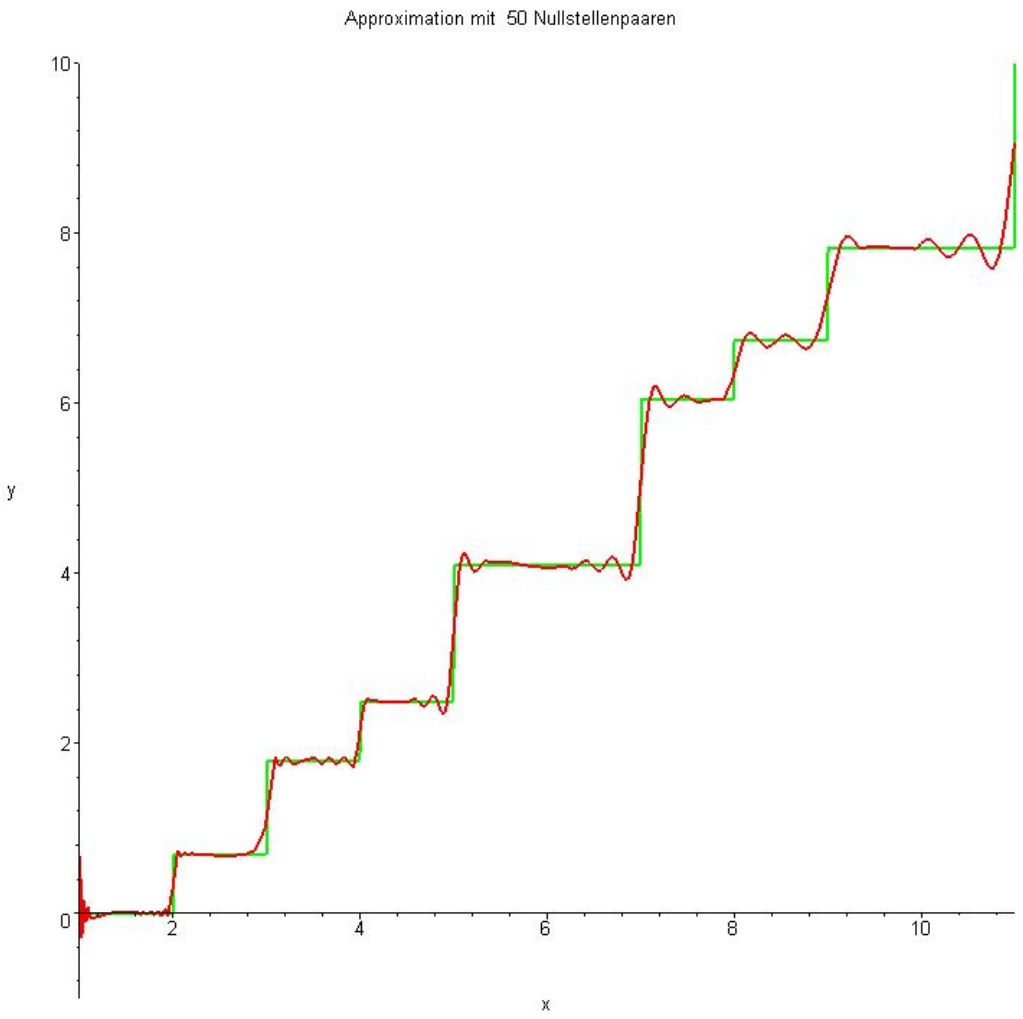
Stepwise Approximation of $\psi(x)$



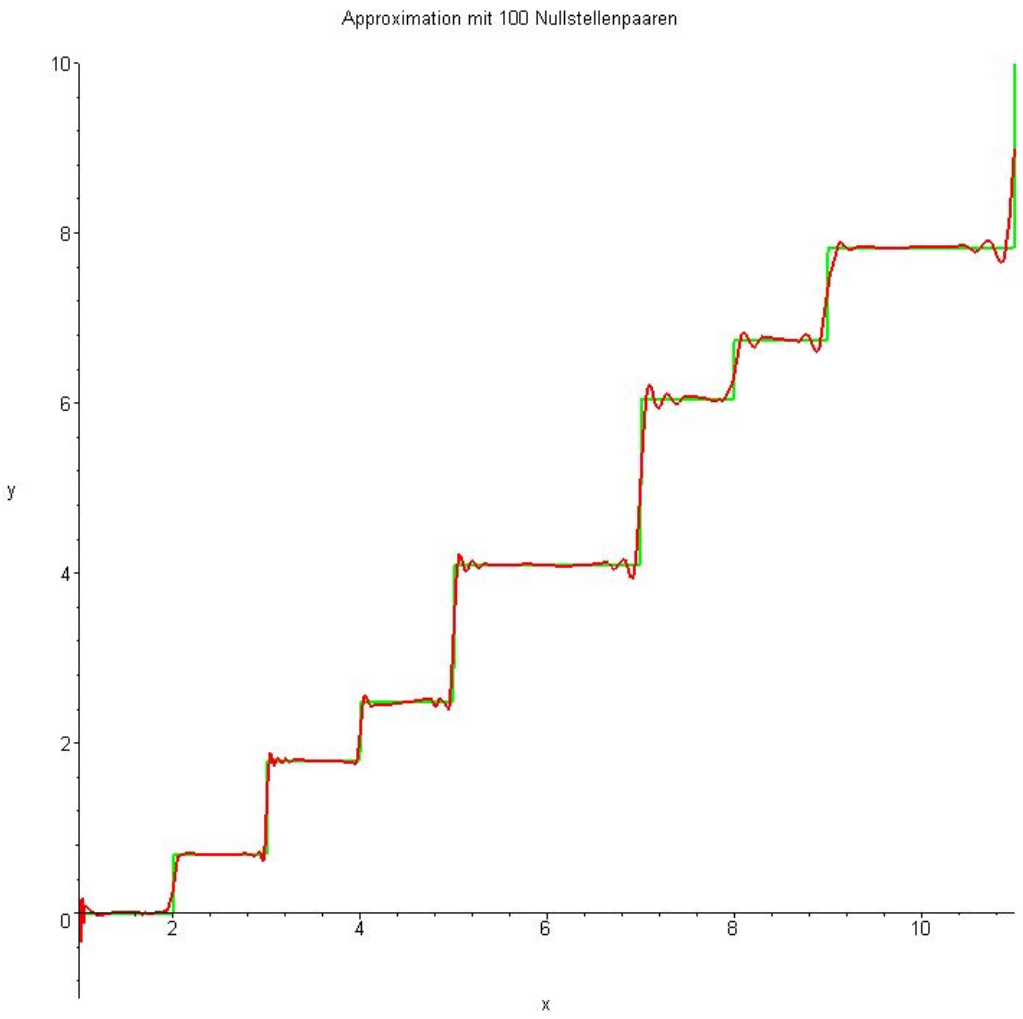
Stepwise Approximation of $\psi(x)$



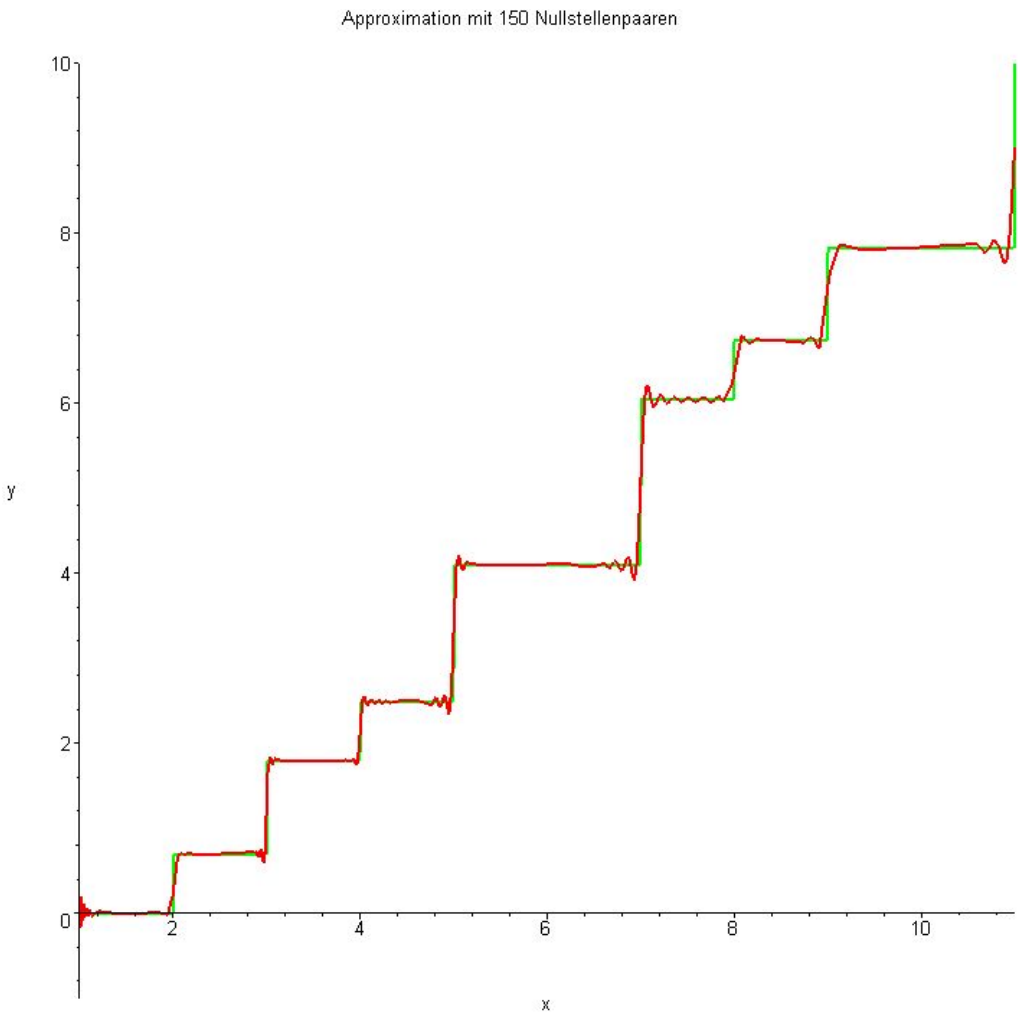
Stepwise Approximation of $\psi(x)$



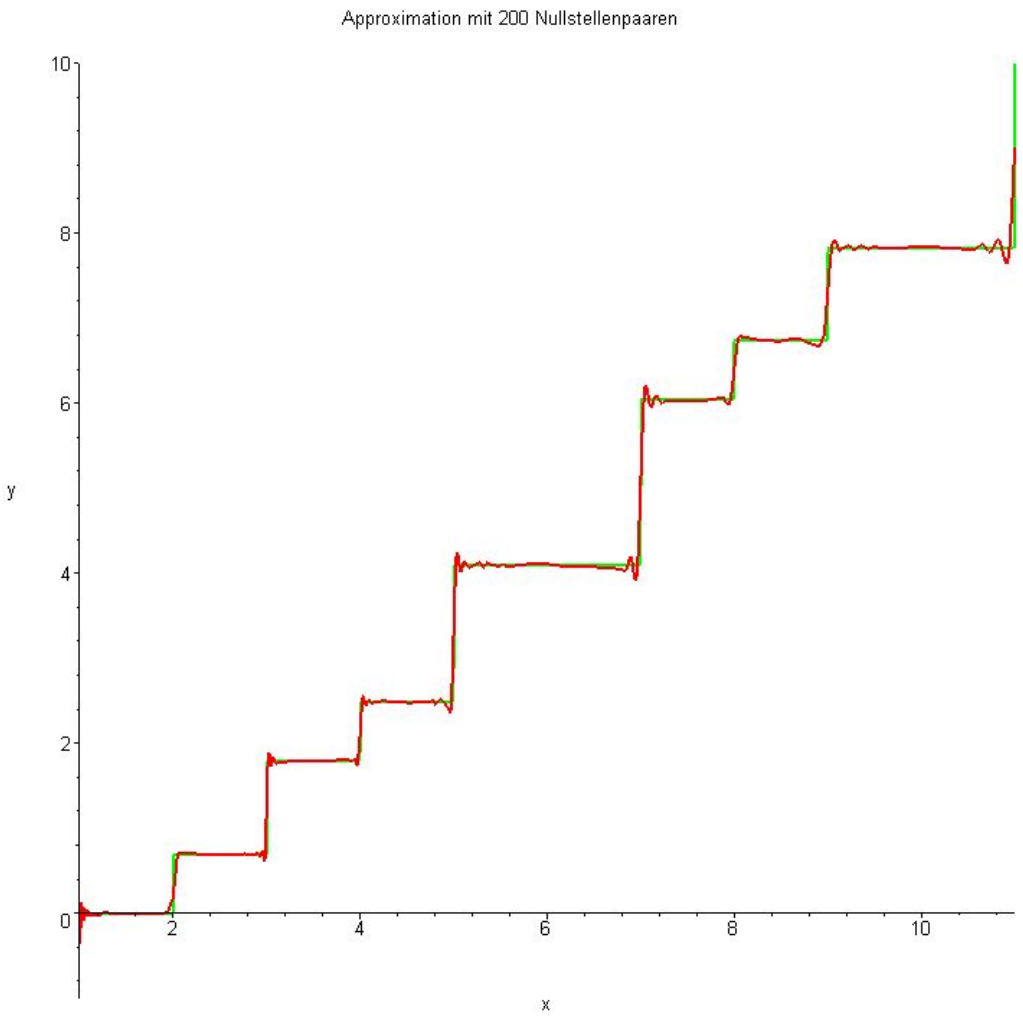
Stepwise Approximation of $\psi(x)$



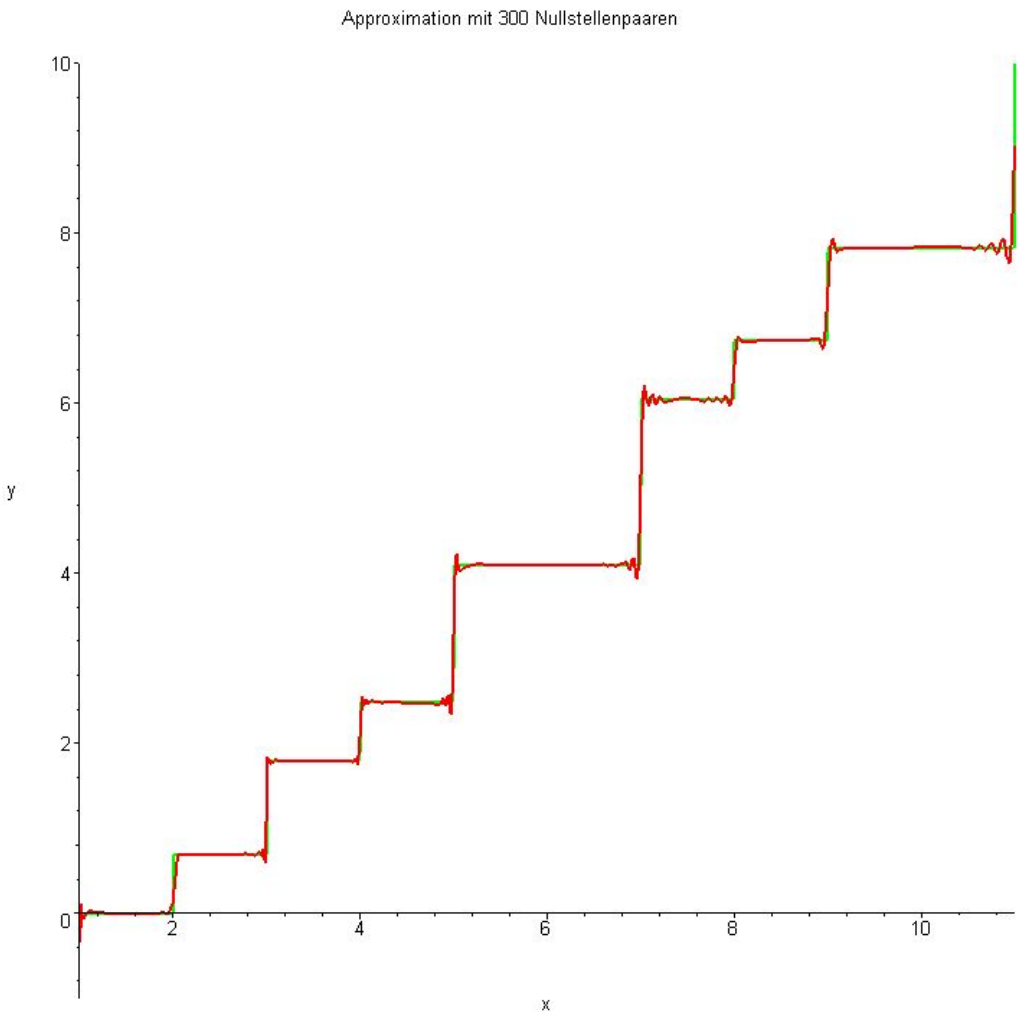
Stepwise Approximation of $\psi(x)$



Stepwise Approximation of $\psi(x)$



Stepwise Approximation of $\psi(x)$



The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann's conjecture has not been proved or disproved.
The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant C such that $\pi(x)$ does not deviate from $\mathrm{Li}(x)$ *by more than* $C \cdot \sqrt{x}\log(x)$.

The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann's conjecture has not been proved or disproved.

The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant C such that $\pi(x)$ does not deviate from $\mathrm{Li}(x)$ *by more than* $C \cdot \sqrt{x}\log(x)$.

The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann's conjecture has not been proved or disproved.

The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant C *such that* $\pi(x)$ *does not deviate from* $\mathrm{Li}(x)$ *by more than* $C \cdot \sqrt{x}\log(x)$.

The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann's conjecture has not been proved or disproved.

The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant C such that $\pi(x)$ does not deviate from $\mathrm{Li}(x)$ *by more than* $C \cdot \sqrt{x}\log(x)$.

The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann's conjecture has not been proved or disproved.

The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant C such that $\pi(x)$ does not deviate from $\mathrm{Li}(x)$ *by more than* $C \cdot \sqrt{x}\log(x)$.

The Riemann Hypothesis

"'. . . Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien. . . . "'

Riemann

Following Riemann's Pioneering Work . . .





. . . von Mangoldt . . . Hadamard . . . de la Vallée Poussin
. . . Hardy . . . Littlewood . . . Selberg . . . Montgomery . . .

The End

Many Thanks
for your kind attention

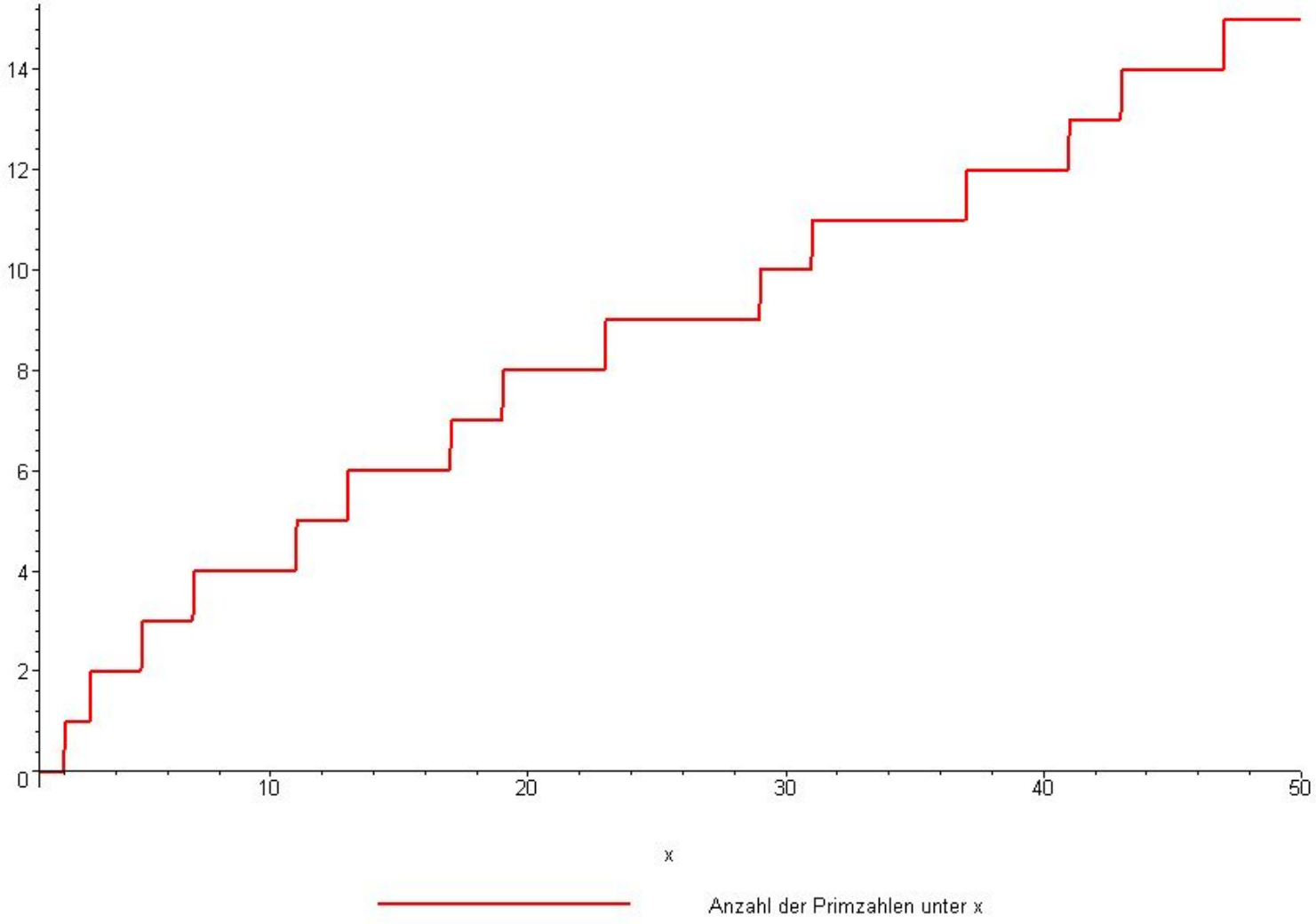
and credits to
Tobias Ebel
for the computer-technical assistance

Appendix with Pictures and Graphics

There are infinitely many prime numbers

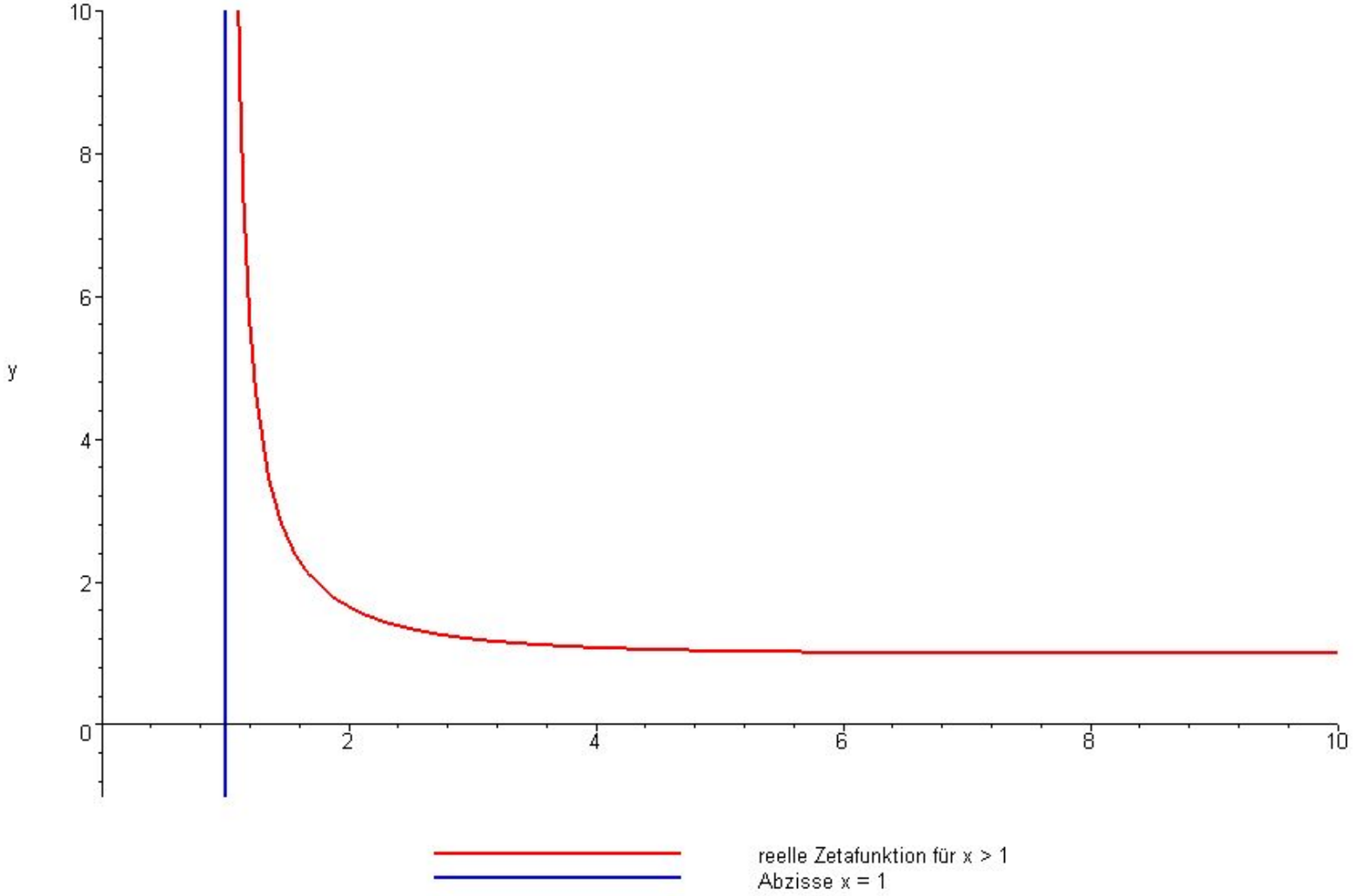
Number of Primes up to a Given Size

Number of Primes up to a Given Size

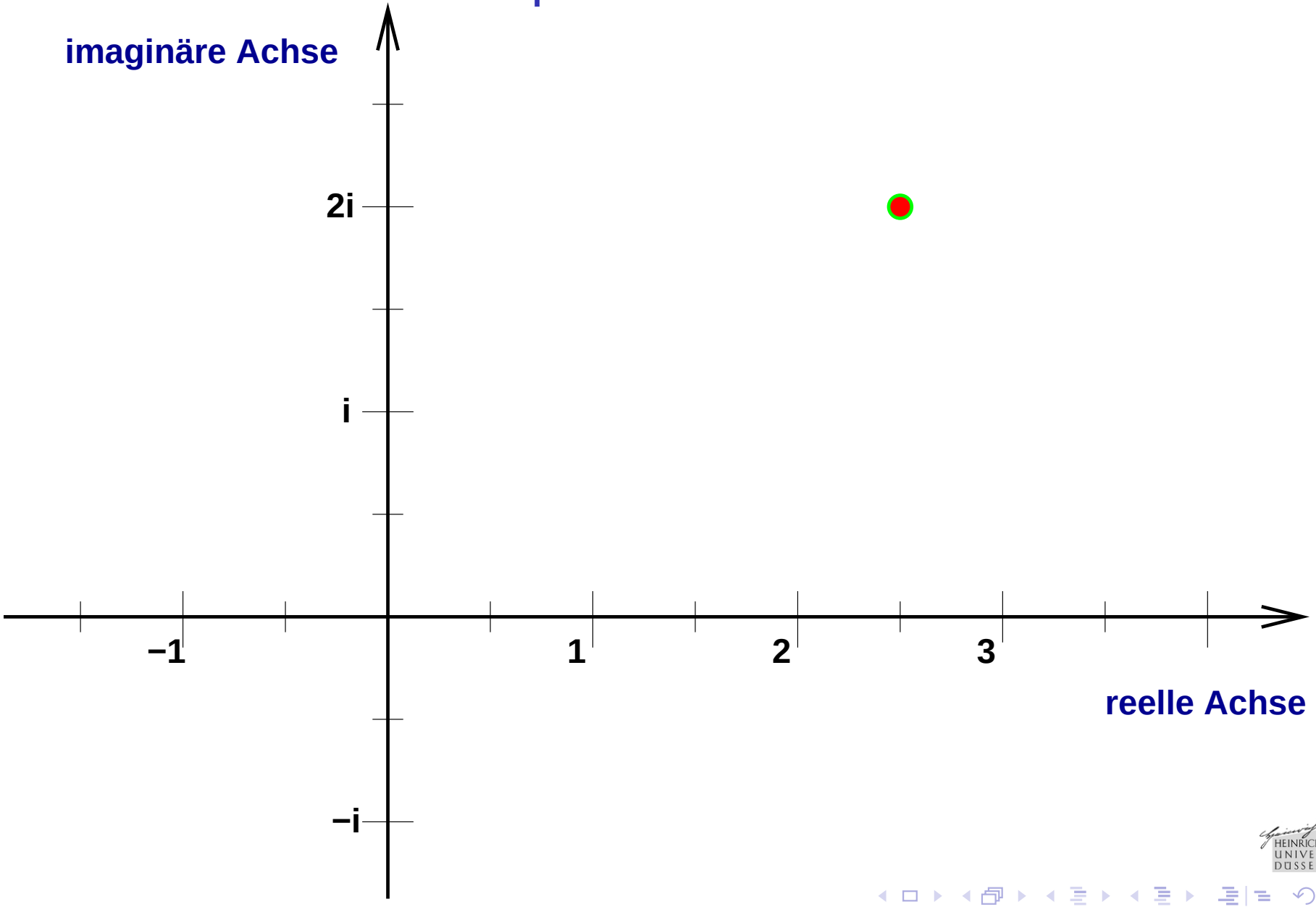


Euler and the ‘Real Zeta Function’

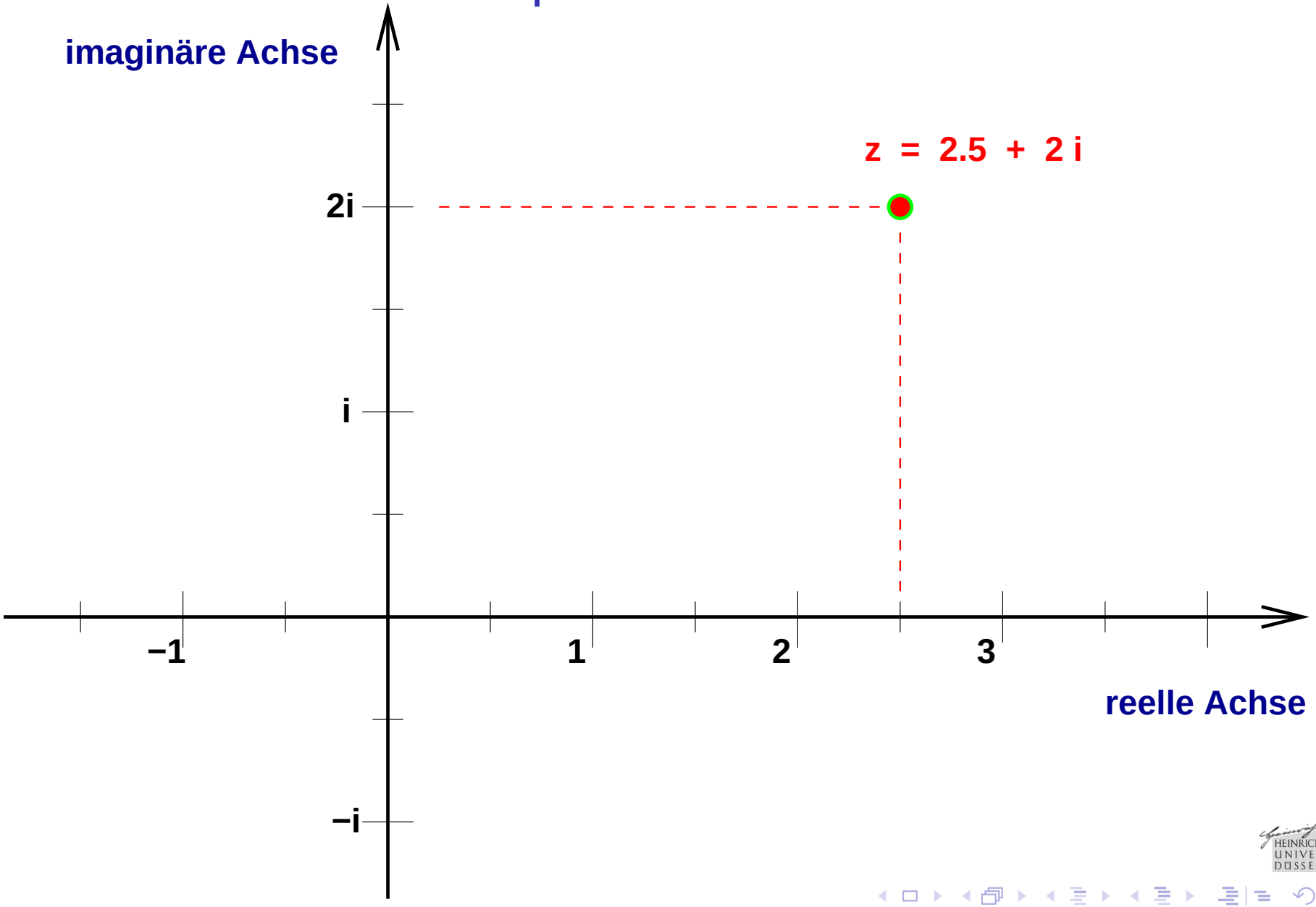
Euler and the ‘Real Zeta Function’



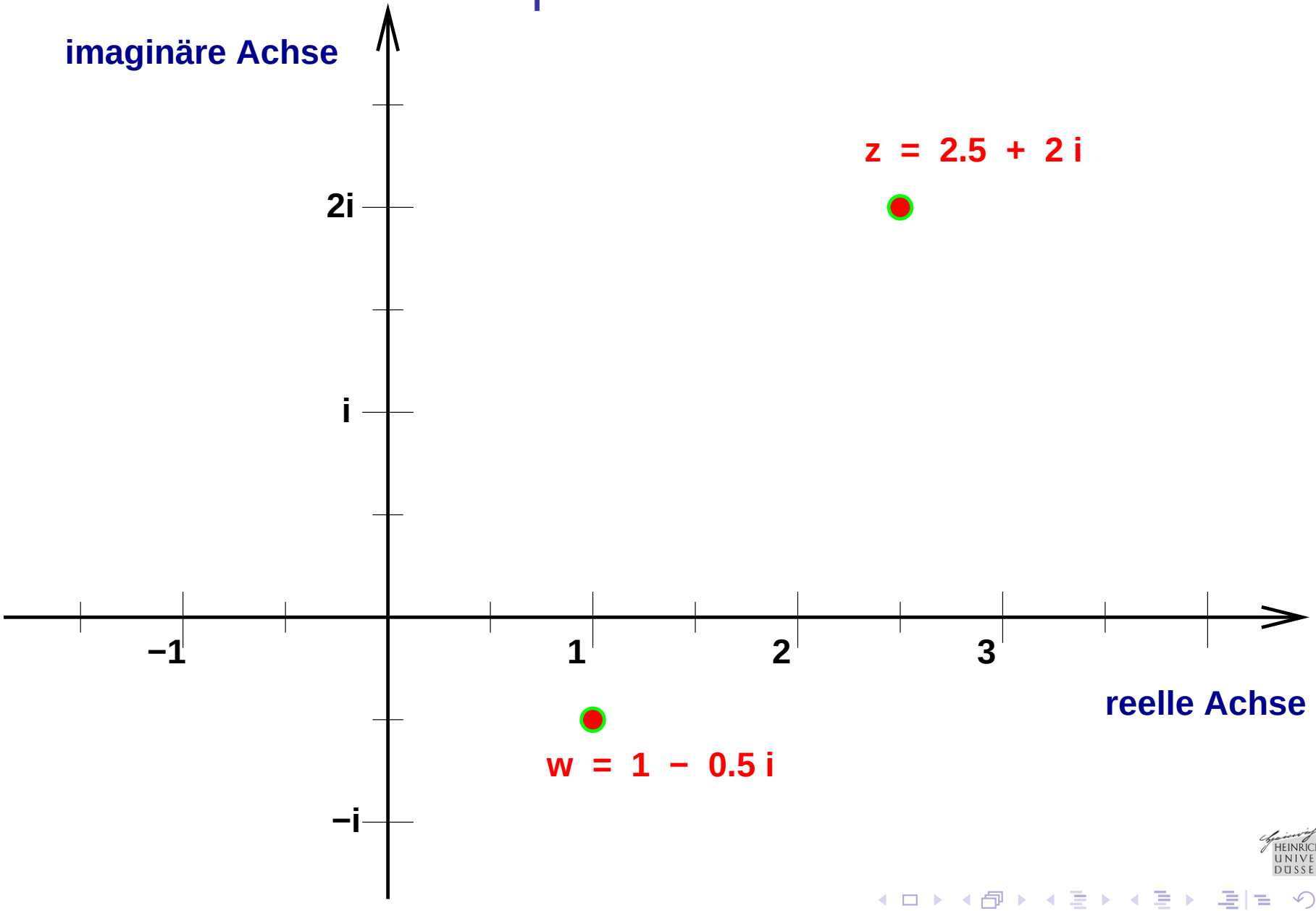
The Complex Number Plane



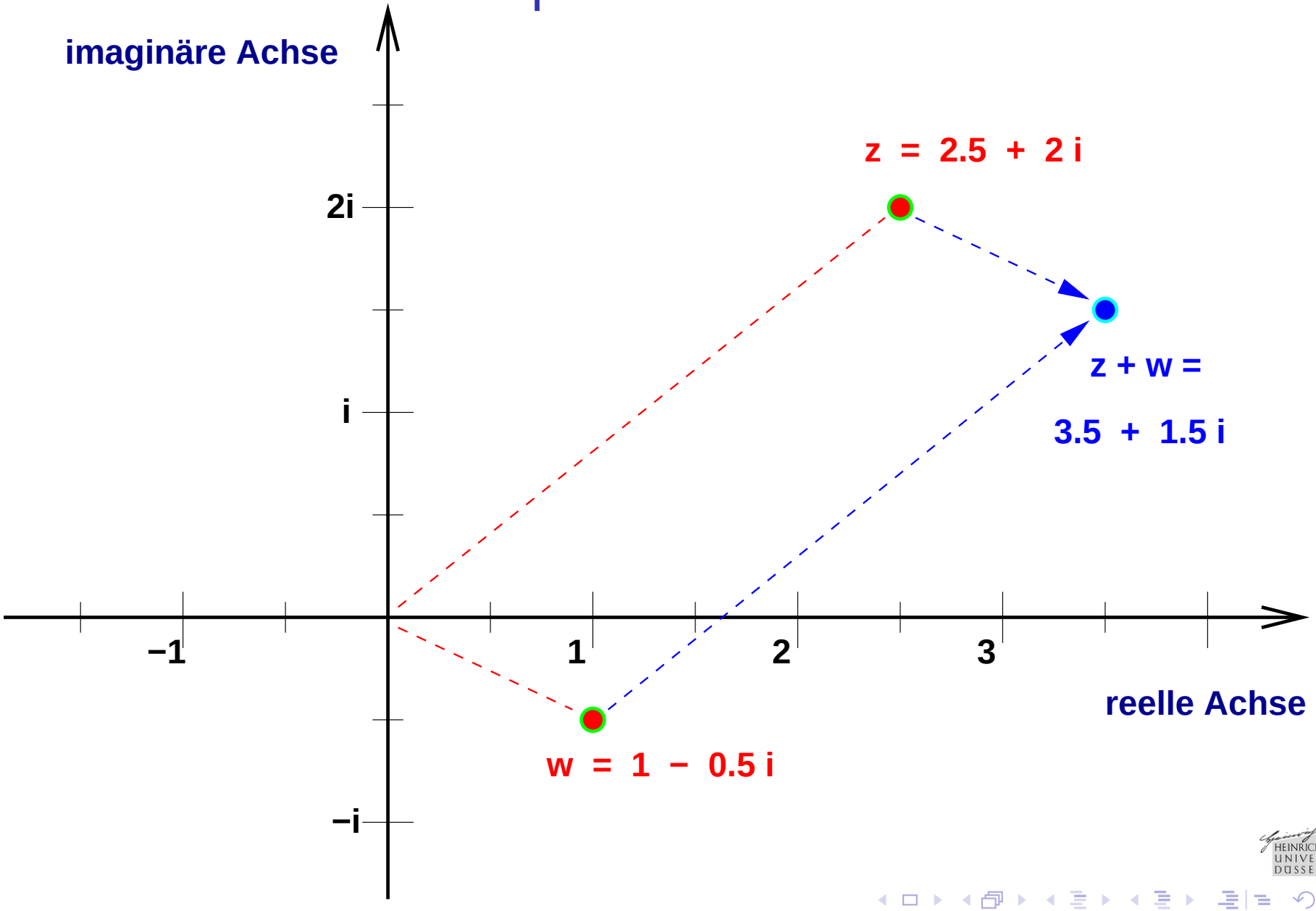
The Complex Number Plane



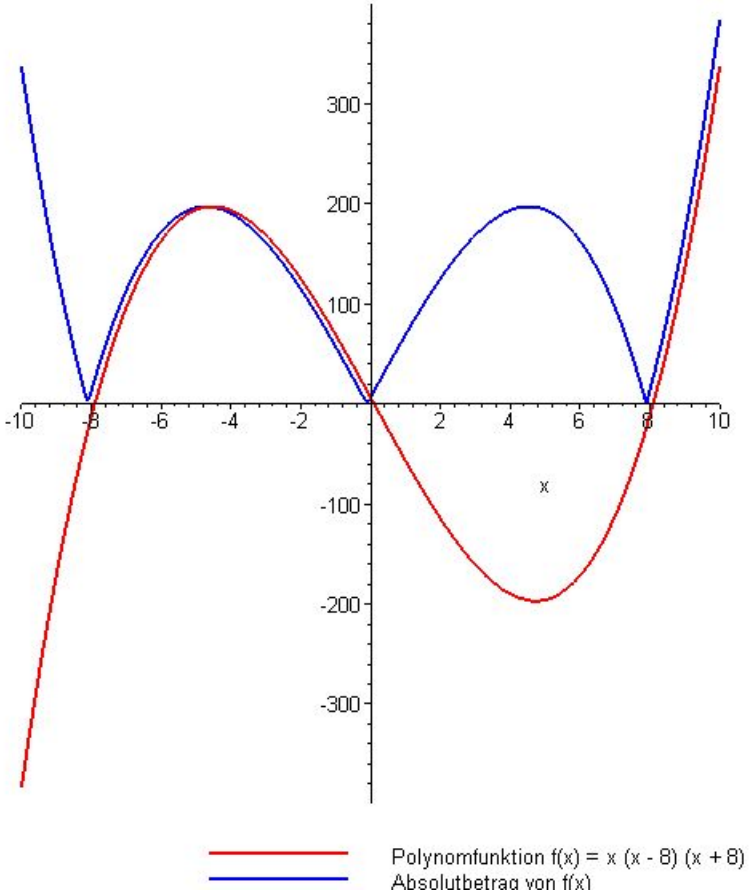
The Complex Number Plane

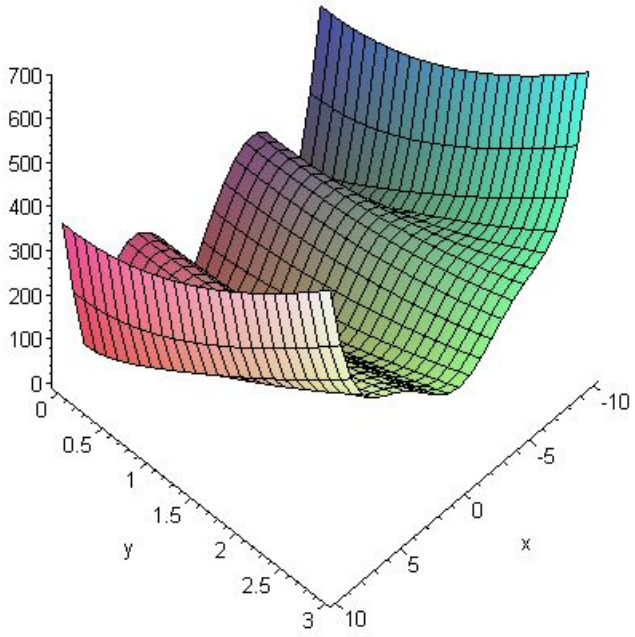


The Complex Number Plane

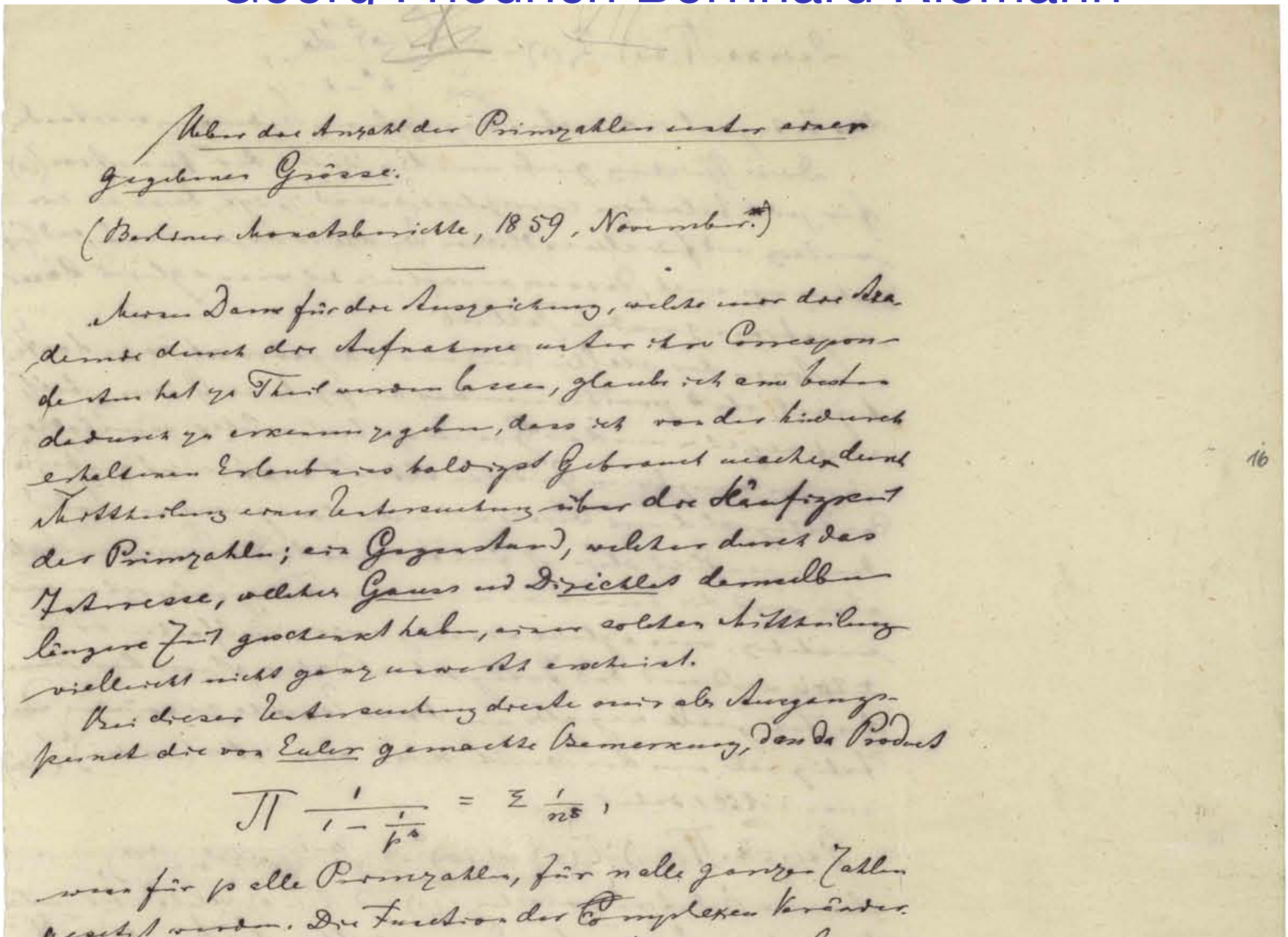


Complex Functions





Georg Friedrich Bernhard Riemann



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859, November.)

Wenn Dann für die Auszeichnung, welche unser der Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss bald zum Gebrauch machen durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit gewidmet haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

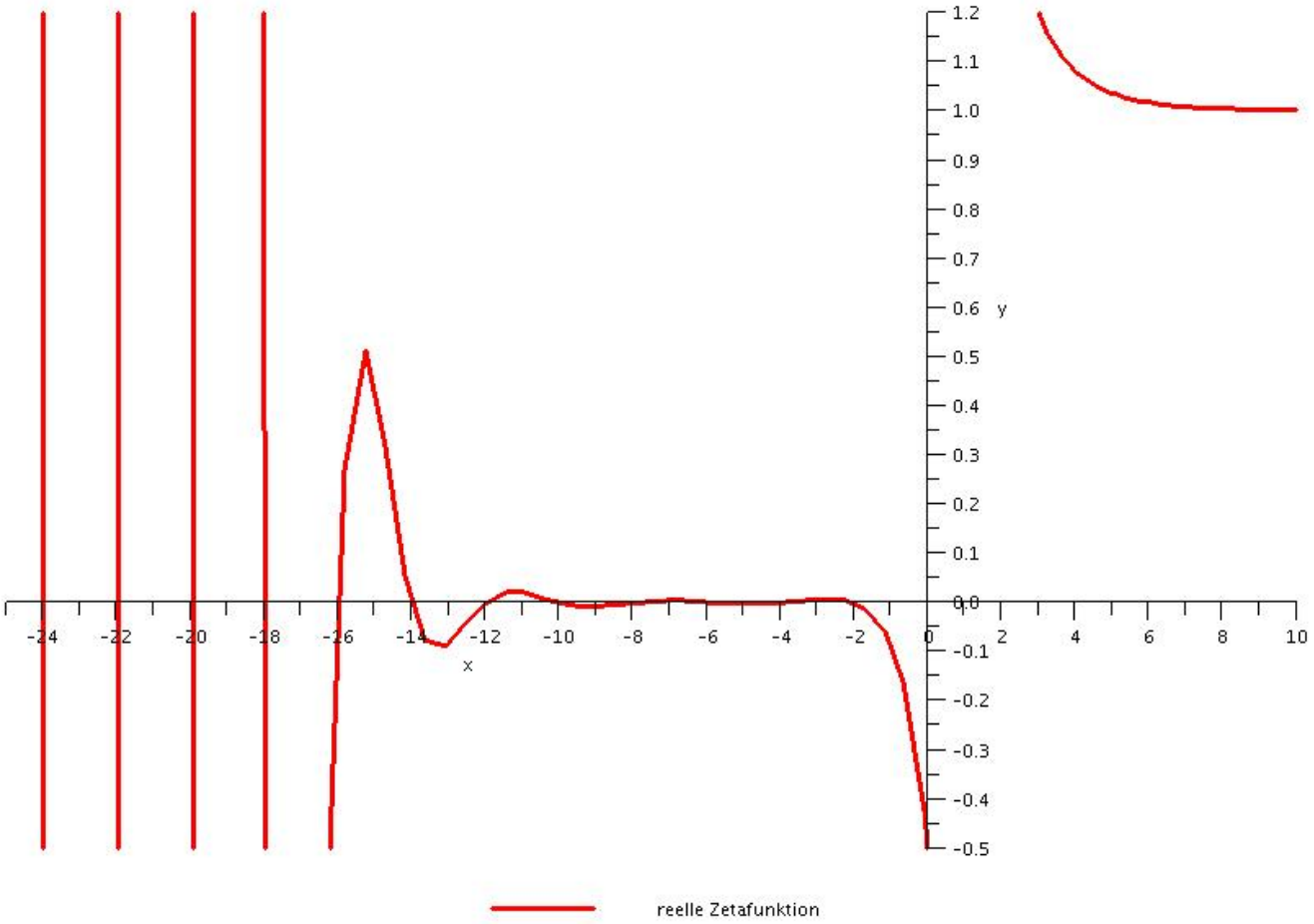
Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

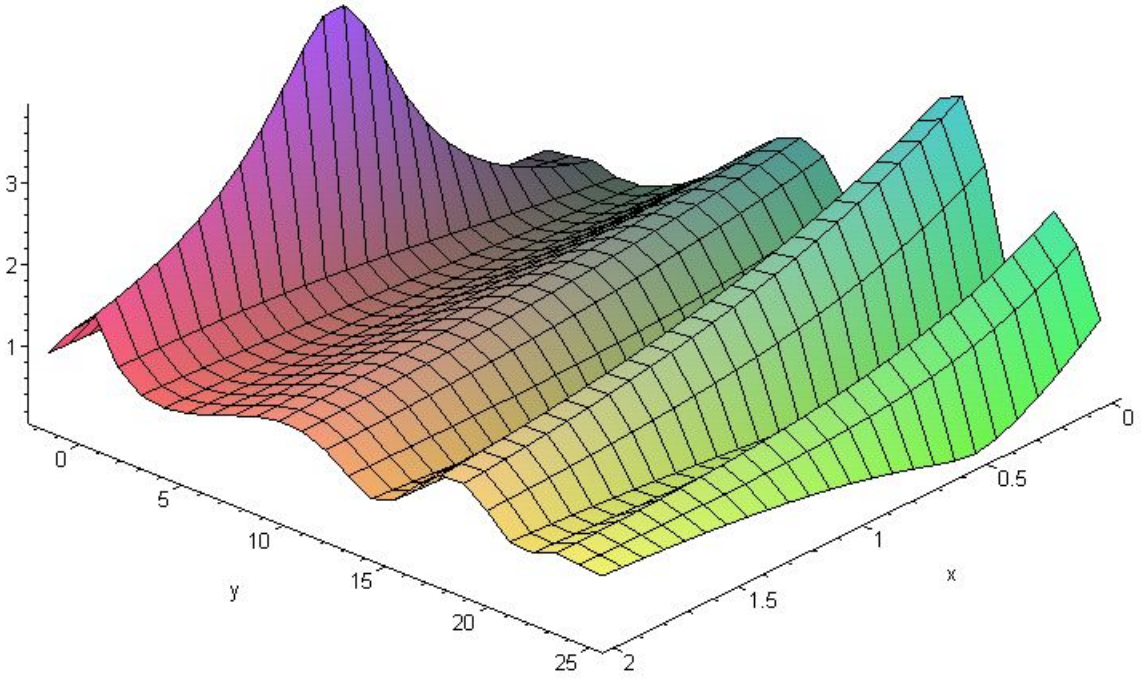
wenn für p alle Primzahlen, für n alle ganzen Zahlen genommen werden. Die Function der complexen Veränder-

16

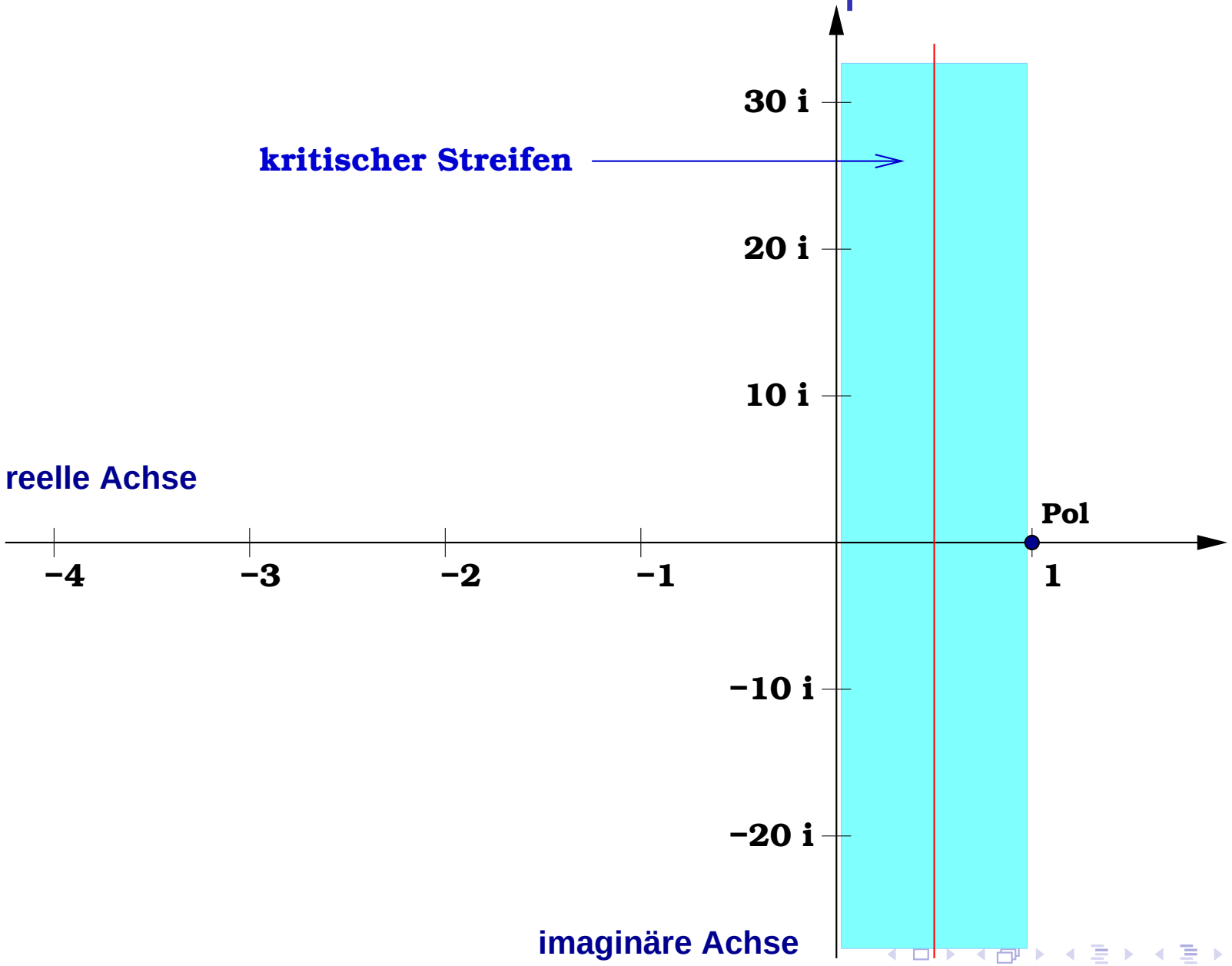
The Riemann Zeta Function



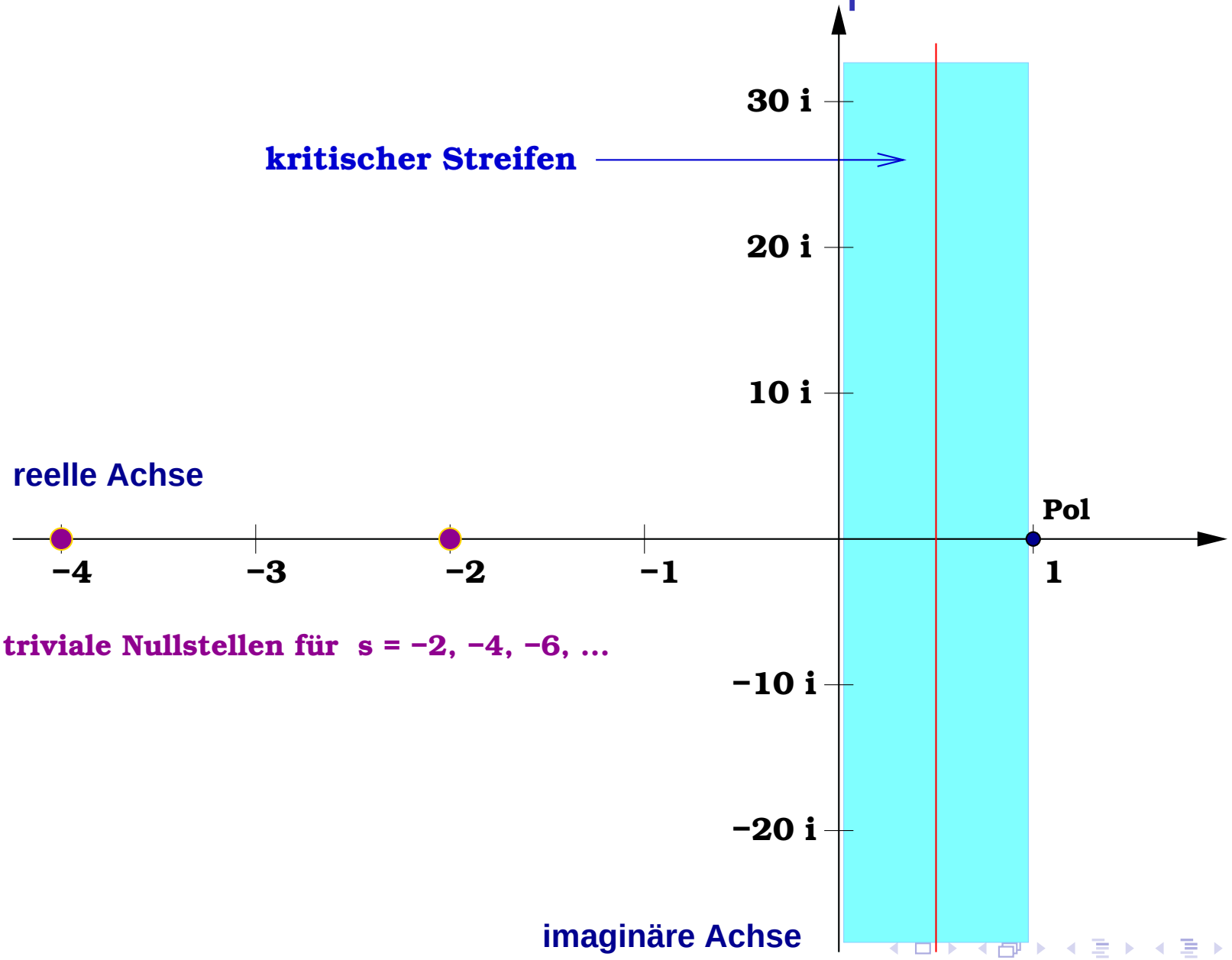
The Riemann Zeta Function



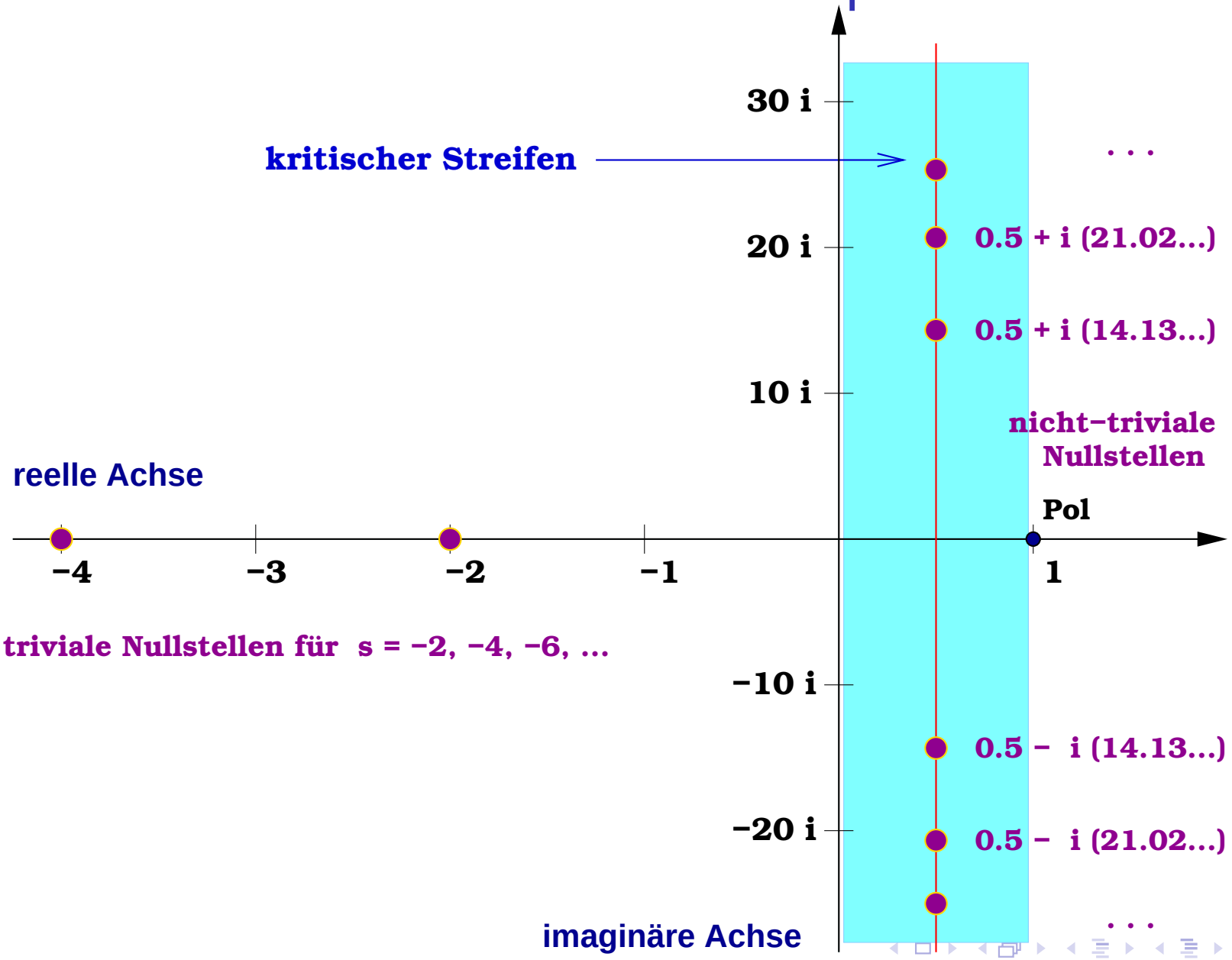
The Critical Strip



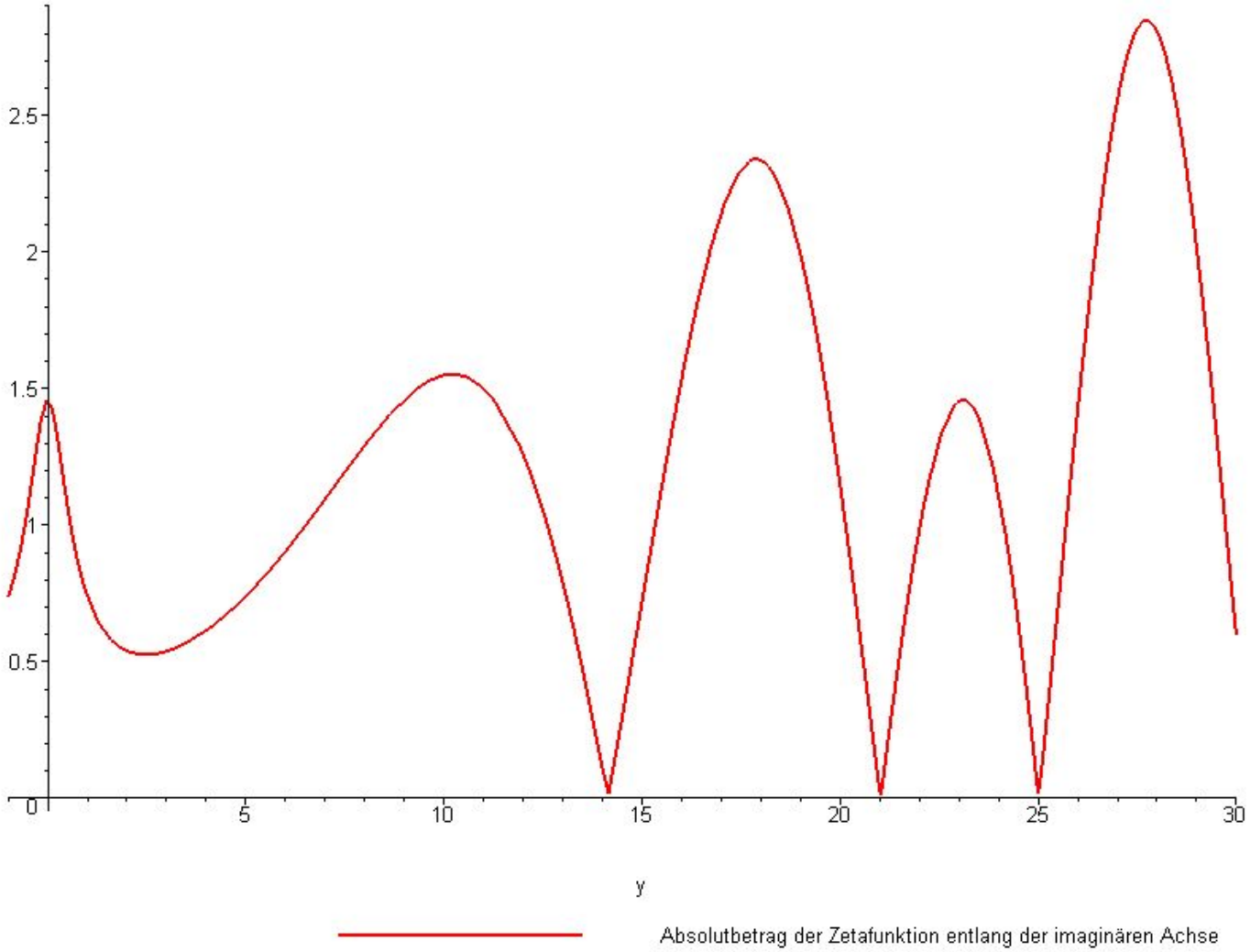
The Critical Strip



The Critical Strip



The Critical Strip



The Chebyshev ψ-Function