

# SUBSTITUTION GROUPS, SUBGROUP GROWTH AND OTHER TOPICS

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*A thesis submitted in partial fulfilment of the requirements  
for the degree of Doctor of Philosophy*

Jesus College, Oxford

Trinity Term 1999

## ABSTRACT

In the first and longest part of the thesis we study substitution groups of formal power series; the best known example of such a group is the so-called Nottingham group, a finitely generated pro- $p$  group with remarkable properties. Let  $R$  be a commutative ring with 1, and let  $\mathcal{N}(R)$  be the group of normalised formal power series over  $R$  under substitution. We investigate the connection between the ideal structure of  $R$  and the normal subgroup structure of  $\mathcal{N}(R)$ . In particular, we show that for every finite field  $K$  with  $\text{char}(K) \neq 2$  the Nottingham group  $\mathcal{N}(K)$  is just-infinite (as an abstract group). As a further application we consider the profinite completion of the group  $\mathcal{N}(R)$ . We show that, if  $R$  is saturated with ideals of finite index, then  $\widehat{\mathcal{N}(R)} \cong \mathcal{N}(\widehat{R})$ . For every finite field  $K$  with  $\text{char}(K) \geq 5$  we determine the automorphism group of the Nottingham group  $\mathcal{N}(K)$ , and we give a complete description of the conjugacy classes of elements of order  $p$ . A result of Camina shows that every countably based pro- $p$  group embeds as a closed subgroup into the Nottingham group  $\mathcal{N}(\mathbb{F}_p)$ . Despite this fact it seems very difficult to give an explicit description of closed subgroups of  $\mathcal{N}(\mathbb{F}_p)$  which are not pro-cyclic. We introduce the class of so-called index subgroups and study some of their properties. In particular, we determine which Hausdorff dimensions occur for index subgroups and prove that the Hausdorff spectrum of  $\mathcal{N}(\mathbb{F}_p)$  begins with an interval. Finally, we show that a small extension of Camina's original construction embeds countably based pro- $p$  groups directly into the Nottingham group.

In the second part of the thesis we present two independent results about subgroup growth. For every residually-finite virtually soluble minimax group we give upper and lower linear bounds for the degree of subgroup growth in terms of the Hirsch length. For every integer  $d > 1$  we construct a finitely generated metabelian group whose subgroup growth is approximately  $c^{n^{(d-1)/d}}$  for some constant  $c \in \mathbb{R}_{>1}$ .

In the third and final part of the thesis we put forward a precise definition of wreath products in the category of profinite groups and begin a systematic study of finitely generated pro- $p$  standard wreath products. In particular, we give a linearity criterion for such groups, and we construct finitely generated pro- $p$  groups with full Hausdorff spectrum. In the latter context we also touch upon profinite branch groups, certain profinite groups which act naturally on spherically homogeneous rooted trees.