Prime Numbers and the Riemann Hypothesis

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Tag der Forschung  ○  November 2005


“Investigation of the Distribution of Prime Numbers”

Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

Bernhard Riemann

[Monatsberichte der Berliner Akademie, November 1859.]

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch die Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwürdig erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Produkt

$$\prod \frac{1}{1 - \frac{1}{p^s}} - \sum \frac{1}{n^s},$$

wenn für \( p \) alle Primzahlen, für \( n \) alle ganzen Zahlen gesetzt werden. Die Funktion der complexen Veränderlichen \( s \), welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch \( \zeta(s) \). Beide convergiren nur, so lange der reelle Theil von \( s \) größer als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx}x^{s-1}dx - \frac{\Pi(s-1)}{n^s},$$

erhält man zunächst

$$\Pi(s-1)\zeta(s) = \int_0^\infty \frac{x^{s-1}}{e^x-1} dx.$$
What will we encounter in the next 45 minutes?

The ‘Building Blocks of Numbers’
What are prime numbers?
How many prime numbers are there?

Riemann – a Pioneer in Complex Analysis
Infinite Series and Complex Functions
Riemann and the Zeta Function $\zeta(s)$

A ‘Wonderful Formula’
Riemann’s Formula
The Riemann Hypothesis
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What are prime numbers?

Definition
A prime number is a natural number $p$ admitting precisely two distinct divisors, namely 1 and $p$.

Thus the first seven prime numbers are 2, 3, 5, 7, 11, 13, 17.

Theorem (Fundamental Theorem of Arithmetic)
Every natural number can written uniquely as a product of prime numbers, up to re-ordering the factors.

For instance, the fairy tale number satisfies

$$1001 = 3^2 \cdot 11 \cdot 31.$$
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\[ 1001 = 3^2 \cdot 5 \cdot 19 \cdot 131. \]
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The Sieve of Eratosthenes

Among the first 100 numbers there are 25 prime numbers.
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25 26 27 28 29 30 31 32 33 34 35 36
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61 62 63 64 65 66 67 68 69 70 71 72
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85 86 87 88 89 90 91 92 93 94 95 96
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The Sieve of Erathosthenes

Among the first 100 numbers there are 25 prime numbers.

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There are infinitely many prime numbers

Theorem (Euclid, around 300 BC)

There are infinitely many prime numbers.

Proof.

- Assume, \( p_1 = 2, p_2 = 3, \ldots, p_r \) are all the prime numbers.
- Set \( N := p_1 \cdot p_2 \cdots p_r + 1 \).
- Then \( N \geq 2 \), but \( N \) is not divisible by any prime number.
- Contradiction!

Question: How many is “infinitely many”?
There are infinitely many prime numbers.

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For every real number \( x \) let \( \pi(x) \) denote the number of primes between 1 and \( x \).

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Number of Primes up to a Given Size

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The Logarithmic Integral Function

Observation: Viewed from a distance, the function $\pi(x)$ is surprisingly smooth. The ‘density’ of primes around a large number $x$ is about $1/\log(x)$.

Remark: $\log(x) \approx 2.3 \cdot (\# \text{ digits before the decimal point of } x)$. Thus one has approximately $\pi(x) \approx x/\log(x)$.

Gauß suggested the following logarithmic integral $\text{Li}(x)$ as a more precise approximation:

$$\text{Li}(x) := \int_2^x \frac{1}{\log(t)} dt \approx \frac{1}{\log(2)} + \frac{1}{\log(3)} + \ldots + \frac{1}{\log[x]}.$$
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Anzahl der Primzahlen unter $x$  
Integrallogarithmus $\text{Li}(x)$  
$x/\log(x)$
The Logarithmic Integral Function

Observation: Viewed from a distance, the function \( \pi(x) \) is surprisingly smooth. The 'density' of primes around a large number \( x \) is about \( \frac{1}{\log(x)} \).

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$$\text{Li}(x) := \int_{2}^{x} \frac{1}{\log(t)} \, dt \approx 1 \log(2) + 1 \log(3) + ... + 1 \log(\lfloor x \rfloor).$$
We build infinite series
We build infinite series

- The harmonic series diverges to infinity:

\[
\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots = \infty.
\]
We build infinite series

For

\[ \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) + \ldots \]
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$$\geq 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) + \ldots$$
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\[ = 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \ldots \]
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- In contrast, the sum of quadratic reciprocals

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = 1 + \frac{1}{4} + \frac{1}{9} + \ldots \]

is bounded above and converges.
We build infinite series

For

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2}
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For

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + \sum_{n=2}^{\infty} \frac{1}{(n - 1)n}
\]
We build infinite series

For

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\]

\[
= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots
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\[ = 1 + 1 + 0 + 0 + 0 + \ldots = 2. \]
We build infinite series

- The harmonic series diverges to infinity:
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is bounded above and converges.

**Question:** What is the sum of all the quadratic reciprocals?
Euler and the ‘Real Zeta Function’

One has

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449,$$

where $\pi = 3.1415\ldots$.

Euler considered more generally the real function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{for } s > 1$$

and he computed $\zeta(2m)$ for all even numbers 2, 4, 6, 8, $\ldots$
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The Euler Product Formula

The **Euler Product Formula** yields an interesting connection between the function \( \zeta(s) \) and prime numbers:

\[
\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \frac{1}{6^s} + \ldots \\
= \left(1 + \frac{1}{2^s} + \frac{1}{2^{2s}} + \frac{1}{2^{3s}} + \ldots\right) \cdot \left(1 + \frac{1}{3^s} + \frac{1}{3^{2s}} + \frac{1}{3^{3s}} + \ldots\right) \\
\cdot \left(1 + \frac{1}{5^s} + \frac{1}{5^{2s}} + \frac{1}{5^{3s}} + \ldots\right) \cdots \\
= \prod_{p \text{ prime}} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \ldots\right) \\
= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.
\]
The Complex Number Plane

The familiar real number line \( \mathbb{R} \) extends to the so-called complex number plane \( \mathbb{C} = \mathbb{R} + i\mathbb{R} \).

The complex number \( i \), called the imaginary unit, has the property that its square equals \( -1 \).

The basic arithmetic operations addition, subtraction, multiplication and division extend from the real numbers to the complex numbers.
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Complex Functions

Simple transformations of the complex number plane to itself, such as the polynomial function

\[ f(z) = f(x + iy) := z^3 - 64z, \]

can be visualised as landscapes.
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From Euler to Riemann

A ‘Wonderful Formula’

The End
The Riemann Zeta Function

The infinite series

\[ \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots \]

converges for all complex numbers \( s = x + yi \) with real part \( x > 1 \).
Moreover, the function \( \zeta(s) \), extends uniquely to the entire complex number plane, apart from a pole at \( s = 1 \).
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The Critical Strip

The ‘real’ function $\zeta(s)$ can be extended to the entire complex number plane by means of the Functional Equation

$$\Lambda(s) := \pi^{-s/2} \Gamma(s/2) \zeta(s) = \Lambda(1 - s).$$

Of particular importance are the zeros of $\Lambda(s)$. They correspond to the non-trivial zeros of $\zeta(s)$ and lie in the critical strip $S := \{x + yi \mid 0 \leq x \leq 1\}$.

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The Chebyshev $\psi$-Function

Recall: We are to show that, for large $x$, the number $\pi(x)$ of prime numbers less than or equal to $x$ is closely approximated by the logarithmic integral $\text{Li}(x)$.

More handy than the prime counting function $\pi(x) = \sum_{p \leq x} 1$ is a variant, introduced by Chebyshev,

$$\psi(x) := \sum_{p^k \leq x} \log(p),$$

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Loosely speaking, the approximation $\pi(x) \approx \text{Li}(x)$ is equivalent to $\psi(x) \approx x$. 
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The step function $\psi(x)$ can be expressed precisely in terms of the complex zeros of the Riemann zeta function:

$$\psi(x) = x - \log(2\pi) - \sum_{\rho} \frac{x^\rho}{\rho} - \frac{1}{2} \log \left(1 - \frac{1}{x^2}\right).$$

The following graphics visualise how the prime counting function $\psi(x)$ is stepwise approximated using the first 300 pairs of zeros of the Riemann zeta function.
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The Riemann Hypothesis

Riemann Hypothesis

All non-trivial zeros of the Riemann zeta function lie on the line $\mathcal{G} = \{x + yi \mid x = 1/2\}$.

To this day, Riemann’s conjecture has not been proved or disproved.

The Riemann Hypothesis is equivalent to the following assertion.

Equivalent Formulation of the Riemann Hypothesis

There is a constant $C$ such that $\pi(x)$ does not deviate from $\text{Li}(x)$ by more than $C \cdot \sqrt{x} \log(x)$. 
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"... Man findet nun in der That etwa so viel reelle Wurzeln innerhalb dieser Grenzen, und es ist sehr wahrscheinlich, dass alle Wurzeln reell sind. Hiervon wäre allerdings ein strenger Beweis zu wünschen; ich habe indess die Aufsuchung desselben nach einigen flüchtigen vergeblichen Versuchen vorläufig bei Seite gelassen, da er für den nächsten Zweck meiner Untersuchung entbehrlich schien. ..."
Following Riemann’s Pioneering Work . . .

...von Mangoldt ...Hadamard ...de la Vallée Poussin
...Hardy ...Littlewood ...Selberg ...Montgomery ...
The End

Many Thanks
for your kind attention

and credits to
Tobias Ebel
for the computer-technical assistance
Appendix with Pictures and Graphics
There are infinitely many prime numbers.
Number of Primes up to a Given Size

Definition
For every real number $x$, let $\pi(x)$ denote the number of primes between 1 and $x$.

$x/\pi(x) \approx 6.0, 8.1, 10.4, 12.7, 15.0, 17.4, \ldots$
Number of Primes up to a Given Size

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For every real number $x$, let $\pi(x)$ denote the number of primes between 1 and $x$. 

\[
\frac{x}{\pi(x)} \approx 6.
\]

\[
\frac{1000}{168} \approx 6.
\]

\[
\frac{10{,}000}{1{,}229} \approx 8.
\]

\[
\frac{100{,}000}{9{,}592} \approx 10.
\]

\[
\frac{1{,}000{,}000}{78{,}498} \approx 12.
\]

\[
\frac{1{,}000{,}000}{57{,}721} \approx 15.
\]
Euler and the ‘Real Zeta Function’

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449 \]

Euler considered more generally the real function \( \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \) for \( s > 1 \) and he computed \( \zeta(2m) \) for all even numbers 2, 4, 6, 8, …
Euler and the ‘Real Zeta Function’

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449, \]

where \( \pi \approx 3.1415 \ldots \)

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\[ z = 2.5 + 2i \]
\[ w = 1 - 0.5i \]

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$z = 2.5 + 2\mathrm{i}$

$w = 1 - 0.5\mathrm{i}$

$z + w = 3.5 + 1.5\mathrm{i}$

The basic arithmetic operations addition, subtraction, multiplication and division extend from the real numbers to the complex numbers.
Complex Functions

Polynomial function $f(x) = x (x - 8) (x + 8)$

Absolute value of $f(x)$
Über die Anzahl der Primzahlen unter einer gegebenen Größe.
(Berliner Correspondenz, 1859, November)


Bei dieser Einrückung diente mir als Ausage der von Euler gemachte Bemerkung, dass die Pseudo-

\[ \prod \frac{1}{1 - \frac{1}{\nu^k}} = \frac{\pi^2}{\nu^{2k}} \]

war für alle Primzahlen, für alle ganzen Zahlen gebrochen. Die Funktion der Complexen Verhältnis
The Riemann Zeta Function

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10 i
20 i
30 i
−10 i
−20 i
1
−1
−2
−3
−4

reelle Achse
imaginäre Achse
kritischer Streifen
Pol

Of particular importance are the zeros of $\Lambda(s)$. They correspond to the non-trivial zeros of $\zeta(s)$ and lie in the critical strip $S := \{x+i|0 \leq x \leq 1\}$. All known zeros of $\Lambda(s)$ lie on the line $G := \{x+i|\ x = 1/2\}$. 

0.5 + i (14.13...)
0.5 + i (21.02...)
0.5 − i (21.02...)
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. . .
. . .
Nullstellen
Pol
nicht−triviale
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triviale Nullstellen für $s = -2, -4, -6, ...$
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Absolutbetrag der Zetafunktion entlang der imaginären Achse
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